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Mode Classification of Linear Adiabatic Oscillations of Spherically Symmetric Stars

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Abstract

We propose a mathematical scheme to classify the eigenmodes of linear adiabatic oscillations of spherically symmetric stars, with fully taking account of the perturbation to the gravitational potential. The scheme allows us to allocate a unique and continuous integral index (the radial order) to each eigenmode with a positive squared eigenfrequency of any stellar structure, which is invariant against any continuous change in the equilibrium structure. Based on this index, we can unambiguously define the three kinds of eigenmodes of adiabatic stellar oscillations, p modes, g modes, and f modes.

Key words: stars: oscillations — Sun: oscillations — methods: analytical

1. Introduction

The eigenmode analysis is one of the most fundamental problems in theory of stellar oscillations. In this paper, we concentrate on the simplest case, linear and adiabatic oscillations of spherically symmetric stars, which are free from the effects of rotation and the magnetic field. In this case, the angular dependence of perturbed physical quantities is described by spherical harmonics. Each spherical harmonic is characterized by the two indexes, the spherical degree l and the azimuthal order m. The former is equal to the total number of nodal lines of the angular pattern on a sphere specified by the spherical harmonic, while the latter is equal to the number of nodal lines of the pattern that pass through both of the poles.

It was Cowling (1941) who empirically classified the nonradial eigenmodes (with $l \geq 2$) by examining the physical character of oscillations in the case of polytropic models. He divided the eigenmodes into three categories: the high frequency branch, the low frequency branch, and the single eigenmode with an intermediate frequency for a given l. Following the nature of the restoring force, the high and low frequency branches were named as pressure (p) modes and gravitational (g) modes, respectively, while the mode in the third category was called the fundamental (f) mode, which is interpreted as consisting of surface gravity waves. However, it is not straightforward to justify this classification for general stellar models with mathematical rigor. In fact, our current mathematical understanding of the eigenmode analysis depends on l.

In the case of radial oscillations, which have $l = 0$, the problem is essentially reduced to the Strum–Liouville problem (e.g. Ledoux & Walraven 1958), whose properties have been examined thoroughly in mathematics. We can show that there always exists an eigenmode with the lowest frequency, which is called the radial fundamental mode. Note that it is not related to the fundamental modes of the nonradial case that Cowling (1941) introduced. There is only one node at the center in the radial component of the displacement vector ξ_r of the radial fundamental mode. The eigenmode with the second lowest frequency is termed the first overtone mode, which has two nodes (one at the center and the other between the center and the surface) in ξ_r. Generally speaking, the number of nodes in ξ_r of the eigenmode with the n-th lowest frequency [the $(n-1)$-th overtone mode] is generally equal to n. We can thus tell the order of a given eigenmode in the eigenfrequency spectrum by only counting the number of nodes in ξ_r. It is important that, if we follow the eigenmode with the n-th lowest frequency during continuous changes in the equilibrium structure, the corresponding eigenfunction ξ_r keeps having n nodes. Because, from a physical point of view, the restoring force of radial oscillations is the pressure gradient, all of the radial modes are qualified to be called p modes. The eigenmode with the n-th lowest frequency is then regarded as the n-th p mode (the p_n mode).

The problem of nonradial oscillations, which have $l > 0$, is more complicated, because the buoyancy force takes part in the problem due to inhomogeneity induced in the horizontal direction. From a mathematical point of view, the reason for the complexity is that the governing system of equations is of the fourth order, to which the Strum–Liouville theory is not applicable.

The dipolar modes, which correspond to the case of $l = 1$, are exceptional in this context, because there exists a specific integral coming from momentum conservation, which reduces the order of the governing system of equations from four to two (Takata 2005, 2006a). The resultant system can be analyzed by the Strum theorems (e.g. Arnol'd 1992; Eckart 1960). In general, there does not exist the minimum value in the eigenfrequency. The frequency spectrum usually has an accumulation point at zero, and there is no upper bound in it. We can still identify the origin of the eigenmodes that corresponds to the radial fundamental mode (or, equivalently, the first p mode). The order of each eigenmode relative to the origin in the eigenfre-

quency spectrum is then known from the corresponding eigenfunctions. The p modes are defined as the eigenmodes with frequencies higher than or equal to that of the origin, while we can characterize the g modes as the modes with frequencies lower than that of the origin. The relative order in the frequency spectrum provides the unique and continuous integral index to each mode, which can be identified as the radial order. The radial order is again invariant against continuous change in the equilibrium structure.

