UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Physics 221C **Quantum Field Theory** **Spring 2007**

http://www.kitp.ucsb.edu/~joep/W221C/221C.html

FINAL EXAM SOLUTIONS

Mean (of 5 scores) = 55

1. Use an $SU(2)$ rotation to make $\phi_2 = 0$ and ϕ_1 real and positive. The potential becomes

$$V(\varphi,\chi) = \frac{1}{4}\lambda\varphi_1^4 + \frac{1}{2}\mu\varphi_1^2 + g'\varphi_1^2\chi_3 + \frac{1}{2}M^2\chi_a\chi_a \ .$$

Now minimize with respect to ϕ_a to get $\chi_1 = \chi_2 = 0$, $\chi_3 = -g'\varphi_1^2/M^2$. Inserting this into the potential gives

$$V = \frac{1}{4}\lambda'\varphi_1^4 + \frac{1}{2}\mu\varphi_1^2 \ , \quad \lambda' = \lambda - 2g'^2/M^2 \ .$$

The minimum is therefore at $\varphi_1^2 = -\mu^2/\lambda'$, call this $v^2/2$ as in the usual case. The expectation value of φ is invariant under $U(1)_{\text{EM}} = T^3 + Y$ as in the usual case, and the expectation value of χ^a is invariant under both T^3 and Y, so again $U(1)_{\text{EM}}$ is unbroken.

The vector mass term is

$$\varphi^\dagger(g_2 A_\mu^a\sigma^a/2 - g_1 B_\mu/2)^2\varphi + \frac{1}{2}g_2^2 A_\mu^a A^{\mu b}\chi_c T^a_{cd}T^b_{de}\chi_e$$

The first term is exactly as in the Standard Model and becomes $(g_2 v/2)^2(W^{+\mu}W^-_\mu + \frac{1}{2}Z_\mu Z^\mu/\cos^2\theta_w)$, with $\cos^2\theta_w = g_2^2/(g_1^2+g_2^2)$. In the second term, $\chi_c T^a_{cd}T^b_{de}\chi_e = \chi_3^2$ for $a = b = 1$ and for $a = b = 2$ and zero otherwise (we did this calculation in class). So this is an additional contribution of $g_2^2\chi_3^3$ to M_W^2:

$$M_W^2 = g_2^2(v^2/4 + \chi_3^2) \ , \quad M_Z^2 = g_2^2 v^2/4\cos^2\theta_w \ .$$

$$\frac{M_W^2}{M_Z^2} = \cos^2\theta_w + \frac{4\cos^2\theta_w\chi_3^2}{v^2}$$

The first term is the Standard Model ratio, so

$$\delta\frac{M_W^2}{M_Z^2} = \frac{g'^2 v^2\cos^2\theta_w}{M^4}$$

Also, for a small change, $\delta(x^2) = 2x\delta x$, so

$$\delta\frac{M_W}{M_Z} = \frac{2g'^2v^2\cos\theta_w}{M^4} < 10^{-3} \ .$$

Putting in the numbers $\cos^2\theta_w = 0.77$, $v = 250$ GeV gives

$$M^2 > g' \times 1.05 \times 10^4 \text{ GeV}.$$

If $g' = M$ then $M > 10$ TeV, but if $g' \sim M_Z$ then $M > 1$ TeV.

2. Let α, β run over $0, 1, 2$ and μ, ν over $0, 1, 2, 3$, so that $k^2 \equiv k_\mu k^\mu = k_\alpha k^\alpha + k_3^2$. The gauge field action, for the three nonzero components, is

$$-A_\alpha(-k)(k^2 g^{\alpha\beta} - k^\alpha k^\beta)A_\alpha(k) \ .$$

Then numerator of the propagator, $M_{\alpha\beta}$, is the inverse 3×3 matrix,

$$(k^2 g^{\alpha\beta} - k^\alpha k^\beta)M_{\beta\gamma} = \delta^\alpha{}_\gamma.$$

