UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

**Physics 221C** **Quantum Field Theory** **Spring 2007**

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`http://www.kitp.ucsb.edu/~joep/W221C/221C.html`

# FINAL EXAM

**Open notes (including Coleman), homework, solutions, Srednicki (the text, not the person), Peskin & Schroeder, Weinberg. Please do not discuss the test with anyone but me before the due time. I will check my email regularly, and post any corrections/clarifications on the course web page.**

Two pages, four problems.

Begin: Wednesday, June 13, noon.
Due: Thursday, June 14, noon in my office, KITP 2319.

1. (20 pts) Consider the Higgs sector of the Standard Model with a real scalar field $\chi^a$ in the $SU(2)\times U(1)$ representation $(3,0)$, in addition to the usual Higgs doublet $\varphi$. The Higgs potential is

$$V(\varphi,\chi) = \frac{1}{4}\lambda(\varphi^\dagger\varphi)^2 + \frac{1}{2}\mu^2\varphi^\dagger\varphi + g'\varphi^\dagger\sigma^a\varphi\chi^a + \frac{1}{2}M^2\chi^a\chi^a,$$

where $\sigma^a$ are the Pauli matrices as usual, and $\mu^2 < 0$, $M^2 > 0$. (I have left out some terms that would be required for renormalization but would make the problem here messier). The scalar fields couple to the $SU(2)\times U(1)$ gauge fields through the usual covariant derivatives in the kinetic terms.

a) Find the minimum of the potential (for simplicity you can rotate $\varphi$ into the usual direction). What is the unbroken symmetry? (Note: if $g'$ gets too big the potential is unbounded below; don't consider this case).

b) Find the masses of the gauge bosons. If we require that the ratio $M_W/M_Z$ not change from its Standard Model value by more than $10^{-3}$, and assume that $g' = g_2$, what bound does this imply on the mass $M$ (express in GeV)?

2. (20 pts) Find the propagator of the photon in the gauge $A^3 = 0$. Writing the propagator as $-iM_{\mu\nu}(k)/(k^2 - i\epsilon)$, what are the eigenvalues of $M_{\mu\nu}$ when $k^2 = 0$? (Assume that $k^\mu$ is not pointed exactly along the 3-direction).

3. (30 pts) This problem is in *two* spacetime dimensions. For a massless Dirac spinor with

Lagrangian density $i\bar{\psi}\gamma^\mu\partial_\mu\psi$, the energy-momentum tensor is

$$T_{\mu\nu} = \frac{i}{2}\left\{\bar{\psi}\gamma_\mu\partial_\nu\psi - (\partial_\nu\bar{\psi})\gamma_\mu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma_\nu\psi\right\} \ .$$

a) Calculate the expectation value

$$\langle 0|j_\rho(k)T_{\mu\nu}(k')|0\rangle \ ,$$

where $j_\rho = \bar{\psi}\gamma_\rho\psi$. (This is a one loop graph - do it in dimensional regularization).

b) If we add to the action the term $\int d^2x\, h_{\mu\nu}(x)T^{\mu\nu}(x)$ (that is, we couple the energy-momentum tensor to some fixed background field), use your result from part (a) to calculate

$$\langle 0|j_\rho(k)|0\rangle_h \ .$$

That is, the expectaction value with $h$ turned on, to first order in $h$ only.

c) Use the relation between $j_\rho$ and $j_{5\rho} = \bar{\psi}\gamma_\rho\gamma_5\psi$ to obtain $\langle 0|j_{\rho 5}(k)|0\rangle_h$.

d) Calculate the divergences $k^\rho\langle 0|j_\rho(k)|0\rangle_h$ and $k^\rho\langle 0|j_{5\rho}(k)|0\rangle_h$ using the results of parts b and c.

4. (30 pts) Let $\phi_{ij}$ be an $N \times N$ traceless Hermitean matrix, with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) - g\mathrm{Tr}(\phi^4) - g'\mathrm{Tr}(\phi^2)\mathrm{Tr}(\phi^2) \ . \qquad (1)$$

I'll give you the propagator, since I already more-or-less gave it in class:

$$\langle 0|\phi_{ij}(k)\phi_{kl}(k')|0\rangle = (2\pi)^4\delta(k+k')\frac{-i}{k^2 - i\epsilon}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

Do *not* assume that $N$ is large.

a) Calculate the one loop beta functions for $g$ and $g'$. Notice that I have not included any symmetry factors in the action: you will have to figure out what is needed (you are welcome to redefine $g$ and $g'$ by numerical factors if you find it convenient, but be sure that you tell me what you are doing). This problem is a bit challenging to organize, do the best you can.

b) Define new couplings $\lambda$ and $\lambda'$ by $g = \lambda N^a$ and $g' = \lambda' N'^b$: what are the largest values of $a$ and $b$ such that the one loop renormalization group equations, written in terms of $\lambda$ and $\lambda'$, have a well-behaved large-$N$ limit? Write the RG equations in this limit.