For the quadrupolar and higher multipolar modes (the modes with $l \geq 2$), there has been no mathematically exact scheme to classify the eigenmodes. What has often been done is to ignore the perturbation to the gravitational field caused by oscillations, following Cowling (1941). Once we accept this treatment (the Cowling approximation), the order of the governing system of equations is decreased from four to two. We can readily apply Strum's analyses to the second order system, so that the radial order of each mode can be calculated from the eigenfunctions (Eckart 1960; Scuflaire 1974; Osaki 1975). It is known that this scheme appears to give valid radial orders to almost all eigenmodes (with $l \geq 2$) of any realistic stellar models. However, it would be valuable if we could establish a complete scheme of the mode classification without assuming the Cowling approximation to understand the structure of the problem deeply.

In this paper, we focus on the problem of the mode classification of adiabatic stellar oscillations, fully taking account of the perturbation to the gravitational field. The key point of our analysis is that it is based on the mathematical framework developed in the field of seismology (Woodhouse 1988), which provides an original view of eigenvalue problems of a specific type. We not only reproduce the framework in an elementary way, but also make extensions in a few points. We explicitly discuss the case of degeneracy in which two eigenmodes with the same spherical degree happen to have the same eigenvalue. In addition, we devise a scheme of the mode classification that is specific to linear Hamiltonian systems with two degrees of freedom.

The structure of the paper is as follows: we summarize the formulation of the problem in section 2; a scheme of the mode classification is proposed in section 3; section 4 is devoted to discussion; we finally give conclusion in 5. The preliminary results of the present study are found in Takata (2012).

2. Formulation of the Problem

2.1. Standard Formulation Based on the Fitting Method

Adiabatic oscillations of stars are generally governed by four kinds of physical equations: the (perturbed) continuity equation, the (perturbed) Euler equations, the (perturbed) Poisson equation of the gravitational potential, and the adiabatic relation between the pressure and the density (e.g. Unno et al. 1989). These equations can be arranged to form a fourth order system of ordinary differential equations,

$$\frac{d\boldsymbol{y}}{dx} = \mathsf{A}\boldsymbol{y}\,, \tag{1}$$

in which x means the fractional radius, $\boldsymbol{y}$ is a four component vector composed of perturbed physical quantities, and A is a four by four matrix. The elements of A are not divergent except at the center ($x = 0$) and the surface ($x = 1$). The matrix A depends on not only x but also the dimensionless squared angular frequency λ and the spherical degree l. We assume $\lambda > 0$ throughout this paper.

There are two boundary conditions set at the center, which guarantee that the perturbed physical quantities are regular at $x = 0$, while the other two boundary conditions are set at the surface. As one of the surface boundary conditions, we require that the Lagrangian perturbation to the pressure vanishes, which means that no force operates on the star from the outside. The other surface condition is that the gravitational potential and its first derivative are matched with the appropriate solutions in the outside of the star at the surface.

We therefore find that only two of the four linearly independent solution vectors of equation (1) satisfy the central boundary conditions. We denote such solution vectors by $\boldsymbol{y}^{\mathrm{c},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{c},(2)}(x)$. Any solution vector that satisfies the central boundary conditions can be expressed as a linear combination of $\boldsymbol{y}^{\mathrm{c},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{c},(2)}(x)$. In other words, the whole solution vectors that satisfy the central boundary conditions constitute a two dimensional linear space $L^{\mathrm{c}}(x)$ at each radius x. Similarly, two linearly independent solution vectors selected by the surface boundary conditions are indicated by $\boldsymbol{y}^{\mathrm{s},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{s},(2)}(x)$. The corresponding linear space is expressed by $L^{\mathrm{s}}(x)$. Note that all of $\boldsymbol{y}^{\mathrm{c},(i)}(x)$ and $\boldsymbol{y}^{\mathrm{s},(i)}(x)$ ($i = 1, 2$) are also functions of λ (and l). Introducing the discriminant $D(\lambda)$, which is defined by

$$D(\lambda) = \left| \begin{array}{cccc} \boldsymbol{y}^{\mathrm{c},(1)}(x_{\mathrm{f}}) & \boldsymbol{y}^{\mathrm{c},(2)}(x_{\mathrm{f}}) & \boldsymbol{y}^{\mathrm{s},(1)}(x_{\mathrm{f}}) & \boldsymbol{y}^{\mathrm{s},(2)}(x_{\mathrm{f}}) \end{array} \right|, \tag{2}$$

we can express the condition of an eigenmode as $D(\lambda) = 0$. Note that the fitting point x_{f} is included in the interval $(0, 1)$. Thus, each eigenvalue λ is identified as a root of $D(\lambda)$.

