Lecture 20

Conformal invariance

Operator renormalization

Operator product expansion

$$\int d^4x \quad -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4!}\lambda\phi^4 \qquad \lambda \text{ is dimensionless}$$

so classically there is no scale: symmetry

scale invariance: $\quad \delta\phi(x) = x^\mu\partial_\mu\phi + c\phi$

constant to be determined.

$$\delta\mathcal{L} = -\partial_\mu\phi\,\partial^\mu(x^\nu\partial_\nu\phi) - \partial_\mu\phi\,\partial^\mu(c\phi)$$

$$-\frac{1}{3!}\lambda\left(\phi^3 x^\nu\partial_\nu\phi + \phi^4 c\right)$$

second line $= -\frac{\lambda}{4!}\partial_\nu(x^\nu\phi^4)$ <u>if</u> $c=+1$

$(\partial_\nu x^\nu = 4)$ $\qquad$ (ϕ has dimension 1).

$$-\partial_\mu\phi\,\partial_\mu\phi - \partial_\mu\phi\,\partial^\mu\phi - \partial_\mu\phi\, x^\nu\partial_\nu\partial_\mu\phi)$$

$$-\frac{1}{2}x^\nu\partial_\nu(\partial_\mu\phi\,\partial^\mu\phi)$$

$$= -\frac{1}{2}\partial_\nu(x^\nu\partial_\mu\phi\,\partial^\mu\phi) \quad \checkmark$$

Noether current: for translations, it's $-a_\mu T^{\mu\nu}$

so let's try $\quad -x_\mu T^{\mu\nu}$.

$$\partial^{\nu}(\underbrace{-x_\mu}\, T^{\mu\nu}) = -g_{\mu\nu}T^{\mu\nu} \pm x_\mu \cancel{\partial_\nu T^{\mu\nu}}^{\,0} = \underline{\underline{-T_\mu{}^\mu}}$$

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi + \underline{g^{\mu\nu}}\left(-\tfrac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \tfrac{1}{4!}\lambda\phi^4\right)$$

$$T^\mu{}_\mu = -\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{3!}\lambda\phi^4$$

Equation of motion: $\partial_\mu\partial^\mu\phi = +\dfrac{\lambda}{3!}\phi^3$

$$T^\mu{}_\mu = -\partial^\mu\phi\,\partial_\mu\phi - \phi\,\partial_\mu\partial^\mu\phi$$

$$= \partial_\mu(-\phi\,\partial^\mu\phi) = -\tfrac{1}{2}\partial^2\phi^2$$

$$\boxed{j^\nu = -x_\mu T^{\mu\nu} + \phi\,\partial^\nu\phi} \quad \text{is conserved}$$

Following step not necessary, but helpful.

Now, consider $T_{\mu\nu} + A\left(g_{\mu\nu}\partial_\alpha\partial^\alpha\phi^2 - \partial_\mu\partial_\nu\phi^2\right) \equiv \tilde{T}_{\mu\nu}$

$$\partial^\mu\left(\qquad \right)$$

$$= \partial_\nu\partial^2\phi^2 - \partial_\nu\partial^2\phi^2 = 0$$

$$\tilde{T}_\mu{}^\mu = T_\mu{}^\mu + 3A\,\partial^2\phi^2$$

If $A = \frac{1}{6}$, $\tilde{T}_\mu{}^\mu = 0$ and $j^\nu = -x_\mu\tilde{T}^{\mu\nu}$ is conserved.

Significance of $\tilde{T}_{\mu\nu} - T_{\mu\nu}$? It's trivial, not a new symmetry. Clue #1: we didn't have to use the equations of motion.