The most general form invariant under Lorentz transformations of the $0, 1, 2$ directions (leaving the 3-direction fixed) is $M_{\beta\gamma} = a g_{\beta\gamma} + b k_\beta k_\gamma$. Inserting this into the previous equation gives $k^2 a = 1$, $k^2 b - a - k^\beta k_\beta b = 0$, so $a = 1/k^2$ and $b = 1/(k^2 k_3^2)$. The propagator is

$$\frac{-i}{k^2}\left(g_{\alpha\beta} + \frac{1}{k_3^2}k_\alpha k_\beta\right) \ .$$

If we look at the residue of the $k^2 = 0$ pole, then by using $k^2 = k_\alpha k^\alpha + k_3^2$ we get

$$M_{\alpha\beta} = g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k_\gamma k^\gamma} \ .$$

This is precisely a projection operator onto the directions perpendicular to k_α. The vector k_α is timelike (because k^2 is null and k_3 is nonzero). We can then boost in the 0-1-2 directions to the 'rest frame' where $k_1, k_2 = 0$ (this boost does not affect A_3), and then $M_{\alpha\beta} = \text{diag}(0, 1, 1)$. Also, the 33 component of the propagator is of course zero.

By the way, we can also write this propagator in a more covariant form. Let n_μ be a unit four-vector in the 3-direction. Then the previous equation becomes, in terms of $\mu = (\alpha, 3)$,

$$\frac{-i}{k^2}\left(g_{\mu\nu} - n_\mu n_\nu + \frac{1}{(n\cdot k)^2}(k_\mu - n_\mu n\cdot k)(k_\nu - n_\nu n\cdot k)\right) \ .$$

The factors of n just act to project away the 3-components. Also, by regrouping terms we can write this as

$$\frac{-i}{k^2}\left(g_{\mu\nu} + \frac{1}{(n\cdot k)^2}\left[k_\mu k_\nu - (n_\mu k_\nu + k_\mu n_\nu)\, n\cdot k\right]\right) \ .$$

In this form it is clear that it is the Feynman gauge propagator plus pure gauge pieces (proportional to k_μ or k_ν, as must be true in any gauge.

3. To make this problem more interesting (as will be explained), I am going to consider the more general energy-momentum tensor

$$T_{\mu\nu} = \frac{i}{2}\left\{\alpha\bar{\psi}\gamma_\mu\partial_\nu\psi - \beta(\partial_\nu\bar{\psi})\gamma_\mu\psi + \alpha\bar{\psi}\gamma_\nu\partial_\mu\psi - \beta(\partial_\mu\bar{\psi})\gamma_\nu\psi\right\} ,$$

where α and β are some constants (which must add up to 2, by the way). This is still symmetric, and also conserved: it can be interpreted as assigning a different spin to ψ (in string theory this is known as the bc system). Now evaluate the T-ordered 2-point function (I apologize for omitting the 'T' on the exam). Consider first the first two terms in $T_{\mu\nu}$; the last two are then given by symmetry. This is essentially the same as the vacuum polarization, except that the extra derivatives in $T_{\mu\nu}$ give an extra factor of $i\alpha(k+l)^\nu + i\beta l^\nu = i\alpha k^\mu + il^\mu$:

$$-\frac{i}{2}i^2\int\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}(\gamma_\rho \not{l}\gamma_\mu[\not{k}+\not{l}])(\alpha k + 2l)_\nu}{l^2(k+l)^2} .$$

Now, introducing Feynman parameters, shift the loop momentum, continue to Euclidean signature, and use rotation invariance (odd powers of q integrate to zero, and $q_\mu q_\nu$ becomes $g_{\mu\nu}q^2/d$) to get

$$-\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\,\frac{N_{\rho\mu\nu}}{(q^2+D)^2}$$

with $D = (x-x^2)k^2$ and $N_{\rho\mu\nu} =$

$$k_\nu g_{\mu\rho}(\alpha-2x)[(\tfrac{2}{d}-1)q^2+x(1-x)k^2]+(k_\rho g_{\mu\nu}+k_\mu g_{\nu\rho})2(1-2x)q^2/d-2x(1-x)(\alpha-2x)k_\rho k_\mu k_\mu .$$