2.2. Formulation in Terms of the Exterior Products

We can regard the condition for an eigenmode [$D(\lambda) = 0$] as a specific relation among the four solution vectors $\boldsymbol{y}^{\mathrm{c},(i)}(x)$ and $\boldsymbol{y}^{\mathrm{s},(i)}(x)$ ($i = 1, 2$). In order to evaluate D, we do not need to know all of these vectors separately. In fact, if we introduce the exterior products between two vectors $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ by

$$y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = \left| \begin{array}{cc} y_i^{(1)} & y_i^{(2)} \\ y_j^{(1)} & y_j^{(2)} \end{array} \right| = y_i^{(1)} y_j^{(2)} - y_i^{(2)} y_j^{(1)} \tag{3}$$

(e.g. Arnold 1989), D can be expanded as

$$\begin{aligned} D(\lambda) = {} & (y_1 \wedge y_2)^{\mathrm{c}}(y_3 \wedge y_4)^{\mathrm{s}} - (y_1 \wedge y_3)^{\mathrm{c}}(y_2 \wedge y_4)^{\mathrm{s}} \\ & + (y_1 \wedge y_4)^{\mathrm{c}}(y_2 \wedge y_3)^{\mathrm{s}} + (y_2 \wedge y_3)^{\mathrm{c}}(y_1 \wedge y_4)^{\mathrm{s}} \\ & - (y_2 \wedge y_4)^{\mathrm{c}}(y_1 \wedge y_3)^{\mathrm{s}} + (y_3 \wedge y_4)^{\mathrm{c}}(y_1 \wedge y_2)^{\mathrm{s}} \end{aligned} \tag{4}$$

(an example of the Laplace expansion of the determinant), where

$$\left(y_i \wedge y_j\right)^{\mathrm{b}} = y_i \wedge y_j\left(\boldsymbol{y}^{\mathrm{b},(1)}, \boldsymbol{y}^{\mathrm{b},(2)}\right) \quad \text{for } \mathrm{b} = \mathrm{c}, \mathrm{s}\,. \tag{5}$$

As fundamental properties of exterior products, we note the following relations among any vectors $\boldsymbol{y}^{(i)}$ ($i = 1, 2, 3$) and any scalars a and b:

$$\begin{aligned} y_i \wedge y_j\left(a\boldsymbol{y}^{(1)} + b\boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}\right) = {} & a y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(3)}\right) \\ & + b y_i \wedge y_j\left(\boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}\right), \end{aligned} \tag{6}$$

$$y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = -y_j \wedge y_i\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right), \tag{7}$$

and

$$y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = -y_i \wedge y_j\left(\boldsymbol{y}^{(2)}, \boldsymbol{y}^{(1)}\right) . \tag{8}$$

We particularly have

$$y_i \wedge y_i\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = 0 \tag{9}$$

and

$$y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)}\right) = 0 . \tag{10}$$

In addition, we can show

$$\begin{aligned}&(y_1 \wedge y_2)(y_3 \wedge y_4) - (y_1 \wedge y_3)(y_2 \wedge y_4)\\ &+ (y_1 \wedge y_4)(y_2 \wedge y_3) = 0 ,\end{aligned} \tag{11}$$

by expanding the determinant of the singular matrix,

$$\left(\begin{array}{cccc} \boldsymbol{y}^{(1)} & \boldsymbol{y}^{(2)} & \boldsymbol{y}^{(1)} & \boldsymbol{y}^{(2)} \end{array} \right) . \tag{12}$$

Equation (4) suggests that all we need to evaluate D are the exterior products only. We therefore consider the differential equations that the exterior products satisfy. If we introduce the four by four matrix $\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)$ whose (i, j) element is given by

$$\left[\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)\right]_{i,j} = y_i \wedge y_j\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) , \tag{13}$$

it is straightforward to show that Y satisfies the linear system of equations,

$$\frac{d\mathsf{Y}}{dx} = \mathsf{A}\mathsf{Y} + \mathsf{Y}\mathsf{A}^T , \tag{14}$$

