UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Physics 221C **Quantum Field Theory** **Spring 2007**

Prof: Joe Polchinski joep@kitp.ucsb.edu
Office hours: Kohn 2319, Th 3:30-4:30 (or email/see me after class to set up a time)

TA: Richard Eager

ASSIGNMENT #7
Due: Friday, May 25, 5pm in TA's mailbox

This homework may look long, but it's mostly because the problems take a long time to describe. Some parts (e.g. 3.b,c) may be challenging.

1. The connection between the π^0 decay rate and the anomaly: Consider QCD with two flavors of massless quark u, d, and consider the symmetry

$$\delta u = i\delta\chi\gamma_5 u \quad \delta d = -i\delta\chi\gamma_5 d \ . \tag{1}$$

a) This symmetry has no anomaly (why?). Now introduce the electromagnetic interaction, where the up quark has charge $\frac{2}{3}$ and the down quark has charge $-\frac{1}{3}$. The symmetry (1) now has an anomaly with two photons: what is the divergence of the current? (Not a long calculation).

b) Now consider the low energy effective Lagrangian, with the $\pi^\pm$ fields set to zero since they won't enter. How does the symmetry (1) act on the π^0 field?

c) What is the Noether current for the symmetry (1), in terms of π^0?

d) Write the anomalous conservation equation in terms of the π^0 field - it should look like a field equation with a source term.

e) Add an interaction term to the effective Lagrangian so as the obtain the field equation above.

f) Use this to calculate the π^0 lifetime and compare with the actual value. Referring back to (a), the π^0 lifetime effectively measures the quark charges.

2. The Ising model is the simplest lattice theory. Its degrees of freedom are spins, which take only the values $+1$ and -1, living at the sites of a lattice. The action is $-\frac{1}{g^2}s_i s_{i'}$, summed over all links (nearest-neighbor pairs); this wants to align the spins. It can have any dimension, but let us focus on the case of two dimensions. The 'path integral' (sum, actually) is

$$\left(\prod_{\text{sites}} \sum_{s_i=\pm 1}\right) \exp\left(\sum_{\text{links}} s_i s_{i'}/g^2\right) \ .$$

a) In the strong-coupling limit, evaluate the two-point function $\langle s_i s_j \rangle$ where i and j are m sites apart along some coordinate axis.

b) Evaluate $\langle s_i s_j \rangle$ again, in the weak coupling limit: here the path integral is dominated by the configuration(s) of lowest action.

In one case you should find that the correlation goes to zero at long distance, and in the other not: the latter is the sign of spontaneous breaking of the $s \to -s$ symmetry. The Ising model thus has two phases (and by changing the sign of the action, you can get to a third, 'antiferromagnetic,' phase).

c) The Ising model has a duality symmetry, which is a prototype for the dualities in more complicated systems. First, define a spin σ_l living on links, which is just the product $s_i s_{i'}$ of the spins on the ends. We can replace the sum over configurations s_i with the sum over configurations of the link variables σ_l, except that there is a constraint that the product of σ_l around any plaquette p is 1 (why?). Then the path integral can be written

$$\left(\prod_{\text{links}} \sum_{\sigma_l = \pm 1} \right) \left(\prod_{\text{plaquettes}} \delta_{\sigma_p, 1} \right) \exp \left(\sum_{\text{links}} \sigma_l / g^2 \right) ,$$

where $\sigma_p = \pm 1$ is the product of the σ_l around the plaquette. Now introduce the 'discrete Fourier transform,'

$$\delta_{\sigma_p, 1} = \frac{1}{2} \sum_{\lambda_p = 0,1} \sigma_p^{\lambda_p} ,$$

(verify). We can replace the product of delta-functions with a configuration sum over this new field λ_p. We can think of this field as living on a *dual lattice*, whose sites are the plaquettes of the original lattice. Now, focus on a given link, identify all terms involving that σ_l, and show that the sum over that σ_l can be written as $f(\lambda_p + \lambda_{p'})$ for some function f. Here λ_p and $\lambda_{p'}$ are the two plaquettes containing l: these are nearest neighbors on the dual lattice. Now, define $\tilde{s}_p = (-1)^{\lambda_p}$, taking values ± 1, and show that

$$f(\lambda_p + \lambda_{p'}) = Z e^{\tilde{s}_p \tilde{s}_{p'} / \tilde{g}^2}$$

for some constants Z and $\tilde{g}^2$. Taking the product over all links, conclude that the Ising path integral with coupling g can be rewritten as an Ising path integral with coupling $\tilde{g}$, times an overall normalization $Z^{\text{number of links}}$. Graph $\tilde{g}^2$ versus g^2.

There are many more interesting phenomena, and extensions, here. See John B. Kogut Rev.Mod.Phys.51:659,1979.

3. Consider a theory with $SO(3)$ symmetry, with two real scalar fields ϕ^a, ρ^a, each transforming as the **3** representation.

a) Write down the most general renormalizable Lagrangian that is invariant under (1) simultaneous $SO(3)$ rotations of the two triplets, (2) the discrete symmetry that interchanges the two triplets, $(\phi^a, \rho^a) \to (\rho^a, \phi^a)$, and (3) the discrete symmetry that reflects one scalar but not the other, $(\phi^a, \rho^a) \to (\phi^a, -\rho^a)$ Hint: it is not symmetric under independent SO(3) rotations of the two scalars.

b) Determine the conditions on the quartic couplings such that the potential is positive as and go to infinity (in all directions in field space).

c) For negative mass-squared, find the minimum of the potential and determine the unbroken subgroup of $SO(3)$. Hint: there are three phases. One way to approach the problem is to use the $SO(3)$ symmetry to rotate the fields into convenient directions.

d) Gauge the $SO(3)$ symmetry, and find the masses of the gauge bosons in each phase.