Notice that if α were 1, this would be odd under $x \to 1-x$ and so vanish on integration: this is a consequence of C invariance. But no one got zero, due to algebraic mistakes, so some were able to do the remaining parts. So now do the q^2 integral to get

$$-\frac{1}{2\pi}(k_\nu g_{\mu\rho} - k_\rho k_\mu k_\mu)\int_0^1 dx\,(\alpha-2x)$$

Integrating over x, and adding in the other two terms from $T_\mu\nu$, gives the final result

$$-\frac{\alpha-1}{2\pi}(k_\nu g_{\mu\rho} + k_\mu g_{\nu\rho} - 2k_\rho k_\mu k_\mu/k^2) .$$

b) We just get

$$-i\frac{\alpha-1}{2\pi}h^{\mu\nu}(k)(k_\nu g_{\mu\rho} + k_\mu g_{\nu\rho} - 2k_\rho k_\mu k_\mu/k^2) :$$

the same graph with i times the background field attached.

c) From $j_{5\rho} = \epsilon_\rho{}^\sigma j_\sigma$ one gets

$$-i\frac{\alpha - 1}{2\pi}\epsilon_\rho{}^\sigma h^{\mu\nu}(k)(k_\nu g_{\mu\sigma} + k_\mu g_{\nu\sigma} - 2k_\rho k_\mu k_\mu/k^2) \ :$$

the same graph with i times the background field attached.

d) Contracting with k^ρ, part b gives 0 (the vector current has no anomaly) while part c gives

$$-i\frac{\alpha - 1}{2\pi}h^{\mu\nu}(k)k^\rho(\epsilon_{\rho\mu}k_\nu + \epsilon_{\rho\mu}k_\mu) \ .$$

This is an anomaly proportional to the Riemann tensor, it is analogous to the anomaly that requires $\mathrm{Tr}_{\mathrm{LH\,fermions}}(Q)$ to vanish in the Standard Model for consistent coupling to gravity.

4. (Graphs scanned in separate file, labelled by letters). Let us start by drawing all graphs with two g-vertices (fig. a). I will leave off the arrows henceforth, but because they are there a vertex can be rotated but not flipped over. First connect one line on each vertex by a propagator (fig. b). There are 16 ways to do this. Now connect a second line. There are 9 choices, but only four distinct graphs (figs. c1, c2, c3, c4), which arise in 2, 4, 2, and 1 way respectively. There is also a factor of $\frac{1}{2}$ from the expansion of e^{iS} to second order, and a factor of $\frac{1}{2}$ because we have constructed each graph twice by distinguishing the 'first propagator' and the 'second propagator'. The loop integral and propagators give $1/8\pi^2\epsilon$, so contribute a factor of $1/8\pi^2$ in the β function. There is a 4! from all ways to connect the four external lines to the four lines in the graph, but the same factor is present in the tree amplitude and so divides out in the β function. Also, as in $g\phi^4$ the propagator corrections do not contribute to one loop.

Thus, figure c1 contributes

$$\frac{g^2}{8\pi^2} \times 16 \times 2 \times \frac{1}{4} \times N = \frac{8Ng^2}{8\pi^2}$$

to the β function for $\mathrm{Tr}(\phi^4)$. The N is from the inner index loop, and the $\mathrm{Tr}(\phi^4)$ is because all the external lines are attached cyclically to the out index loop. Perhaps a more familiar way to obtain this result would be to define $g = \tilde{g}/4$, where the 4 represents the cyclic symmetry of the vertex. The β function for $\tilde{g}$ has a 2 in the numerator in place of an 8: this is just from the two distinct graphs of figure c1* for a given cyclic ordering of the external lines (vs. the single tree graph shown). There is no further factor in the denominator because the double-line graph has no symmetries: if you try to permute lines it won't look the same.

