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Instantons on Noncommutative $\mathbb{R}^4$, and $(2, 0)$ Superconformal Six Dimensional Theory

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Abstract: We show that the resolution of moduli space of ideal instantons parameterizes the instantons on noncommutative $\mathbb{R}^4$. This moduli space appears to be the Higgs branch of the theory of k $D0$-branes bound to N $D4$-branes by the expectation value of the B field. It also appears as a regularized version of the target space of supersymmetric quantum mechanics arising in the light cone description of $(2, 0)$ superconformal theories in six dimensions.

1. Introduction

The appearence of noncommutative geometry [1] in the physics of D-branes has been anticipated with the very understanding of the fact that the gauge theory on the world-volume of N coincident D-branes is a non-abelian gauge theory [2], [3]. In this theory the scalar fields X_i in the adjoint representation are the non-abelian generalizations of the transverse coordinates of the branes.

It is known that the compactifications of Matrix theory [4], [5] on tori $\mathbf{T}^d$ exhibit richer structures as the dimensionality d of the tori increases [6], [7]. The compactification on a torus implies that certain constraints are imposed on the matrices X_i:

$$X_i + 2\pi R_i \delta_{ij} = U_j X_i U_j^{-1}. \tag{1.1}$$

It seems natural to study all possible solutions U_i to the consistency equations for the compactification of the matrix fields[1][9].

Recently, the noncommutative torus emerged as one of the solutions to (1.1), [9]. It has been argued that the parameter of noncommutativity is related to the flux of the B-field through the torus. It has been further shown in [10], that the compactification on a noncommutative torus can be thought of as a T-dual to a limit of the conventional

[1] The conjecture of [8] is that the non-abelian tensor fields in six dimensions would also appear as such solutions.

compactification on a commutative torus. See [11] for further developments in the studies of compactifications on low-dimensional tori.

On the other hand, the modified self-duality equations on the matrices in the Matrix description of fivebrane theory has been used in [12] in the study of quantum mechanics on the instanton moduli space. The modification is most easily described in the framework of ADHM equations. It makes the moduli space smooth and allows to define a six dimensional theory decoupled from the eleven-dimensional supergravity and all other M-theoretic degrees of freedom. The heuristic reason for the possibility of such decoupling is the fact that the Higgs branch of the theory is smooth and there is no place for the Coulomb branch to touch it.

In this paper we propose an explanation of the latter construction in terms of noncommutative geometry. We show that the solutions to modified ADHM equations parameterize (anti-)self-dual gauge fields on noncommutative $\mathbb{R}^4$.

2. Instantons on a Commutative Space

Let X be a four dimensional compact Riemannian manifold and P a principal G-bundle on it, with $G = U(N)$. The connection A is called anti-self-dual (ASD), or an instanton, if its curvature obeys the equation:

$$F^+ := \tfrac{1}{2}(F + *F) = 0,$$

where $* : \Omega^k \longrightarrow \Omega^{4-k}$ is the Hodge star. The importance of ASD gauge fields in physics stems from the fact that they minimize the Yang–Mills action in a given topological sector, i.e. for fixed $k = -\frac{1}{8\pi^2}\int \mathrm{Tr} F \wedge F$. In supersymmetric gauge theories the instantons are the configurations of gauge fields which preserve some supersymmetry, since the self-dual part of F appears in the right-hand side of susy transformations. For the same reason they play a major role in Matrix theory.

The space of ASD gauge gauge fields modulo gauge transformations is called the moduli space of instantons $\mathcal{M}_k$ and in a generic situation it is a smooth manifold of dimension

$$4Nk - \frac{N^2-1}{2}(\chi + \sigma),$$

where χ and σ are the Euler characteristics and signature of X respectively.

The moduli space $\mathcal{M}$ is non-compact. The lack of compactness is due to the so-called point-like instantons. What can happen is that for a sequence of ASD connections A_i the region D_i where some topological charge $-\frac{1}{8\pi^2}\int_{D_i} \mathrm{Tr} F \wedge F$ is concentrated can shrink to zero size. There exists a compactification $\bar{\mathcal{M}}_k$ due to Uhlenbeck [13] which simply adds the centers of the point-like instantons:

$$\bar{\mathcal{M}}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1} \times X \cup \ldots \cup \mathcal{M}_{k-l} \times \mathrm{Sym}^l X \ldots$$

which is suitable for certain purposes but not for all. In particular, the space $\bar{\mathcal{M}}_k$ has orbifold singularities.

One can also study non-compact spaces, $X = \mathbb{R}^4$ being the first example. In posing a moduli problem one has to specify the conditions on the behavior of the gauge fields at infinity. The natural condition is : $A_\mu \sim g^{-1}\partial_\mu g + O(\frac{1}{r^2})$ as $r \longrightarrow \infty$. One may also restrict the allowed gauge transformations to those which tend to 1 at infinity.

2.1. Review of ADHM construction. Atiyah–Drinfeld–Hitchin–Manin describe [14] a way of getting the solutions obeying the asymptotics stated above to the instanton equations on $\mathbb{R}^4$ in terms of solutions to some quadratic matrix equations. More specifically, in order to describe charge N instantons with gauge group $U(k)$ one starts with the following data:

1. A pair of complex hermitian vector spaces $V = \mathbb{C}^N$ and $W = \mathbb{C}^k$.
2. The operators $B_0, B_1 \in Hom(V,V)$, $I \in Hom(W,V)$, $J \in Hom(V,W)$, which must obey the equations $\mu_r = 0, \mu_c = 0$, where:

$$\begin{aligned}\mu_r =&[B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J,\\ \mu_c =&[B_0, B_1] + IJ.\end{aligned} \tag{2.1}$$

There is also a non-degeneracy condition which must be imposed by hand, namely, the set (B, I, J) should have trivial stabilizer in the $U(V)$ group.

For $z = (z_0, z_1) \in \mathbb{C}^2 \approx \mathbb{R}^4$ define an operator $\mathcal{D}_z^\dagger : V \oplus V \oplus W \to V \oplus V$ by the formula:

$$\begin{gathered}\mathcal{D}_z^\dagger = \begin{pmatrix}\tau_z\\ \sigma_z^\dagger\end{pmatrix},\\ \tau_z = \begin{pmatrix}B_0 - z_0 & -B_1 + z_1 & I\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}B_1 - z_1\\ B_0 - z_0\\ J\end{pmatrix}.\end{gathered} \tag{2.2}$$

Given the matrices obeying all the conditions above, the actual instanton solution is determined by the following rather explicit formulae:

$$A_\mu = \psi^\dagger \partial_\mu \psi, \tag{2.3}$$

where $\psi : W \to V \oplus V \oplus W$ solves the equations: $\mathcal{D}^\dagger \psi = 0$ and is normalized : $\psi^\dagger \psi = 1$.

The moduli space of instantons with fixed framing at infinity is identified with

$$\mathcal{M} = \left(\mu_r^{-1}(0) \cap \mu_c^{-1}(0)\right) / U(V), \tag{2.4}$$

where $U(V)$ is the group of unitary transformations of V acting on the matrices B, I, J in a natural way.