where the superscript T indicates a transpose matrix. Note that only five out of sixteen elements of equation (14) are independent because Y is antisymmetric [cf. equation (7)], and there is an identity given by equation (11). Since the boundary points ($x = 0$ and $x = 1$) are the singular points of equation (14), we need to construct the series expansion of Y near the boundaries to start integration from the boundaries. These expansions are clearly calculated from those of $\boldsymbol{y}^{\mathrm{c},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{c},(2)}(x)$ near the center, and those of $\boldsymbol{y}^{\mathrm{s},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{s},(2)}(x)$ near the surface. We can thus integrate equation (14) from both of the center and the surface. Let us introduce Y^{b} by

$$\mathsf{Y}^{\mathrm{b}} = \mathsf{Y}\left(\boldsymbol{y}^{\mathrm{b},(1)}, \boldsymbol{y}^{\mathrm{b},(2)}\right) \quad \text{for b} = \mathrm{c}, \mathrm{s} . \tag{15}$$

We can evaluate D to find eigenvalues based on equation (4), the right-hand side of which is calculated from the elements of Y^{c} and Y^{s} at $x = x_{\mathrm{f}}$. We discuss how to determine the associated eigenfunctions (for non-degenerate eigenmodes) in subsection 2.4.

If $\boldsymbol{y}^{(1)} = a\boldsymbol{y}^{(2)}$ for some scalar a, $\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)$ is equal to a zero matrix [cf. equations (6) and (10)]. On the other hand, if $\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)$ is equal to a zero matrix, it is shown that $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ are linearly dependent on each other. In this sense, $\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)$ is a measure of the linear independence of $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$. In general, from a given non-zero antisymmetric matrix $\mathsf{Y}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right)$, whose elements satisfy equation (11), we can construct a pair of linearly independent vectors, which can be adopted as $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$.

When we construct Y^{b} (for each of b = c, s), we may replace $\boldsymbol{y}^{\mathrm{b},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{b},(2)}(x)$ by any pair of their linear combinations, $\tilde{\boldsymbol{y}}^{\mathrm{b},(1)}(x)$ and $\tilde{\boldsymbol{y}}^{\mathrm{b},(2)}(x)$, which are linearly independent of each other. If we describe the transformation by

$$\left(\begin{array}{cc} \tilde{\boldsymbol{y}}^{\mathrm{b},(1)}(x) & \tilde{\boldsymbol{y}}^{\mathrm{b},(2)}(x) \end{array} \right) = \left(\begin{array}{cc} \boldsymbol{y}^{\mathrm{b},(1)}(x) & \boldsymbol{y}^{\mathrm{b},(2)}(x) \end{array} \right) \mathsf{M}^{\mathrm{b}} , \tag{16}$$

where M^{b} is an arbitrary two by two constant regular matrix, equations (6) and (8) tell us

$$\begin{aligned}&y_i \wedge y_j\left(\tilde{\boldsymbol{y}}^{\mathrm{b},(1)}(x), \tilde{\boldsymbol{y}}^{\mathrm{b},(2)}(x)\right)\\ &= \left|\mathsf{M}^{\mathrm{b}}\right| \, y_i \wedge y_j\left(\boldsymbol{y}^{\mathrm{b},(1)}(x), \boldsymbol{y}^{\mathrm{b},(2)}(x)\right) .\end{aligned} \tag{17}$$

We thus find that Y^{b} is only multiplied by the non-zero constant $\left|\mathsf{M}^{\mathrm{b}}\right|$, even if we adopt $\tilde{\boldsymbol{y}}^{\mathrm{b},(1)}$ and $\tilde{\boldsymbol{y}}^{\mathrm{b},(2)}$ in place of $\boldsymbol{y}^{\mathrm{b},(1)}$ and $\boldsymbol{y}^{\mathrm{b},(2)}$, respectively. This means that Y^{b} is unique up to a (non-zero) constant factor as long as L^{b} is fixed.

The two linear spaces $L^{\mathrm{c}}(x)$ and $L^{\mathrm{s}}(x)$ are identical to each other for $0 \le x \le 1$, if and only if the equality $\mathsf{Y}^{\mathrm{c}} = a\mathsf{Y}^{\mathrm{s}}$ holds for a non-zero constant a at $x = x_{\mathrm{f}}$. In this case, any solution vector of equation (1) that satisfies the central boundary conditions also satisfies the surface boundary conditions. This is the case of degeneracy, in which two linearly independent eigenfunction vectors are associated with the same eigenvalue.

