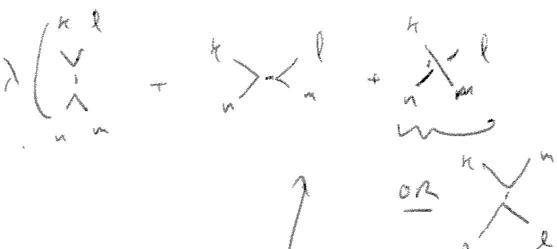
Lecture 19

QFT → ∞ # of Feynman graphs. Sometimes can find limit where things are simpler and we can sum them all.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi_i \phi_i - \frac{1}{8}\lambda(\phi_i\phi_i)^2$$

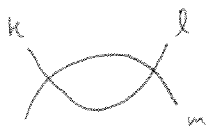
$i = 1, \ldots, N$. $\qquad\qquad \phi_i\phi_i\phi_j\phi_j$

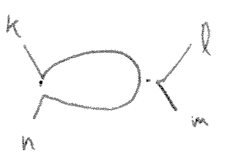
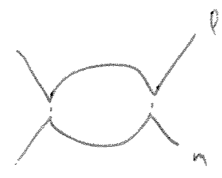
Vertex: $\underbrace{i \times}_{}$  $= \lambda(\delta_{kl}\delta_{mn} + \delta_{kn}\delta_{lm} + \delta_{km}\delta_{ln})$

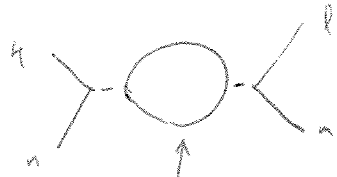
$i V_{4klmn} \leftarrow \quad \equiv \lambda\Big($  $\Big)$

OR

Now look at one-loop correction:

 = sum of 9 terms (3^2)

e.g.  $\propto \delta_{kn}\delta_{lm}$  $\delta_{kl}\delta_{mn}$

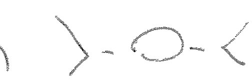
one of the 9 is special:  $= \delta_{kn}\delta_{lm} \times N$

sum over internal index.

 $\sim \lambda$

$\sim \lambda^2 N$

Take $N \to \infty$ with λN fixed. ② ~~Leading terms~~ go

 $\sim \frac{1}{N}(\lambda N)$  $\sim \frac{1}{N}\cdot(\lambda N)^2$

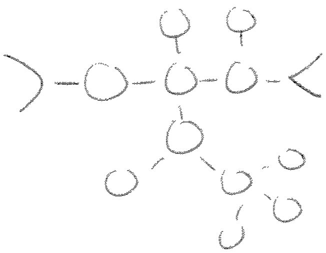
trick: $e^{-\frac{i}{8}\int \lambda(\phi_i\phi_i)^2 d^4x} = \int \mathcal{D}\sigma \, e^{\frac{i}{2}\int \frac{1}{\lambda}\sigma^2 + \sigma\phi_i\phi_i}$

Leading graphs have # of index loops = # of loops

 $\sim \lambda^4 N^3 = \lambda(\lambda N)^3$

geometric series

Also,



← summable, but contains nontrivial physics.

can sum cactii iteratively

[Large-N vectors straightforward.

Large N matrices ← double line

hard but interesting]

Want to use this to solve a theory in which some fields are massless in $\mathcal{L}$, and to all orders of perturbation theory, but get a mass non-perturbatively.

Consider $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i$, $i = 1, 2, 3$,

but with constraint $\phi_i\phi_i = \frac{1}{\lambda}$

$$\phi_3 = \sqrt{\frac{1}{\lambda} - \phi_1^2 - \phi_2^2} \qquad \partial_\mu\phi_3 = \frac{-\sqrt{\lambda}(\phi_1\partial_\mu\phi_1 + \phi_2\partial_\mu\phi_2)}{\left(1 - \lambda(\phi_1^2 + \phi_2^2)\right)^{1/2}}$$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{\lambda}{2}\frac{(\phi_1\partial_\mu\phi_1 + \phi_2\partial_\mu\phi_2)^2}{1 - \lambda(\phi_1^2 + \phi_2^2)}$$

$$\frac{1}{\text{denominator}} = \left(1 + \lambda(\phi_1^2 + \phi_2^2) + \lambda^2(\phi_1^2 + \phi_2^2)^2 + \dots\right)$$

λ = coupling

"Nonlinear sigma model": scalar field theory with field-dependent kinetic term.

(Field space = sphere).