Clue #2 : You can check the ~~0 component of~~

$$\tilde{T}_\mu{}^0 - T_\mu{}^0 = \text{total derivative in spatial direction}$$

$$\rightarrow \quad \underline{\tilde{P}_\mu = P_\mu}$$

Now, consider a more general symmetry

$$-V_\mu(x) T^{\mu\nu}$$

$$\partial_\nu(-V_\mu T^{\mu\nu}) = -\partial_\nu V_\mu T^{\mu\nu} = -\frac{1}{2}(\partial_\nu V_\mu + \partial_\mu V_\nu) T^{\mu\nu}$$

(because T is symmetric)

$$= 0 \ \underline{\text{if}} \quad \partial_\mu V_\nu + \partial_\nu V_\mu = g_{\mu\nu} X$$

any X

in fact, taking trace

$$2\partial_\mu V^\mu = 4X$$

$$\partial_\mu V_\nu + \partial_\nu V_\mu - \frac{1}{2} g_{\mu\nu} \partial_\sigma V^\sigma = 0$$

Solution: translation
Lorentz transformation
scale transformation
→ special conformal transformation

$$V_\nu = x_\nu a_\sigma x^\sigma + B a_\mu x_\sigma x^\sigma$$

x_σ = any fixed 4-vector

$$\partial_\mu V_\nu = g_{\mu\nu} a_\sigma x^\sigma + x_\nu a_\mu + 2B a_\mu x_\nu$$

$$\partial_\nu V_\mu = \underbrace{g_{\mu\nu} a_\sigma x^\sigma}_{\propto g_{\mu\nu}} + \underbrace{a_\nu x_\mu + 2B\, a_\nu x_\mu}_{\text{cancel if } B = -\frac{1}{2}}$$

$$\delta x^\mu = x^\mu a_\sigma x^\sigma - \frac{1}{2} a^\nu x_\sigma x^\sigma$$

What is this? Don't confuse it with a local symmetry (which would have an arbitrary position dependence), it's just a modest enlargement of the spacetime symmetry.

Eg:

preserves angles but not lengths. (takes small squares into small squares, but sizes rescaled)

For Euclidean metric (+ + + +)

~ position dependent rescaling (but position dependence not arbitrary).

Arises when $T_\mu{}^\mu = 0$, or more generally when

$$T_\mu{}^\mu = \partial_\mu \partial_\nu X^{\mu\nu}$$

$\leftarrow$ any operator

If $T_{\mu\alpha}{}^{\mu} = 0$, conf. inv.

If $T_{\mu\alpha}{}^{\mu} = \partial_\alpha^i X$ (any X) $\exists\ \tilde{T}_{\mu\nu}$ s.t. $\tilde{T}_{\mu\nu} = 0$ inv.

If $T_\mu{}^\mu = \partial_\mu K^\mu$ but $K^\mu \neq$ total der.,

$\exists$ scale but not conformal inv.

A nice way to think about conformal symmetry in Minkowski space: $\delta(x^2) = 2x_\mu^a \delta x^\mu = 2x^2 a\cdot x - x^2 a\cdot x$

$= 0$ if $x^2 = 0$

Conformal symmetry = most general mapping of spacetime that preserves null vectors.

1+1: $ds^2 = -dt^2 + dx^2 = -2\,dx^+ dx^-$

$x^\pm = t \pm x$

Light rays: $x^+ =$ const or $x^- =$ const

$x^{+\prime} = f(x^+)$ $x^{-\prime} = g(x^-)$ ← ~~preserved~~ preserves light cones. ~~of~~ for any f, g (must be monotonic).

Still not a local symmetry! (This requires $\delta x^\pm = v^\pm(x^+, x^-)$).

Just a (very big) global symmetry.

(In $d=2$: $\mathcal{L} = -\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j$

is classically scale invariant for any $g_{ij}(\phi)$.