2.3. *Application of the Properties of Hamiltonian Systems*

It is shown that equation (1) can be reformulated as the system of canonical equations of a linear Hamiltonian system with two degrees of freedom (cf. appendix 2). We may therefore set

$$\boldsymbol{y} = \left(\begin{array}{cccc} p_1 & p_2 & q_1 & q_2 \end{array} \right)^T , \tag{18}$$

where p_i and q_i (i = 1, 2) are canonical variables. The corresponding coefficient matrix A is expressed in terms of the derivatives of the Hamiltonian H as

$$\mathsf{A} = \left(\begin{array}{cc} -\mathsf{R}^T & -\mathsf{Q} \\ \mathsf{P} & \mathsf{R} \end{array} \right) , \tag{19}$$

in which we have introduced the three two by two matrixes P, Q, and R by

$$(\mathsf{P})_{i,j} = \frac{\partial^2 H}{\partial p_i \partial p_j} , \tag{20}$$

$$(\mathsf{Q})_{i,j} = \frac{\partial^2 H}{\partial q_i \partial q_j} , \tag{21}$$

and

$$(\mathsf{R})_{i,j} = \frac{\partial^2 H}{\partial p_i \partial q_j} , \tag{22}$$

respectively. One of the most fundamental properties of linear Hamiltonian systems with two degrees of freedom is that the symplectic form,

$$\mathcal{S}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = \sum_{i=1}^{2} p_i \wedge q_i\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = \boldsymbol{y}^{(1)T} \mathsf{J} \boldsymbol{y}^{(2)} , \tag{23}$$

is a constant of integration for any pair of solution vectors, $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$, of equation (1) (e.g. Arnold 1989). Here we have introduced the four by four antisymmetric constant matrix J by

$$\mathsf{J} = \left(\begin{array}{cc} \mathsf{O}_2 & \mathsf{I}_2 \\ -\mathsf{I}_2 & \mathsf{O}_2 \end{array} \right) , \tag{24}$$

in which O_2 and I_2 are a two by two zero matrix and a two by two unit matrix, respectively. There exists a trivial relation,

$$\mathcal{S}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)}\right) = 0 , \tag{25}$$

for any four component vector $\boldsymbol{y}^{(1)}$ [cf. equation (10)]. The constancy of $\mathcal{S}$ is directly proved by differentiating it as

$$\frac{d}{dx}\left[\boldsymbol{y}^{(1)T}\mathsf{J}\boldsymbol{y}^{(2)}\right] = \boldsymbol{y}^{(1)T}\left(\mathsf{JA} + \mathsf{A}^T\mathsf{J}\right)\boldsymbol{y}^{(2)} = 0 , \tag{26}$$

because we can verify from equations (19) and (24)

$$\mathsf{JA} = \begin{pmatrix} \mathsf{P} & \mathsf{R} \\ \mathsf{R}^T & \mathsf{Q} \end{pmatrix} = -\mathsf{A}^T\mathsf{J} , \tag{27}$$

which means JA is symmetric. Particularly, the boundary conditions of the problem require

$$\mathcal{S}\left(\boldsymbol{y}^{\mathrm{b},(1)}, \boldsymbol{y}^{\mathrm{b},(2)}\right) = 0 \quad \text{for b} = \mathrm{c}, \mathrm{s} , \tag{28}$$

as we show in equations (A66) and (A91), respectively.

Another general property of linear Hamiltonian systems with two degrees of freedom is that the determinant of any four by four matrix whose columns are composed of four solution vectors $\boldsymbol{y}^{(i)}(x)$ $(i = 1, 2, 3, 4)$ is a constant of integration (Liouville's theorem). This is because there is an algebraic relation,

$$\begin{aligned} &\left| \boldsymbol{y}^{(1)}(x) \quad \boldsymbol{y}^{(2)}(x) \quad \boldsymbol{y}^{(3)}(x) \quad \boldsymbol{y}^{(4)}(x) \right| \\ &= \mathcal{S}^{(1,3)}\mathcal{S}^{(2,4)} - \mathcal{S}^{(1,4)}\mathcal{S}^{(2,3)} - \mathcal{S}^{(1,2)}\mathcal{S}^{(3,4)} , \end{aligned} \tag{29}$$

in which we have introduced the notation,

$$\mathcal{S}^{(i,j)} = \mathcal{S}\left(\boldsymbol{y}^{(i)}(x), \boldsymbol{y}^{(j)}(x)\right) \tag{30}$$

(cf. appendix 1.1). Therefore, the discriminant D, which is defined by equation (2), does not depend on x_{f}, as long as we adopt the canonical variables as the elements of $\boldsymbol{y}$ [cf. equation (18)].