I wlll continue to use my counting, but you should compare with your way (note that in the double-trace operator the natural coupling is $g' = \tilde{g}'/8$). Figure c2 would contribute to $\mathrm{Tr}(\phi^3)\mathrm{Tr}(\phi)$ (one line on the inner loop and three on the outer) but this vanishes because

ϕ is traceless. The remaining two graphs are

$$\text{c3}: \frac{8g^2}{8\pi^2}(\text{Tr}(\phi^2))^2 \ , \qquad \text{c4}: \frac{4g^2}{8\pi^2}(\text{Tr}(\phi^2))^2 \ .$$

After grading the exams, I find that most people don't follow how the indices work. In c1, label the external indices (going clockwise from upper left) ij, kl, mn, pq: this graph is proportional to $\delta_{jk}\delta_{lm}\delta_{np}\delta_{qi}$, the same structure as the g tree vertex, and then there is a factor of N from the sum over the internal loop. In c2, label the indices ij, kl, mn around the outside, and pq for the interior propagator. This graph is proportional to $\delta_{jk}\delta_{lm}\delta_{ni}\delta_{pq}$, which you would get from the tree interaction $\text{Tr}(\phi^3)\,\text{Tr}(\phi)$. In c3 and c4, if we label the outer lines ij, kl and the inner lines mn, pq then both graphs go as $\delta_{jk}\delta_{li}\delta_{np}\delta_{qm}$., which is what you would get from $\text{Tr}(\phi^2)\,\text{Tr}(\phi^2)$.

Let me go through c1 carefully. Label the external lines $k_1 ij, k_2 kl, k_3 mn, k_4 pq$. There are 4! ways to attach the external lines to this graph; let us focus on the cyclic order 1234. For the tree graph this contributes $-4ig\delta_{jk}\delta_{lm}\delta_{np}\delta_{qi}$, since there are four cyclic permutations. For the graph of figure c1, there are $16 \times 2 \times \frac{1}{2}$ ways to build it, a $\frac{1}{2}$ from expanding e^{iS}, and the same 4 cyclic ways to attach the external lines to the graph - so the structure is the same as the tree g vertex and the divergence can be absorbed in Z_g. The rest of the factors are

$$g^2 \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \frac{1}{(q^2 + D)^2} = \frac{ig^2}{8\pi^2 \epsilon} \ .$$

So this contributes

$$\frac{g}{8\pi^2\epsilon} \times \times 16 \times 2 \times \frac{1}{4} \times N$$

to Z_g, and so make the indicated contribution to the β function.

We must also include the second term in the propagator, so each graph has three other terms, as shown for c1. Each doubled-back propagator brings in a factor $-1/N$, so we have

$$\text{c1}' : -\frac{8g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c1}'' : -\frac{8g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c1}''' : \frac{8g^2}{8\pi^2 N^2}(\text{Tr}(\phi^2))^2 \ ,$$

where in c1$'$ and c1$''$ the external lines hook into a single index loop, and in c1$'''$ two hook into each of two loops. In c1$'''$, of the 4! perturbations, 8 have the index structure $\delta_{jk}\delta_{li}\delta_{np}\delta_{qm}$, 8 have $\delta_{jm}\delta_{ni}\delta_{lp}\delta_{qk}$, and 8 have $\delta_{jp}\delta_{qi}\delta_{nk}\delta_{lm}$. This is the same as the 4! ways to attach the external lines to $(\text{Tr}(\phi^2))^2$, so we just get $8g/8\pi^2 N^2$ in Z_g.