2.2. Regularization of ADHM data. As we noted above the compactification $\bar{\mathcal{M}}_k$ is a singular manifold. One may resolve it to a smooth variety by deforming the equations $\mu_r = \mu_c = 0$ to $\mu_r = \zeta_r \mathrm{Id}, \quad \mu_c = 0$. One may add a constant to μ_c as well but this modification is equivalent to the one already considered by a linear transformation of the data $B_{0,1}, B_{0,1}^\dagger, I, I^\dagger, J, J^\dagger$. The modification has been studied mathematically by various people and we recommend the beautiful lectures by H. Nakajima [15] for a review. The deformed data form a moduli space:

$$\mathcal{M}_\zeta = \left(\mu_r^{-1}(\zeta \mathrm{Id}) \cap \mu_c^{-1}(0)\right) / U(V) \tag{2.5}$$

and they parameterize the torsion free sheaves on $\mathbb{CP}^2$ with fixed framing at the line at infinity. As we shall show in the next section, the deformed moduli space parameterizes the instantons on the noncommutative $\mathbb{R}^4$.

3. Instantons on Noncommutative Spaces

The paradigm of noncommutative geometry is to describe the geometry of ordinary spaces in terms of the algebra of (smooth, continuous, ...) functions and then generalizing to the noncommutative case. In this sense, the noncommutative $\mathbb{R}^4$ is the algebra $\mathcal{A}_\omega$ generated by $x_\alpha, \alpha = 1, 2, 3, 4$ which obeys the relations:

$$[x_\alpha, x_\beta] = \omega_{\alpha\beta}, \tag{3.1}$$

where $\omega_{\alpha\beta}$ is a constant antisymmetric matrix. There are three distinct cases one may consider:

1. ω has rank 0. In this case $\mathcal{A}_\omega$ is isomorphic to the algebra of functions on the ordinary $\mathbb{R}^4$.
2. ω has rank 2. In this case $\mathcal{A}_\omega$ is the algebra of functions on the ordinary $\mathbb{R}^2$ times the noncommutative $\mathbb{R}^2 = \{(p, q)|[p, q] = -i\}$.
3. ω has rank 4. In this case $\mathcal{A}_\omega$ is *the* noncommutative $\mathbb{R}^4$. Let $\pi^{\alpha\beta}$ be the inverse matrix to $\omega_{\alpha\beta}$ and $x^\alpha = \pi^{\alpha\beta} x_\beta$. Let $\vec{x} = (x^\alpha)$ and $\vec{x}^\vee = (x_\alpha)$.

The algebra depends essentially on one number ζ (which can be scaled away, but we shall keep it in order to be able to take a limit to the commutative case). We shall denote it as $\mathcal{A}_\zeta$. Introduce the generators: $z_0 = x^1 + ix^2, z_1 = x^3 + ix^4$, then[2]

$$[z_0, \bar{z}_0] = [z_1, \bar{z}_1] = -\frac{\zeta}{2}. \tag{3.2}$$

The commutation relations (3.2), have an obvious group of automorphisms of the form $x_\alpha \mapsto x_\alpha + \beta_\alpha \cdot 1, \beta_\alpha \in \mathbb{R}$. We denote the Lie algebra of this group by $\underline{\mathbf{g}}$. For the algebra (3.2) to represent the algebra of real-valued functions (and be represented by Hermitian operators) we need $\zeta \in \mathbb{R}$. (In the language of mathematics we can say that this condition means that the algebra at hand has an involution.) We choose $\zeta > 0$. Of course, the algebra of polynomials in $z, \bar{z}$ should be completed in some way. We propose to start with the algebra End$\mathcal{H}$ of operators acting in the Fock space $\mathcal{H} = \sum_{(n_0,n_1)\in\mathbb{Z}^2_+} \mathbb{C}|n_0, n_1\rangle$, where $z, \bar{z}$ are represented as creation-annihilation operators:

$$\begin{aligned} z_0|n_0, n_1\rangle &= \tfrac{\zeta}{2}\sqrt{n_0 + 1}|n_0 + 1, n_1\rangle \ \bar{z}_0|n_0, n_1\rangle = \tfrac{\zeta}{2}\sqrt{n_0}|n_0 - 1, n_1\rangle, \\ z_1|n_0, n_1\rangle &= \tfrac{\zeta}{2}\sqrt{n_1 + 1}|n_0, n_1 + 1\rangle \ \bar{z}_1|n_0, n_1\rangle = \tfrac{\zeta}{2}\sqrt{n_1}|n_0, n_1 - 1\rangle. \end{aligned} \tag{3.3}$$

The algebra End$\mathcal{H}$ has a subalgebra End$_0\mathcal{H}$ of operators A which have finite norm; we will take the Hilbert–Schmidt norm: $\mathrm{Tr}_{\mathcal{H}}(AA^\dagger)^{\frac{1}{2}}$. We consider an algebra $\mathcal{A}_\zeta$ defined as a subalgebra of End$_0\mathcal{H}$ which consists of smooth operators, i.e. those A for which the function $f_A : \underline{\mathbf{g}} \to \mathrm{End}_0\mathcal{H}$, $f_A(\vec{t}) = Ad_{\vec{t}\cdot\vec{x}}A$ is smooth. Notice that 1 does not belong to this algebra. This is a consequence of non-compactness of $\mathbb{R}^4$. One can represent the elements of the Fock space $\mathcal{H}$ as L^2 functions in two variables $q_0 = x^1$ and $q_1 = x^3$ (oscillator wave functions in real polarization). It is actually possible to prove using the results of [16] that smooth operators are those whose matrix elements $\mathbf{A}(q_0', q_1'; q_0, q_1)$ belong to the Schwartz space $\mathcal{S}(\mathbb{R}^2 \times \mathbb{R}^2)$. We consider representations of the involutive algebra $\mathcal{A}_\zeta$ by means of operators in Hilbert space (Hilbert modules with involution in mathematical terminology). A vector bundle over a noncommutative space $\mathcal{A}$ is a

[2] So, $\pi^{\mu\nu}$ is proportional to ζ, while $\omega_{\alpha\beta}$ is proportional to $\frac{1}{\zeta}$ in accordance with the standard quasiclassical limits.

projective module $\mathcal{E}$, i.e. such a module for which another module $\mathcal{E}^\perp$ exists with the property $\mathcal{E} \oplus \mathcal{E}^\perp = \mathcal{A} \oplus \ldots \oplus \mathcal{A}$. (The last module is called free. In the commutative case free modules correspond to trivial bundles.) In our description of instantons over $\mathbb{R}^4$ we shall be dealing with free modules only.

