UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Physics 221C **Quantum Field Theory** **Spring 2007**

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ASSIGNMENT #5
Due: Friday, May 11, 5pm in TA's mailbox

1. Srednicki 73.3.

2. Consider the two-loop contributions to the QCD beta function drawn in the file HW6.figs on the course home page. For each, write just the group theory factors, and show that (after manipulations) each can be written as the tree level factor T^a_{Rij} times a function of the various trace T and Casimir C invariants as we did at one loop. Let the gauge group and representation be arbitrary. In the vacuum polarization graph, the representation R' in the loop might be different from the one R on the external line.

3. a) In a $U(1)$ gauge theory we define a phase $U_P(x,y)$, where P is a path from y to x (the notation is redundant because P determines x and y – but it's convenient). Define

$$U_P(x,y) = \exp\left(ie \int_P A_\mu dx^\mu \right) \ ,$$

where the integral is from y to x. How does $U_P(x,y)$ transform under the gauge transformation $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \theta$? How should ϕ transform to make $\phi^\dagger(x) U_P(x,y)\phi(y)$ gauge invariant? How does $U_C(x,x)$ transform, where C is a closed path? How does $U_P(x,y)\phi(y) - \phi(x)$ transform? Evaluate the latter for $y = x + v$, where v is an infinitesimal vector and y is a straight line, to first order in v.

b) The non-Abelian version of same. Define the matrix $U_P(x,y)$ iteratively as follows. We require that $U_P(x,y) = U_{P_2}(x,z)U_{P_1}(z,y)$, where P is the path defined by going first along P_1 to z and then along P_2 to x; the order of matrices in the product matters. Define also that for $x = y + v$ and P a straight line,

$$U_P(x,y) = I + igv^\mu A^a_\mu(x) T^a_R + O(v^2) \ ,$$

for arbitrary gauge group and representation (we should probably put a label R on U_P). This enables us to define $U_P(x,y)$ for any path, by breaking it up into short segments and using the product rule. Show that under $A^a_\mu T^a_R \to U A^a_\mu T^a_R U^\dagger + (i/g) U \partial_\mu U^\dagger$, $U_P(x+v,y)$ has a gauge transformation analogous to that found in part (a). Using this, how does $U_P(x,y)$ transform for a general path? How does $\mathrm{Tr}(U_C(x,y))$ transform for a closed path?

c) The non-Abelian phase can be written formally as

$$U_P(x,y) = \mathrm{P}\exp\left(ig\int_P dx^\mu A^a_\mu T^a_R\right)$$

where the P denotes *path-ordered product.* To understand this, use the definitions in part (b) to expand $U_P(x,y)$ through second order in A.

4. A baby version of BRST cohomology. Consider a system with six states, $|\mu\rangle$ for $\mu = 0,1,2,3$, $|c\rangle$, and $|\bar{c}\rangle$. Let

$$Q|\mu\rangle = k_\mu|c\rangle\ , \quad Q|c\rangle = 0\ , \quad Q|\bar{c}\rangle = k^\mu|\mu\rangle \ \ \text{(summed on } \mu\text{)}\ .$$

What is the condition on k such that $Q^2 = 0$? Take a general linear combination $|\psi\rangle$ of the states, so there are six parameters. What conditions follow from $Q|\psi\rangle = 0\rangle$? If we define an equivalence relation $|\psi\rangle \cong |\psi\rangle + Q|\chi\rangle$ for arbitrary χ, how many additional parameters can be set to zero? How many are left? (It might be useful to choose a frame for k^μ).