$d=4 \quad [\phi] = m$

$[\phi\phi\partial\phi\partial\phi] = m^{6>d}$: non-renormalizable

(however, low energy self-interactions of pions are described by just such an $\mathcal{L}$)

$d=2 \quad [\phi] = m^0 \qquad [\phi\phi\partial\phi\partial\phi] = m^2 = d$: renormalizable.

Let's study it!

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_i \partial_\mu \phi_i \qquad \phi_i \phi_i = 1/\lambda \equiv g\text{uess } N/g \qquad i = 1, \dots, N$$

$$\int \mathcal{D}\phi|_{\phi\cdot\phi=1} \; e^{i\int -\frac{1}{2}\partial_\mu\phi_i \partial^\mu \phi_i \, d^2x}$$

how to deal with this?

Integral representation. $\delta(x) = \int_{-\infty}^{\infty} dy \; e^{2\pi i y x}$

Let us write

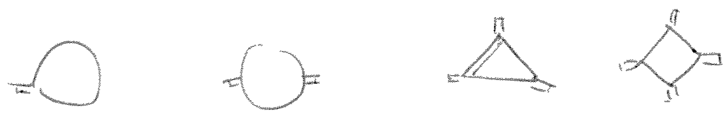
$$\prod_x \text{"}\delta(\phi\phi-1)\text{"} = \int_{-\infty}^{\infty} d\sigma \; e^{-\frac{i}{2}\sigma(\phi_i\phi_i - 1/\lambda)}$$

convenient normalization of σ.

$$\int \mathcal{D}\phi\, \mathcal{D}\sigma \; e^{-\frac{i}{2}\int \partial_\mu \phi_i\, \partial^\mu \phi_i \,+\, \sigma(\phi_i \phi_i - 1/\lambda)}$$

- Do path $\int$ over ϕ first. It will turn out that σ-path integral is <u>highly-peaked</u> on one (constant) configuration.

Expand in powers of σ: Connected graphs



$$= N \sum_{n=1}^{\infty} \uparrow \left[(-i) \frac{-\sigma i}{k^2 - i\varepsilon} \right]^n \cdot \frac{1}{2n}$$

$\uparrow$ symmetry factor

$\int \frac{d^2k}{(2\pi)^2}$

$$= -\frac{N}{2} \ln\left(1 + \frac{\sigma}{k^2 - i\varepsilon}\right)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = -\ln(1+x) \qquad -\frac{N}{2}\int_0^{\Lambda} \frac{d^2\bar{k}}{(2\pi)^2} \ln\left(1 + \frac{\sigma}{\bar{k}^2}\right)$$

$$- \int \frac{iN}{4\pi} \int_0^{\Lambda^2} d(\bar{k}^2) \ln\left(1 + \frac{\sigma}{\bar{k}^2}\right) =$$

$\uparrow$ put finite cutoff

$$= -\frac{iN}{4\pi}\left(\sigma \ln \Lambda^2/\sigma - \sigma\right) \quad \leftarrow \text{exponentiate}$$

$$\int \mathcal{D}\sigma \; e^{i S_{eff}(\sigma)}$$

$$S_{eff}(\sigma) = \frac{N\sigma}{2g} - \frac{N}{4\pi}\left(\sigma \ln \Lambda/\sigma - \sigma\right) \quad \text{for } \sigma = \text{const}$$

#1 renormalize

$$\frac{1}{g_0} - \frac{1}{2\pi} \ln \frac{\Lambda}{\mu} \equiv \frac{1}{g(\mu)}$$

$$\frac{N\sigma}{2g(\mu)} - \frac{N}{4\pi}\left(\sigma \ln \mu/\sigma - \sigma\right)$$

N is large → dominated by stationary points.

$$\frac{\partial S}{\partial \sigma} = 0 = \frac{N}{2g(\mu)} - \frac{N}{4\pi} \ln \frac{\mu}{\sigma}$$

$$\sigma = \mu e^{-2\pi/g(\mu)} \leftarrow \quad < g^n \text{ for any } n, \text{ as } n \to 0.$$

Now insert $\phi_i(k_1)\,\phi_j(k_2')$ before $\int \mathcal{D}\phi$ get

extra factor of $\dfrac{-i}{k^2+\sigma-i\varepsilon}\,(2\pi)^2\,\delta(k+k')$

σ = mass.

Note: also, illustrates Coleman's theorem:

ϕ_i is a Goldstone boson, massless in pert. thy.,

but quantum fluctuations restore symmetry, give

mass to ϕ_i