The notion of connection in the bundle $\mathcal{E}$ has several definitions. The most convenient for us at the moment will be the following one. Let $\underline{\mathbf{g}} \subset \mathrm{Aut}(\mathcal{A})$ be a Lie sub-algebra of the algebra of automorphisms of $\mathcal{A}$. Then the connection ∇ is an operator $\nabla : \underline{\mathbf{g}} \times \mathcal{E} \to \mathcal{E}$ which obeys the Leibnitz rule:

$$\nabla_\xi(f \cdot s) = f \cdot \nabla_\xi s + \xi(f) \cdot s, \qquad f \in \mathcal{A}, \quad \xi \in \underline{\mathbf{g}}, \quad s \in \mathcal{E}. \tag{3.4}$$

The curvature of ∇ is an operator:

$$F(\nabla) : \Lambda^2\underline{\mathbf{g}} \times \mathcal{E} \to \mathcal{E}, \quad F(\nabla)(\xi, \eta) = [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}. \tag{3.5}$$

3.1. Instantons on noncommutative $\mathbb{R}^4$. Just like in the commutative case one may define the Yang–Mills action, the instanton equations, etc. See [1] for the definitions and [9] for recent discussion. We are looking at the solutions of the ASD conditions

$$F^+ = 0$$

on the connection in the module $\mathcal{E}$ over $\mathcal{A}_\zeta$, where the + sign means the self-dual part with respect to the natural Hodge star operator acting on the $\Lambda^2\underline{\mathbf{g}}$ where $\underline{\mathbf{g}} \approx \mathbb{R}^4$ is the abelian Lie algebra of automorphisms of $\mathcal{A}_\zeta$.

Now we are going to show that the resolution (2.5) gives rise to the ADHM description of instantons on the noncommutative $\mathbb{R}^4$.

Indeed, the core of the ADHM construction are the equations

$$\tau_z \sigma_z = 0, \quad \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z, \tag{3.6}$$

where the operators σ_z, τ_z are constructed as above. Now suppose that the matrices $B, B^\dagger, I^\dagger, I, J, J^\dagger$ obey the modified equations (2.5). Then (3.6) are no longer valid but they will be valid if the coordinate functions $z_i, \bar{z}_i$ will not commute! In fact, by imposing the commutation law $[z_0, \bar{z}_0] = [z_1, \bar{z}_1] = -\frac{\zeta}{2}$ we fulfill (3.6) as the term with ζ from commutators of B's is now compensated by the commutators of z's! We now follow the steps of the ordinary ADHM construction. We define an operator:

$$\mathcal{D}_z^\dagger : (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta \tag{3.7}$$

by the same formula (2.2). We look for the solution to the equation

$$\mathcal{D}_z^\dagger \psi = 0, \quad \psi : W \otimes \mathcal{A}_\zeta \to (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \tag{3.8}$$

which is again normalized: $\psi^\dagger \psi = \mathrm{Id}_{W \otimes \mathcal{A}_\zeta}$. These are defined up to unitary gauge transformations g acting on ψ on the right. Again, $A_\mu = \psi^\dagger \partial_\mu \psi$, where the derivative is understood as the action of $\underline{\mathbf{g}} = \mathbb{R}^4$ on $\mathcal{A}_\zeta$ by translations. We may now derive the formula for the curvature of the gauge field A. The derivation is very similar to the commutative case and it yields:

$$F = \psi^\dagger \left(d\mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} d\mathcal{D}_z^\dagger \right) \psi. \tag{3.9}$$

The operator $\mathcal{D}_z^\dagger \mathcal{D}_z : (V \oplus V) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta$ is again block-diagonal:

$$\mathcal{D}_z^\dagger \mathcal{D}_z = \begin{pmatrix} \Delta_z & 0 \\ 0 & \Delta_z \end{pmatrix}, \quad \Delta_z = \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z. \tag{3.10}$$

Hence, just as in the commutative case:

$$\begin{aligned} F =& \psi^\dagger \left(d\tau_z^\dagger \frac{1}{\Delta_z} d\tau_z + d\sigma_z \frac{1}{\Delta_z} d\sigma_z^\dagger \right) \psi = \\ =& \psi^\dagger \begin{pmatrix} d\bar{z}_0 \frac{1}{\Delta_z} dz_0 + dz_1 \frac{1}{\Delta_z} d\bar{z}_1 & -d\bar{z}_0 \frac{1}{\Delta_z} dz_1 + dz_1 \frac{1}{\Delta_z} d\bar{z}_0 & 0 \\ -d\bar{z}_1 \frac{1}{\Delta_z} dz_0 + dz_0 \frac{1}{\Delta_z} d\bar{z}_1 & d\bar{z}_1 \frac{1}{\Delta_z} dz_1 + dz_0 \frac{1}{\Delta_z} d\bar{z}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi \end{aligned} \tag{3.11}$$

which is anti-self-dual. We should warn the reader that here $dz_i, d\bar{z}_i$ are just the generating anti-commuting parameters for representing the matrix $F_\nabla(\xi, \eta)$ in short print. They commute with $z_i, \bar{z}_i$.

3.2. Commutative interpretation of the noncommutative equations. The construction above is not yet completely rigorous. We must prove that ψ exists and that Δ_z is invertible. A useful technique is to represent the equations over $\mathcal{A}_\zeta$ in terms of the (perhaps differential) equations on ordinary functions. In this way the multiplication of operators $a \cdot b$ is mapped to

$$a \star b(x) = e^{\frac{1}{2}\pi^{\mu\nu} \frac{\partial^2}{\partial\xi^\mu \partial\eta^\nu}} a(x+\xi) b(x+\eta)|_{\xi=\eta=0}. \tag{3.12}$$

Now we can study the questions we posed in the beginning of the section.

It follows from (3.10) that the condition that v is a zero mode of Δ_z is equivalent to the conditions: $\tau_z^\dagger v = 0$, $\sigma_z v = 0$, where v is an element of our algebra that is considered as an algebra of Hilbert–Schmidt operators in Fock space.

Let us remark that an operator equation $\mathbf{KL} = 0$ is equivalent to the condition that the image of the operator $\mathbf{L}$ is contained in the kernel of $\mathbf{K}$. In other words if we can solve the equation $\mathbf{Kv} = 0$, where $\mathbf{v}$ is a vector then we can also solve the operator equation $\mathbf{KL} = 0$, where $\mathbf{L}$ is an unknown operator. This remark allows us to say that in the conditions above we can consider v as an element of Fock space; if a non-zero solution does not exist in this new setting it does not exist in the old setting either. The elements of the Fock space $\mathcal{H}$ can be represented either as polynomials in z_0, z_1 (this representation we use below) or as L^2 functions on $\mathbb{R}^2$ with coordinates q_0, q_1.

The equations for the vector v can be written in holomorphic representation as follows:

$$\begin{aligned} B_0 v = z_0 v,\ & B_0^\dagger v = \frac{\zeta}{4} \frac{\partial}{\partial z_0} v, \\ B_1 v = z_1 v,\ & B_1^\dagger v = \frac{\zeta}{4} \frac{\partial}{\partial z_1} v, \\ J v = 0, \qquad & I^\dagger v = 0. \end{aligned} \tag{3.13}$$

The right column suggests that:

$$v \sim e^{\frac{4 z_0 B_0}{\zeta} z_0 B_0} v_0(z_1),$$

which is inconsistent with the left column equations. We also say that the left column equations imply that the creation operators have a finite-dimensional invariant subspace which is impossible. In the case $N = 1$ the operator Δ_z is explicitly positive definite.