In a similar way,

$$\text{c2}' : -\frac{16g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c2}'' : -\frac{16g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c2}''' : \frac{16g^2}{8\pi^2 N^2}(\text{Tr}(\phi^2))^2 \ ,$$

$$\text{c3}' : -\frac{8g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c3}'' : -\frac{8g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c3}''' : \frac{8g^2}{8\pi^2 N^2}(\text{Tr}(\phi^2))^2 \ ,$$

$$\text{c4}' : -\frac{4g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c4}'' : -\frac{4g^2}{8\pi^2 N}\text{Tr}(\phi^4) \ , \quad \text{c4}''' : \frac{4g^2}{8\pi^2 N^2}(\text{Tr}(\phi^2))^2 \ ,$$

None of these graphs have closed index loops: the N's are just from the propagator. In all, the g^2 terms are:

$$\beta_g : \frac{g^2}{8\pi^2}(8N - 72/N) \ , \quad \beta_{g'} : \frac{g^2}{8\pi^2}(12 + 36/N^2) \ .$$

At order gg' there are the two vertices of fig. e, which can be joined in 16 ways to make fig. f, and then in (2, 4, 1, 2) ways to make figs. g1, g2, g3, g4. There is a $\frac{1}{2}$ from building each graph twice, but no $\frac{1}{2}$ from e^{iS} because all the propagators are different. Then the graphs contribute

$$\text{g1} : \frac{16Ngg'}{8\pi^2}(\text{Tr}(\phi^2))^2 - \frac{16gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 - \frac{16gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 + \frac{16gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 \ ,$$

$$\text{g2} : \frac{32gg'}{8\pi^2}\text{Tr}(\phi^4) + 0 + 0 + 0 \ ,$$

$$\text{g3} : 0 - \frac{8gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 - \frac{8gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 + \frac{8gg'}{8\pi^2 N}(\text{Tr}(\phi^2))^2 \ ,$$

$$\text{g4} : 0 + 0 + 0 + 0 \ ,$$

where we have given $\text{g1} + \text{g1}' + \text{g1}'' + \text{g1}'''$, etc. The zeros represent terms that we don't have. For example, g3 gives a *triple trace* term $\text{Tr}(\phi)\text{Tr}(\phi)\text{Tr}(\phi^2)$. Note also that $\text{g1}'''$ and $\text{g3}'''$ end up with index loops that contribute a factor of N.

Now repeat for g'^2 in fig. h, i, j. There are only three essentially different graphs, which have weight $16 \times (1, 4, 4) \times \frac{1}{4}$, and so we get:

$$\text{j1} : \frac{4N^2 g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 - \frac{4g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 - \frac{4g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 + \frac{4g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 \ ,$$

$$\text{j2} : \frac{16g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 + 0 + 0 + 0 \ ,$$

$$\text{j3} : \frac{16g'^2}{8\pi^2}(\text{Tr}(\phi^2))^2 + 0 + 0 + 0 \ .$$

In all,

$$\mu\partial_\mu g = (2N - 18/N)\frac{g^2}{2\pi^2} + 8\frac{gg'}{2\pi^2} \ ,$$

$$\mu\partial_\mu g' = (3 + 9/N^2)\frac{g^2}{2\pi^2} + (4N - 6/N)\frac{gg'}{2\pi^2} + (N^2 + 7)\frac{g'^2}{2\pi^2} \ .$$

In order to get a smooth limit we need $g = \lambda/N$, $g' = \lambda'/N^2$, and then for large N

$$\mu\partial_\mu \lambda = 2\frac{\lambda^2}{2\pi^2} \ ,$$

$$\mu\partial_\mu\lambda' = 3\frac{\lambda^2}{2\pi^2} + 4\frac{\lambda\lambda'}{2\pi^2} + \frac{\lambda'^2}{2\pi^2} \ .$$

Thus, in the large N limit the single trace terms drive the double trace terms, but not vice versa: we can solve for λ independent of λ', and then solve for λ'.