~~One-loop β-function = 0 if~~ $R_{ij}(\phi) = 0$

~~curvature tensor built of $g_{ij}(\phi)$~~

One loop correction:

$$T_{\mu\nu} = R_{ij}(\phi)\, \partial_\mu \phi^i \partial_\nu^\mu \phi^j$$

curvature tensor built out of g_{ij}

Scale: $R_{ij} = \nabla_i v_j + \nabla_j v_i$ (any vector fn $v_i(\phi)$)

Conformal: $R_{ij} = \nabla_i \nabla_j F$ (any function $F(\phi)$))

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Where arises? (CFT = QFT w/ conformal symmetry)

- most condensed matter systems at critical points
- approximate sym of QCD at high energy
- string world sheet ~ = 1+1 QFT
- gravity in AdS spacetime dual to CFT

in $d=4$ Lorentz: $SO(3,1)$

Lorentz + P_μ + K_μ + Δ: $SO(4,2)$

AdS_5 = hyperboloid of signature $(4,1)$ in flat space of metric $(--++++)$.

Consequences of conf. inv.

$$\delta\phi(x) = x^\mu \partial_\mu \mathcal{O} + \Delta\mathcal{O} \quad \leftarrow \text{scale}$$

$$= v^\mu \partial_\mu \mathcal{O} + \frac{1}{4}(\partial_\mu v^\mu)\mathcal{O} \quad \leftarrow \text{conf.}$$

$$\delta\mathcal{O}(x) = v^\mu \partial_\mu \mathcal{O} + \frac{1}{4}(\partial_\mu v^\mu)\Delta\mathcal{O} \quad \leftarrow \ = \text{"primary" operator}$$

(non-primary also have $\partial_\mu \partial_\nu v^\rho$ term on right.)

ϕ^n : Δ = (classically) n

scale $\langle 0|T\,\mathcal{O}(x_1)\,\mathcal{O}(x_2)|0\rangle \propto (x_1 - x_2)^{-2\Delta}$

conf: for $\mathcal{O}_i$ primary operators

$$\langle 0|\,\mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2)\,\mathcal{O}_k(x_3)|0\rangle \propto (x_1-x_2)^{\frac{1}{2}(\Delta_3-\Delta_1-\Delta_2)}$$

$$\times (x_1-x_3)^{\frac{1}{2}(\Delta_2-\Delta_1-\Delta_3)} \times (x_2-x_3)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3)}$$

In $d=2$: if # of symmetries $\gtrsim$ # of 'one-particle states' (modes)

can solve using symmetry alone, plus a small amount of physical input ($\equiv$ Rational CFT)

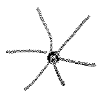
(other such "affine" or "Kac-Moody" symmetries arise, where the symmetry is an arbitrary function of x^+ or x^-

Operator renormalization:

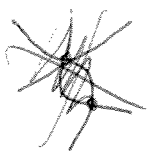
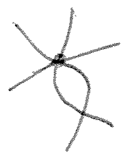
We were define $\langle 0 | T\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle$,

all renormalized. <u>But</u> if we take $x_2 \to x_1$, get <u>new</u>

divergences: to define $\phi^n(x)$ need $Z_{\phi^n}\phi^n(x)$.

Ex:  $\leftarrow n$ lines $\quad\to \gamma_{\phi^n}$ in CZ eq for

$\langle 0 | T\, \phi^n(x_1)\, \phi(x_2) \ldots \phi(x_m)|0\rangle$

  $\leftarrow$ leading divergence.

: $\frac{1}{2}n(n-1)$ such graphs.

For $n=2$ it's just $-2\gamma_m$

$$\gamma_{\phi^n} = n(n-1)\gamma_m + O(\lambda^2) \qquad \Delta_{\phi^n} = n \pm n(n-1)\gamma_m + O(\lambda^2)$$

$$= n(n-1)\frac{\lambda}{16\pi^2}$$

$$\langle 0 | T\, \phi^n(x)\, \phi^n(0)|0\rangle \propto x^{-2n - n(n-1)\lambda_*/16\pi^2 + O(\lambda_*^2)}$$

$(\lambda_*:$ if $\beta(\lambda_*) = 0)$