The question of existence of ψ is addressed similarly: write $\psi = \psi_+ \oplus \psi_- \oplus \xi$, then (3.8) assumes the form:

$$\begin{aligned} &(B_0 + \frac{\zeta}{4}\frac{\partial}{\partial \bar{z}_0} - z_0)\psi_+ - (B_1 + \frac{\zeta}{4}\frac{\partial}{\partial \bar{z}_1} - z_1)\psi_- + I\xi = 0, \\ &(B_1^\dagger - \frac{\zeta}{4}\frac{\partial}{\partial z_1} - \bar{z}_1)\psi_+ + (B_0^\dagger - \frac{\zeta}{4}\frac{\partial}{\partial z_0} - \bar{z}_0)\psi_- + J^\dagger\xi = 0. \end{aligned} \tag{3.14}$$

Here ψ is a function which corresponds to an operator from our algebra. Equation (3.14) can be rewritten in the form:

$$D_A\Psi = -\Xi \tag{3.15}$$

for

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \Xi = \begin{pmatrix} -J^\dagger\xi \\ I\xi \end{pmatrix}$$

and D_A being the Dirac operator in the gauge field $A_\mu dx^\mu$, where

$$\left(A_\mu - \tfrac{1}{2}\omega_{\mu\nu}x^\nu\right) dx^\mu = \frac{4}{\zeta}\left(-B_1^\dagger dz_1 - B_0^\dagger dz_0 + B_0 d\bar{z}_0 + B_1 d\bar{z}_1\right).$$

The gauge field A has constant curvature.

Now let us write down Eq. (3.14) for vectors in the Fock space, this time using the L^2 representation: $\mathbf{v} = \mathbf{v}_+(\mathbf{q}) \oplus \mathbf{v}_-(\mathbf{q}) \oplus \mathbf{w}(\mathbf{q})$,

$$\begin{aligned} &(B_0 + \frac{\zeta}{4}\frac{\partial}{\partial q_0} - q_0)\mathbf{v}_+ - (B_1 - \frac{\zeta}{4}\frac{\partial}{\partial q_1} + q_1)\mathbf{v}_- + I\mathbf{w} = 0, \\ &(B_1^\dagger - \frac{\zeta}{4}\frac{\partial}{\partial q_1} - q_1)\mathbf{v}_+ + (B_0^\dagger - \frac{\zeta}{4}\frac{\partial}{\partial q_0} - q_0)\mathbf{v}_- + J^\dagger\mathbf{w} = 0. \end{aligned} \tag{3.16}$$

It has again the form (3.15) with D_A being a Dirac-like operator in two dimensions. Let us assume that the operator D_A has no zero modes (we hope that this assumption can be justified in the framework of perturbation theory with respect to ζ). Then we can write a solution to (3.15) in terms of the Green's function of the operator D_A. The space of solutions to (3.16) is identified with the space of $\mathbf{w}$'s, i.e. the space of W-valued L^2 functions on $\mathbb{R}^2$. Now, given the solution $\mathbf{v}_+(\mathbf{w}) \oplus \mathbf{v}_-(\mathbf{w}) \oplus \mathbf{w}$ to (3.16) we construct a solution to (3.15) as follows: ψ is supposed to map $W \otimes \mathcal{H} \otimes \mathcal{H} \to (V \oplus V \oplus W) \otimes \mathcal{H} \otimes \mathcal{H}$ (where we used the fact that the algebra of Hilbert–Schmidt operators can be identified with the Hilbert tensor product $\mathcal{H} \otimes \mathcal{H}$ and that $\mathcal{A}_\zeta \subset \mathcal{H} \otimes \mathcal{H}$). Now,

$$\tilde{\psi} = \mathbf{w} \otimes \mathbf{w}' \mapsto \left(\mathbf{v}_+(\mathbf{w}) \oplus \mathbf{v}_-(\mathbf{w}) \oplus \mathbf{w}\right) \otimes \mathbf{w}' g$$

does the job for any non-degenerate $g \in \mathrm{GL}_k(\mathcal{A})$. What remains is to normalize: $\psi = \left(\tilde{\psi}^\dagger\tilde{\psi}\right)^{-\frac{1}{2}}\tilde{\psi}$. This normalization $\psi^\dagger\psi = 1$ reduces the freedom of the choice of solutions to (3.16) to the unitary gauge transformations in $U(k)$.

One can also rewrite the ASD equations for the gauge field on the noncommutative $\mathbb{R}^4$ in terms of commutative functions which are Wick symbols of the gauge fields. Let $A_\mu(x)$ be four functions on $\mathbb{R}^4$ and consider the equations:

$$F^+_{\mu\nu} = 0 \tag{3.17}$$

with

$$F^i_{\mu\nu,j} = \partial_\mu A^i_{\nu,j} - \partial_\nu A^i_{\mu,j} + A^i_{\mu,k} \star A^k_{\nu,j} - A^i_{\nu,k} \star A^k_{\mu,j}. \tag{3.18}$$

Thus, the noncommutative ASD equations can be thought of as the deformation of the ordinary ASD equations. The solutions to (3.17) are automatically the solutions to the deformed Yang–Mills equations:

$$\partial_\mu F_{\mu\nu} + A_\mu \star F_{\mu\nu} = 0. \tag{3.19}$$

3.3. The completeness of the ADHM construction in the noncommutative case. Just like in the case of ordinary $\mathbb{R}^4$ one faces the question – whether the full set of solutions of ASD equations on noncommutative $\mathbb{R}^4$ is enumerated by the solutions to the matrix equation (3.6). It is natural to try to imitate the arguments of [17] and express the matrices $\mathbf{B}_\alpha, I, J$ in terms of solutions to the massless Dirac equations in the instanton background field. We simply sketch the relevant steps of the construction without giving any proofs.

The setup is similar to the commutative case. Given a projective module E over $\mathcal{A}_\zeta$ one studies the associated modules $\mathbf{S}_\pm = E \otimes_{\mathcal{A}_\zeta} S_\pm, S_\pm = \mathbb{C}^2 \otimes \mathcal{A}_\zeta$. Given a connection ∇ in the module E we form the Dirac-Weyl operators

$$D^\dagger : \mathbf{S}_+ \to \mathbf{S}_-, \quad D : \mathbf{S}_- \to \mathbf{S}_+ \tag{3.20}$$

by the standard formulae: $D^\dagger = \nabla_\alpha \otimes \sigma^\alpha$, $D = \nabla_\alpha \otimes \bar{\sigma}^\alpha$, where $\sigma^\alpha : S_+ \to S_-$, $\bar{\sigma}^\mu : S_- \to S_+$ are essentially the ordinary Pauli matrices and $\sigma_4 = \bar{\sigma}_4 = 1$. One proves the identities:

$$DD^\dagger = \Delta_\nabla \otimes \mathbf{1} + F^-_{\mu\nu} \otimes \sigma^{\mu\nu}, \quad D^\dagger D = \Delta_\nabla \otimes \mathbf{1} + F^+_{\mu\nu} \otimes \bar{\sigma}^{\mu\nu}. \tag{3.21}$$

Here Δ_∇ is the covariant Laplacian $E \to E$. Since there are no normalizable (see below) solutions to the equations $\Delta_\nabla \phi = 0$ then for the ASD ∇ one concludes that there are no solutions to the equation $D\psi = 0$. On the other hand, the index arguments predict, just like in the commutative case, the existence of N normalizable (in the sense, described in the next paragraph) zero modes of $D^\dagger$.

