19.1 Fermion number nonconservation in parallel E and B fields.

(a) Show that the Adler-Bell-Jackiw anomaly equation leads to the following law for global fermion number conservation: If N_R and N_L are, respectively, the numbers of right- and left-handed massless fermions, then

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2} \int d^4x \, \mathbf{E} \cdot \mathbf{B}.$$

To set up a solvable problem, take the background field to be $A^\mu = (0, 0, Bx^1, A)$, with B constant and A constant in space and varying only adiabatically in time.

(b) Show that the Hamiltonian for massless fermions represented in the components (3.36) is

$$H = \int d^3x \left[\psi_R^\dagger(-i\boldsymbol{\sigma} \cdot \mathbf{D})\psi_R - \psi_L^\dagger(-i\boldsymbol{\sigma} \cdot \mathbf{D})\psi_L \right],$$

with $D^i = \nabla^i - ieA^i$. Concentrate on the term in the Hamiltonian that involves right-handed fermions. To diagonalize this term, one must solve the eigenvalue equation $-i\boldsymbol{\sigma} \cdot \mathbf{D}\psi_R = E\psi_R$.

(c) The ψ_R eigenvectors can be written in the form

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i(k_2 x^2 + k_3 x^3)}.$$

The functions ϕ_1 and ϕ_2, which depend only on x^1, obey coupled first-order differential equations. Show that, when one of these functions is eliminated, the other obeys the equation of a simple harmonic oscillator. Use this observation to find the single-particle spectrum of the Hamiltonian. Notice that the eigenvalues do not depend on k_2.

(d) If the system of fermions is set up in a box with sides of length L and periodic boundary conditions, the momenta k_2 and k_3 will be quantized:

$$k_i = \frac{2\pi n_i}{L}.$$

By looking back to the harmonic oscillator equation in part (c), show that the condition that the center of the oscillation is inside the box leads to the condition

$$k_2 < eBL.$$

Combining these two conditions, we see that each level found in part (c) has a degeneracy of

$$\frac{eL^2B}{2\pi}.$$

(e) Now consider the effect of changing the background A adiabatically by an amount (19.37). Show that the vacuum loses right-handed fermions. Repeating this analysis for the left-handed spectrum, one sees that the vacuum gains the same number of left-handed fermions. Show that these numbers are in accord with the global nonconservation law given in part (a).