Let $\psi^i \in E \otimes S_+$ be a solution to $D^\dagger \psi^i = 0$, $i = 1, \ldots, N$. Then one may define a projection (just like in the commutative case) of $x_\alpha \psi^i$ onto the space of zero modes of $\mathcal{D}^\dagger$:

$$\left(x_\alpha \delta^i_j - \mathbf{B}^i_{\alpha,j}\right) \psi^j = D(\ldots), \tag{3.22}$$

where $\mathbf{B}_\alpha$ is some matrix (with $\mathbb{C}$-valued entries). We call $B_0 = \mathbf{B}_1 + i\mathbf{B}_2$, $B_1 = \mathbf{B}_3 + i\mathbf{B}_4$, etc.

The matrices I, J are recovered from the large $\vec{x}^2$ asymptotics of ψ^i. In order to explain what it means in the noncommutative setting we represent the coordinates on the noncommutative $\mathbb{R}^4$ as creation-annihilation operators acting in the auxiliary Fock space $\mathcal{H}$. Then the operator ψ^i has the corresponding Wick symbol $\tilde{\psi}^i$ which is simply a function on $\mathbb{R}^4$ whose large $\vec{x}^2$ limit is well-defined and is independent on the ordering prescription since for large $\vec{x}^2 >> \zeta$ one may neglect the noncommutativity of the coordinates.

4. Examples

4.1. Abelian instantons. It is very interesting to study the case $k = 1$. In the commutative case there are no solutions to the abelian instanton equations except for the trivial ones. We shall see that for every N there are non-trivial abelian instantons in the noncommutative case.

We need to solve the equations $\mu_r = \zeta \mathrm{Id}, \mu_c = 0$. It can be shown that for $\zeta > 0$ the solution must have $J = 0$ [0]. Moreover, on a dense open set in $\mathcal{M}_\zeta$ the matrices B_0, B_1 can be diagonalized by a complex gauge transformation[3]:

$$B_\alpha \to \mathrm{diag}\left(\beta_\alpha^1, \ldots, \beta_\alpha^N\right). \tag{4.1}$$

The eigenvalues β_α^i parameterize a set of N points on $\mathbb{R}^4$. The actual moduli space is the resolution of singularities of the symmetric product. Now suppose (B_0, B_1, I) is a solution to the ADHM equations. If we write the element of $V \oplus V \oplus W$ as $\psi = \psi_0 \oplus \psi_1 \oplus \xi$ then the equation $\mathcal{D}^\dagger \psi = 0$ is solved as:

$$\begin{aligned} \psi_\alpha &= -(B_\alpha - z_\alpha)^\dagger \delta^{-1} I \xi, \\ \delta &= \sum_{\alpha=0}^{1} (B_\alpha - z_\alpha)(B_\alpha^\dagger - \bar{z}_\alpha). \end{aligned} \tag{4.2}$$

The operator ξ is determined from the equation $\psi^\dagger \psi = 1$:

$$\xi = \left(1 + I^\dagger \delta^{-1} I\right)^{-\frac{1}{2}}. \tag{4.3}$$

The connection $A = \psi^\dagger d\psi$ has Yang's form:

$$A = \xi^{-1} \bar{\partial} \xi - \partial \xi \xi^{-1}, \tag{4.4}$$

where $\partial = dz_\alpha \frac{\partial}{\partial z_\alpha}$, $\bar{\partial} = d\bar{z}_{\bar{\alpha}} \frac{\bar{\partial}}{\partial \bar{z}_{\bar{\alpha}}}$. Explicitly:

$$A = \xi^{-1} d\xi + \xi^{-1} \alpha \xi,$$

where the "gauge transformed" connection α is equal to:

$$\begin{aligned} &\alpha = \xi^2 \partial \xi^{-2} = \\ &\frac{1}{1 + I^\dagger \delta^{-1} I} I^\dagger \delta^{-1} (B_\alpha^\dagger - \bar{z}_\alpha) dz_\alpha \delta^{-1} I. \end{aligned} \tag{4.5}$$

For $N = 1$: $\xi = \left(\frac{d - \zeta/2}{d + \zeta/2}\right)^{\frac{1}{2}}$, $d = z_0 \bar{z}_0 + \bar{z}_1 z_1$, and the gauge field α is explictly non-singular if the correct ordering is used:

$$\alpha = \frac{1}{d(d + \zeta/2)} \left(\bar{z}_0 dz_0 + \bar{z}_1 dz_1\right). \tag{4.6}$$

One can also compute the curvature:

$$\begin{aligned} &F_A = \frac{\zeta}{(d - \zeta/2) d (d + \zeta/2)} \left(f_3 \left(dz_0 d\bar{z}_0 - dz_1 d\bar{z}_1\right) + f_+ d\bar{z}_0 dz_1 + f_- d\bar{z}_1 dz_0\right), \\ &f_3 = z_0 \bar{z}_0 - z_1 \bar{z}_1, \quad f_+ = 2 z_0 \bar{z}_1, \quad f_- = 2 z_1 \bar{z}_0. \end{aligned} \tag{4.7}$$

The factor $\frac{\zeta}{(d - \zeta/2) d (d + \zeta/2)}$ has a singularity at the state $|0, 0\rangle$ but it is projected out since $f_{3,\pm}$ always have $\bar{z}_0$ or $\bar{z}_1$ on the right. The action density is given by

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[3] We thank D. Bernard for pointing out an error in the earlier version of this paper.

$$\hat{S} = -\frac{1}{8\pi^2}F_A F_A = \frac{\zeta^2}{4\pi^2}\frac{1}{d^2(d-\zeta/2)(d+\zeta/2)}\Pi, \tag{4.8}$$

where $\Pi = 1 - |0,0\rangle\langle 0,0|$. We may define the total action as

$$(\zeta\pi)^2 \mathrm{Tr}_{\mathcal{H}}\hat{S} = 4\sum_{N=1}^{\infty}\frac{1}{N(N+1)(N+2)} = 1.$$

4.2. 't Hooft solutions. It is also relatively easy to describe the noncommutative analogues of 't Hooft solutions for $k = 2$:

$$A_\mu = i\Sigma^{\mu\nu}\Phi^{-1}\partial_\nu\Phi, \tag{4.9}$$

where $\Sigma^{\mu\nu}$ is self-dual in $\mu\nu$ and takes values in traceless two by two Hermitian matrices. In the commutative case the ASD conditions boil down to the Laplace equation on Φ [18]. In the noncommutative case the potential trouble comes from the term in the curvature $F_{\mu\nu}$:

$$\{\Sigma^{\mu\alpha}, \Sigma^{\nu\beta}\}[J_\alpha, J_\beta]$$

with $J_\alpha = \Phi^{-1}\partial_\alpha\Phi$. It is easy to show that the problematic piece is equal to

$$[J_\mu, J_\nu] - \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}[J^\alpha, J^\beta]$$

which is explicitly anti-self-dual. Hence we have shown that the ansatz (4.9) works in the noncommutative case if Φ obeys the Laplace equation which is now to be solved in the noncommutative setting. A solution looks exactly like the commutative ansatz:

$$\Phi = 1 + \sum_{i=1}^{N}\frac{\rho_i^2}{|\vec{x} - \vec{\beta}_i|^2}, \tag{4.10}$$

where now the components of $\vec{x} = (x^\mu)$ represent the noncommuting coordinates.

One might wonder what are the properties of Φ viewed as an operator in a Fock space, where x_α's are realized as the creation and annihilation operators. By acting on a vacuum state $|0,0\rangle$ in the occupation number representation Φ creates a sum of the coherent states. We may represent Φ as follows:

$$\Phi = 1 + \sum_{i=1}^{N}\rho_i^2\upsilon_i,$$

where $\upsilon_i = e^{-\vec{x}^\vee\cdot\vec{\beta}_i}\Upsilon e^{\vec{x}^\vee\cdot\vec{\beta}_i}$, $\Upsilon = \frac{1}{\vec{x}^2}$. Let us now prove that each υ_i obeys the Laplace equation. Since the automorphisms in $\underline{\mathbf{g}} = \mathbb{R}^4$ are internal and they commute it is sufficient to prove that Υ obeys the Laplace equation. For the latter it is enough to check that

$$\vec{x}^2\Upsilon + \Upsilon\vec{x}^2 = 2\vec{x}\Upsilon\vec{x}$$

which is true for $\Upsilon = \frac{1}{\vec{x}^2}$. The operator Υ is diagonal in the occupation number representation and its eigenvalue on a state $|n_0, n_1\rangle$ is equal to $\frac{1}{\frac{\zeta}{2}(n_0+n_1+1)}$.

5. Relation to Matrix Description of Six Dimensional $(2,0)$ Theory

In the recent papers [12], [19] the proposal for Matrix description of six dimensional $(2,0)$ superconformal theory of k coincident fivebranes has been made. The theory which arises on the worldvolume of k coincident fivebranes has the property that when it is compactified on a circle it becomes $U(k)$ gauge theory in 4 + 1 dimension which the coupling which goes to zero as the radius goes to zero. So, by compactifying the theory on a light-like circle one gets the gauge theory with zero coupling. Of course, in a given instanton sector the only surviving gauge configurations are instantons. The theory becomes a quantum mechanics on the moduli space of instantons. But, the latter space has singularities coming from the point-like instantons and at these singularities a second branch of the theory, corresponding to the emission or absorption of $D0$-branes develops. The theory becomes interacting with the bulk degrees of freedom which makes it harder to study. The proposal of [12] was to deform the instanton moduli space by turning on a B field along four dimensions which serves as a FI term ζ in (2.5). The interpretation of our paper is that turning on the B field effectively makes $\mathbb{R}^4$ noncommutative and the instantons live on it.

Now, the problem of instantons shrinking to zero size is cured by quantum fluctuations! Indeed, the position of the point-like instanton is smeared over a region of size $\sim \zeta$ which makes it no longer point-like. So the Higgs branch becomes a smooth hyperkahler manifold and the theory becomes six dimensional (at least decoupled from the bulk degrees of freedom).

6. Future Directions

In this section we briefly sketch a few directions of future research.

6.1. Nahm's transform. Let us define an n-dimensional noncommutative torus $\mathcal{A}_\theta$ as an associative involutive algebra with unit generated by the unitary generators $U_1, ..., U_n$ obeying

$$U_\mu U_\nu = e^{i\theta_{\mu\nu}} U_\nu U_\mu .$$

Here $\theta_{\mu\nu}$ is an antisymmetric tensor; we can also consider it as a 2-form on $\mathbf{R}^n$. The infinitesimal automorphisms $\delta_\alpha U_\beta = \delta_{\alpha\beta} U_\beta$ generate the abelian Lie algebra L_θ acting on $\mathcal{A}_\theta$. As before we use L_θ to define the notion of connection in a $\mathcal{A}_\theta$-module. We always consider modules equipped with Hermitian inner product and Hermitian connections. We will construct a generalization of Nahm's transform [20] relating connections on $\mathcal{A}_\theta$-modules to connections on $\mathcal{A}_{\hat\theta}$-modules [21]. Here $\mathcal{A}_\theta$ and $\mathcal{A}_{\hat\theta}$ are two four-dimensional noncommutative tori. To define a noncommutative generalization of Nahm's transform we need a $(\mathcal{A}_\theta, \mathcal{A}_{\hat\theta})$-module $\mathcal{P}$ with $\mathcal{A}_\theta$ -connection ∇_α and $\mathcal{A}_{\hat\theta}$-connection $\hat\nabla_\mu$. The fact that $\mathcal{P}$ is a $(\mathcal{A}_\theta, \mathcal{A}_{\hat\theta})$-module means that $\mathcal{P}$ is a left $\mathcal{A}_\theta$-module and a right $\mathcal{A}_{\hat\theta}$-module; we assume that $(ax)b = a(xb)$ for $a \in \mathcal{A}_\theta,\ b \in \mathcal{A}_{\hat\theta},\ x \in \mathcal{P}$. In other words, $\mathcal{P}$ can be considered as a $\mathcal{A}_{\theta\oplus\hat\theta}$-module, where $\mathcal{A}_{\theta\oplus\hat\theta}$ is an eight-dimensional noncommutative torus corresponding to the 2-form $\theta \oplus \hat\theta$ on $\mathbf{R}^8$. Assume that the commutators $[\nabla_\alpha, \nabla_\beta],\ [\hat\nabla_\mu, \hat\nabla_\nu],\ [\nabla_\alpha, \hat\nabla_\mu]$ are c-numbers:

$$[\nabla_\alpha, \nabla_\beta] = \omega_{\alpha\beta},\ [\hat\nabla_\mu, \hat\nabla_\nu] = \hat\omega_{\mu\nu},\ [\nabla_\alpha, \hat\nabla_\mu] = \sigma_{\alpha\mu}. \tag{6.1}$$

The condition (6.1) implies that the curvature of connection $\nabla_\alpha \oplus \hat\nabla_\mu$ on $\mathcal{A}_{\theta\oplus\hat\theta}$ is constant. One more assumption is that ∇_α commutes with multiplication by elements of $\mathcal{A}_{\hat\theta}$ and $\hat\nabla_\alpha$ commutes with multiplication by elements of $\mathcal{A}_\theta$. Of course, the module

$\mathcal{P}$ and the connections $\nabla_\alpha, \hat{\nabla}_\mu$ obeying the above requirements exist only under certain conditions on $\theta, \hat{\theta}$. For any right $\mathcal{A}_\theta$-module R with connection ∇^R_α we consider Dirac operator $\mathcal{D} = \Gamma^\alpha(\nabla^R_\alpha + \nabla_\alpha)$ acting in the tensor product

$$(R \otimes_{\mathcal{A}_\theta} \mathcal{P}) \otimes S.$$

(To define Γ-matrices we introduce an inner product in L_θ and in $L_{\hat{\theta}}$.) This operator commutes with multiplication by the elements of $\mathcal{A}_{\hat{\theta}}$, hence the space of zero modes of $\mathcal{D}$ can be regarded as $\mathcal{A}_{\hat{\theta}}$-module; we denote it as $\hat{R}$. The connection $\hat{\nabla}_\mu$ induces a connection $\hat{\nabla}'_\mu$ on

$$(R \otimes_{\mathcal{A}_\theta} \mathcal{P}) \otimes S.$$

We define a connection $\nabla^{\hat{R}}_\mu$ on $\hat{R}$ as $P\hat{\nabla}'_\mu$, where P is the orthogonal projection:

$$(R \otimes_{\mathcal{A}_\theta} \mathcal{P}) \otimes S \to \hat{R}.$$

The above construction can be regarded as a generalized Nahm's transform. To prove that its properties are similar to the properties of the standard Nahm's transform we should impose additional conditions on module $\mathcal{P}$ and connections $\nabla_\alpha,\ \hat{\nabla}_\mu$. It is sufficient to assume that $\sigma_{\alpha\mu}$ determines a non-degenerate pairing between L_θ and $L_{\hat{\theta}}$. Then we can use this pairing to define an inner product in $L_{\hat{\theta}}$. A connection ∇^R_α is an analogue of ASD connection if its curvature $F_{\alpha\beta}$ obeys $F^+_{\alpha\beta} + \omega_{\alpha\beta} \cdot 1 = 0$. (This condition is equivalent to antiselfduality of the connection $\nabla^R_\alpha + \nabla_\alpha$.) Then one can prove that the curvature $\hat{F}_{\mu\nu}$ satisfies $\hat{F}^+_{\mu\nu} - \hat{\omega}_{\mu\nu} \cdot 1 = 0$.

The forms $\omega, \hat{\omega}$ obey certain quantization conditions which depend on $\theta, \hat{\theta}$. We plan to return to this issue elsewhere.

Thus, Nahm's transform maps the modified instantons on one noncommutative torus to the modified instantons on the other noncommutative torus. It can be thought of as an analogue of Morita equivalence. The last notion is usually used in the context of equivalence of algebras and the categories of left- and/or right- modules over them.[4] In the context of gauge theories we study the modules with connections. Nahm's construction gives correspondence between such modules; modules with ASD connections are mapped to each other.

The instantons on the noncommutative torus appear in the problem of $D0$-branes bound to $D4$-branes wrapping the four-torus $A = T^4$ with the B-field turned on. T-duality maps the torus A to its dual and the natural conjecture [22] is that on the level of low-energy fields it maps the instantons on A to those on $A^\vee$. There exist stronger conjectures about the moduli space of instantons on the four-torus which fulfill the base of the constructions of six dimensional interacting string theories [23] and also provide a heuristic test of U-dualities [24]. One assumes that $\mathcal{M}_{N,U(k)} = \mathrm{Sym}^{kN} T^4$ and studies the two dimensional sigma model with target $\mathcal{M}_{N,U(k)}$. Since the symmetric product is an orbifold of the hyperkahler space one may perturb the theory by the marginal operators responsible for blowing up the singularities of the orbifold. As was argued in [25] such a perturbation breaks the natural symmetries of the problem (and cannot be

[4] Two algebras A, B are called Morita equivalent if there exist the (A, B) module E and (B, A) module F s.t. $E \otimes_B F \approx A, \quad F \otimes_A E \approx B$. For example, every algebra A is Morita equivalent to the algebra $\mathrm{Mat}_N(A)$ of matrices with coefficients in A. Given two Morita equivalent algebras one may construct for any A-module $\mathcal{E}$ a B-module $\mathcal{F}$ and vice versa: $\mathcal{F} = F \otimes_A \mathcal{E}$.

The fact that the holomorphic counterpart of Nahm's transform – the Fourier–Mukai transform – can be understood in the framework of generalized Morita equivalence has been pointed out to us by M. Kontsevich.

interpreted as responsible for interactions of little strings). We have argued that given a noncommutative torus (or noncommutative $\mathbb{R}^4$), where the relevant $SU(2)$ symmetry is already broken by the noncommutative deformation, the moduli space of instantons is already smooth, and the decoupling arguments can be applied.

6.2. Instantons on noncommutative ALE spaces. The asymptotically locally euclidean (ALE) manifolds X_Γ are the hyperkahler resolutions of singularities of the orbifold $\mathbb{C}^2/\Gamma$ for $\Gamma \subset SU(2)$ being a discrete subgroup. The subgroups correspond to ADE Lie groups G via McKay's correspondence [26]. The space X_Γ depends on $r = \mathrm{rk}G$ parameters $\vec{\zeta}_i \in \mathbb{R}^3, i = 0, \dots, r$,

$$\sum_i \vec{\zeta}_i = 0.$$

The spaces X_Γ as well as the moduli spaces of $U(N)$ instantons on X_Γ can be constructed via hyperkähler reductions of vector spaces [27][28]. These constructions were interpreted in [29] as originating from the gauge theory on v Dp-branes, put at the orbifold point of $\mathbb{C}^2/\Gamma$ inside the w $D(p+4)$-branes. It has been noticed in [29] that in principle the condition $\sum_i \vec{\zeta}_i = 0$ can be relaxed by turning on a self-dual part of $B_{\mu\nu}$ along $\mathbb{C}^2/\Gamma$. As before, we interpret this as the process of going to the noncommutative ALE space with instantons on it. The ADHM construction of [28] generalizes straightforwardly to this case provided the original vector space $V = \mathrm{Hom}(\mathbb{C}^\Gamma, \mathbb{C}^\Gamma \otimes \mathbb{C}^2)^\Gamma$ whose reduction yields the ALE space is replaced by its noncommutative deformation:

$$V^{\mathrm{q}} = \mathrm{Hom}(\mathbb{C}^\Gamma, \mathbb{C}^\Gamma \otimes \mathcal{A}_\zeta)^\Gamma.$$

6.3. Instantons and holomorphic bundles on noncommutative surfaces. It is well-known that solutions to the instanton equations on the complex surface determine the holomorphic structure in the bundle where the gauge field is defined. In fact, Donaldson-Uhlenbeck-Yau theorem establishes an equivalence between the moduli space of stable in a certain sense holomorphic bundles over a surface S and solutions to the instanton equations on S [30][31]. The holomorphic bundle $\mathcal{E}$ defines a sheaf of its sections, and can be replaced by this sheaf for many purposes. In fact, not every sheaf $\mathcal{F}$ comes from a bundle - for this it must be what is called a locally-free sheaf (the term free means that the sections of $\mathcal{F}$ form a free module over the sheaf $\mathcal{O}$ of holomorphic functions on S). By relaxing the condition of being locally free but insisting on being torsion free, one gets a generalization of the notion of a holomorphic bundle. However, it was not clear how to obtain the corresponding generalization of the notion of instanton. Of course, the consideration of torsion free sheaves allows us to compactify the *moduli space* of instantons [32], [15]. But it is by no means clear whether the compactification can be achieved within a gauge theory. It has been conjectured in [33] that such a compactification occurs in string theory.

We claim that *there exists a generalization of the instanton field* for the torsion free sheaves. This is simply the gauge field on the *noncommutative surface* S.

The importance of the spaces $\mathcal{M}_\zeta$ has been appreciated in the context of string duality and field theory a while ago (see [34], [35], [24], [12]). We hope that the proposed interpretation will help to find further applications as well as justify various assertions needed for establishing string/field dualities.

In our presentation the noncommutativity of $\mathbb{R}^4$ entered only through the construction of the connection in a given bundle. The holomorphic bundle underlying the instanton could be described in purely commutative terms. This relies on the fact that with our choice of the Poisson structure the subalgebra $\mathcal{O}$ of holomorphic functions on $\mathbb{C}^2$ is commutative. It is interesting to study the picture in other complex structures. In fact, a slightly redundant but interesting deformation of ADHM equations is the three-parametric one:

$$\mu_r = \zeta_r \mathrm{Id}, \quad \mu_c = \zeta_c \mathrm{Id}, \tag{6.2}$$

where $\zeta_r \in \mathbb{R}, \zeta_c \in \mathbb{C}$. It is customary to combine $(\zeta_r, \zeta_c, \overline{\zeta_c})$ into a three vector $\vec{\zeta} \in \mathbb{R}^3$.

The specifics of $\mathbb{R}^4$ is that one can always get rid of ζ_c by appropriate rotation in the $SU(2)_R$ global group. Let us not do it but rather look at the complex equation $\mu_c = \zeta_c \mathrm{Id}$:

$$\tau_z \sigma_z = 0, \quad \text{iff} \quad [z_0, z_1] = -\tfrac{1}{2}\zeta_c \tag{6.3}$$

Thus the ζ_c-deformation allows one to construct a coherent sheaf over the noncommutative $\mathbb{C}^2$ as the cohomology of the complex of sheaves:

$$0 \to V \otimes \mathcal{O}_{\zeta_c} \xrightarrow{\sigma_z} \left(V \otimes \mathbb{C}^2 \oplus W\right) \otimes \mathcal{O}_{\zeta_c} \xrightarrow{\tau_z} V \otimes \mathcal{O}_{\zeta_c} \to 0, \tag{6.4}$$

where $\mathcal{O}_{\zeta_c}$ is the associative algebra generated by z_0, z_1 obeying the relation $[z_0, z_1] = -\frac{1}{2}\zeta_c$ and τ_z, σ_z are given by the same formulae (2.2).

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As our paper was ready for publication we have learned about the paper [36] which will adress certain issues concerning gauge theories on noncommutative T^4.

References

1. Connes, A.: *Noncommutative Geometry*. London–New York: Academic Press, 1994
2. Witten, E.: Bound states of strings and p-branes. Nucl. Phys. **B460**, 335 (1990), hep-th/9510135
3. Douglas, M.: Superstring dualities, Dirichlet branes and the small scale structure of space. hep-th/9610041
4. Banks, T., Fischler, W., Shenker, S. and Susskind, L.: M theory as a Matrix Model: A Conjecture. hep-th/9610043, Phys. Rev. **55**, 112 (1997)
5. Banks, T.: Matrix Theory. hep-th/9710231
6. Rozali, M.: Matrix Theory and U Duality in Seven Dimensions. hep-th/9702136, Phys. Lett. **400B**, 260 (1997); Berkooz, M., Rozali, M. and Seiberg, N.: Matrix Description of M theory on T^4 and T^5. hep-th/9704089, Phys. Lett. **408B**, 105 (1997)
7. Seiberg, N.: New Theories in Six Dimensions and Matrix Description of M-theory on T^5 and $T^5/\mathbb{Z}_2$. hep-th/9705221
8. Shatashvili, S.: Unpublished
9. Connes, A., Douglas, M.R. and Schwarz, A.: Noncommutative Geometry and Matrix Theory: Compactification on Tori. hep-th/9711162
10. Douglas, M., Hull, C.: D-branes and noncommutative geometry. hep-th/9711165
11. Leigh, R. and Rozali, M.: hep-th/9712168; Ho, P.-M., Wu, Y.-Y. and Wu, Y.-S.: hep-th/9712201; Ho, P.-M., Wu, Y.-S.: hep-th/9801147; Li, M.: hep-th/9802052
12. Aharony, O., Berkooz, M., Seiberg, N.: Light-cone description of $(2, 0)$ supersonformal theories in six dimensions. hep-th/9712117
13. Uhlenbeck, K.: Removable Singularities in Yang–Mills Fields. Commun. Math. Phys. **83**, 11 (1982)

14. Atiyah, M., Drinfeld, V., Hitchin, N. and Manin, Y.: Construction of Onstantons. Phys. Lett. **65B**, 185 (1978)
15. Nakajima, H.: Heisenberg algebra and Hilbert schemes of points on projective surfaces. alg-geom/9507012; Lectures on Hilbert schemes of points on surfaces. H. Nakajima's homepage
16. Rieffel, M.: Deformation quantization for actions of $\mathbb{R}^d$. Memoirs of AMS, vol. **106**, N 506 (1993)
17. E. Corrigan, Goddard, P.: Constructions of instanton monopole solutions and reciprocity. Ann. Phys. (NY) **154**, 253–279 (1984)
18. Rajaraman, R.: Solitons and Instantons. Amssterdam: North-Holland, 1987
19. Aharony, O., Berkooz, M.,Kachru, S., Seiberg, S., Silverstein, E.: Matrix Description of Interacting Theories in Six Dimensions. hep-th/9707079
20. van Baal, P., Braam, P.: Nahm's transformation for instantons. Commun. Math. Phys. **B 122**, 267–280 (1989)
21. Astashkevich, A., Nekrasov, N., Schwarz, A.: In preparation
22. Douglas, M., Moore, G., Nekrasov, N., Shatashvili, S.: Unpublished
23. Dijkgraaf, R., Verlinde, E., Verlinde, H.: BPS spectrum of the five-brane and black hole entropy. hep-th/9603126; Dijkgraaf, R., Verlinde, R., Verlinde, H.: BPS quantization of the five-brane. hep-th/96040055; Schwarz, J.: Self-dual string in six dimensions. hep-th/9604171; Dijkgraaf, R., Verlinde, E., Verlinde, H.: $5D$ black holes and matrix strings. hep-th/9704018; Losev, A., Moore, G., Shatashvili, S.: M & m's. hep-th/9707250
24. Vafa, C.: Instantons on D-branes. hep-th/9512078, Nucl.Phys. **B463**, 435–442 (1996)
25. Witten, E.: On the conformal field theory of the Higgs branch. hep-th/9707093
26. McKay, J.: Graphs, Singularities and Finite groups. Proc. Sympos. Pure Math. **37**, Providence, RI: AMS 1980, pp. 183–186
27. Kronheimer, P.: The construction of ALE spaces as hyper-kahler quotients. J. Diff. Geom. **28**, 665 (1989)
28. Kronheimer, P. and Nakajima, H.: Yang–Mills instantons on ALE gravitational instantons. Math. Ann. **288**, 263 (1990)
29. Douglas, M., Moore, G.: D-branes, Quivers and ALE Instantons. hep-th/9603167
30. Donaldson, S.: Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. Proc. Lond. Math. Soc, **50**, 1 (1985)
31. Donaldson, S.K. and Kronheimer, P.B.: *The Geometry of Four-Manifolds*. Oxford: Clarendon Press, 1990
32. For Gieseker's compactification, see Okonek, C, Schneider, M., Spindfler, H.: *Vector bundles on complex projective spaces*. Progress in Math., Basel–Boston: Birkhauser, 1980
33. Vafa, C., Witten, C.: A strong coupling test of S-duality. Nucl. Phys. **B431**, 3–77 (1994)
34. Losev, A., Moore, G., Nekrasov, N., Shatashvili, S.: Four-Dimensional Avatars of 2D RCFT. hep-th/9509151, Nucl. Phys. Proc. Suppl. **46**:130–145 (1996)
35. Harvey, J.A., Moore, G.: On the algebras of BPS states. hep-th/9609017
36. Berkooz, M.: To appear

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