2.4. *Eigenfunctions*

The purpose of this subsection is to explain how to determine the eigenfunctions of a non-degenerate eigenmode from the exterior products. We begin with an algebraic identity among any four four-component vectors $\boldsymbol{y}^{(i)}$ $(i = 1, 2, 3, 4)$,

$$\begin{aligned} &\mathcal{S}^{(2,4)}\mathsf{T}^{(1,3)} - \mathcal{S}^{(2,3)}\mathsf{T}^{(1,4)} - \mathcal{S}^{(1,4)}\mathsf{T}^{(2,3)} + \mathcal{S}^{(1,3)}\mathsf{T}^{(2,4)} \\ &= \mathsf{Y}^{(3,4)}\mathsf{J}\mathsf{Y}^{(1,2)} - \mathsf{Y}^{(1,2)}\mathsf{J}\mathsf{Y}^{(3,4)} , \end{aligned} \tag{31}$$

where

$$\mathsf{T}^{(i,j)} = \boldsymbol{y}^{(i)}\boldsymbol{y}^{(j)T} + \boldsymbol{y}^{(j)}\boldsymbol{y}^{(i)T} \tag{32}$$

and

$$\mathsf{Y}^{(i,j)} = \mathsf{Y}\left(\boldsymbol{y}^{(i)}, \boldsymbol{y}^{(j)}\right) \tag{33}$$

(cf. appendix 1.2). For an eigenmode, the condition $D(\lambda) = 0$ implies that there exists at least one vector $\boldsymbol{y}(x)$ for $0 \le x \le 1$ that is expressed as a linear combination (with constant coefficients) of $\boldsymbol{y}^{\mathrm{b},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{b},(2)}(x)$ in the both cases of b = c and s. Such $\boldsymbol{y}(x)$ corresponds to an eigenfunction vector. Without loss of generality, we may set both of $\boldsymbol{y}^{\mathrm{c},(1)}$ and $\boldsymbol{y}^{\mathrm{s},(1)}$ to $\boldsymbol{y}$. If we set

$$\boldsymbol{y}^{(1)} = \boldsymbol{y}^{\mathrm{c},(1)} = \boldsymbol{y} , \tag{34}$$

$$\boldsymbol{y}^{(2)} = \boldsymbol{y}^{\mathrm{c},(2)} = \boldsymbol{y}^{\mathrm{c}} , \tag{35}$$

$$\boldsymbol{y}^{(3)} = \boldsymbol{y}^{\mathrm{s},(1)} = \boldsymbol{y} , \tag{36}$$

and

$$\boldsymbol{y}^{(4)} = \boldsymbol{y}^{\mathrm{s},(2)} = \boldsymbol{y}^{\mathrm{s}} , \tag{37}$$

equation (28) provides

$$\mathcal{S}\left(\boldsymbol{y}, \boldsymbol{y}^{\mathrm{b}}\right) = 0 \quad \text{for b} = \mathrm{c}, \mathrm{s} . \tag{38}$$

Then, we obtain from equation (31)

$$\mu\boldsymbol{y}\boldsymbol{y}^T = \frac{1}{2}\left(\mathsf{Y}^{\mathrm{s}}\mathsf{J}\mathsf{Y}^{\mathrm{c}} - \mathsf{Y}^{\mathrm{c}}\mathsf{J}\mathsf{Y}^{\mathrm{s}}\right) , \tag{39}$$

where μ is independent of x since it is defined by

$$\mu = \mathcal{S}\left(\boldsymbol{y}^{\mathrm{c}}, \boldsymbol{y}^{\mathrm{s}}\right) . \tag{40}$$

As far as $\mu \neq 0$, equation (39) relates the solutions of equation (14) to the eigenfunction vector $\boldsymbol{y}$. In fact, aside from the global normalization factor, we can determine $\boldsymbol{y}(x)$ up to a sign at each x. The sign should be fixed so that $\boldsymbol{y}(x)$ is continuous at each x.

The constant μ is actually a discriminant for degeneracy. In the case of degeneracy, we may regard $\boldsymbol{y}^{\mathrm{c}} = \boldsymbol{y}^{\mathrm{s}}$, so that $\mu = 0$. On the other hand, if $\mu = 0$, $\boldsymbol{y}^{\mathrm{s}}$ belongs to a linear space consisting of all vectors $\boldsymbol{z}$ that satisfy $\mathcal{S}(\boldsymbol{y}^{\mathrm{c}}, \boldsymbol{z}) = 0$ and $\mathcal{S}(\boldsymbol{y}, \boldsymbol{z}) = 0$. Because the two vectors $\boldsymbol{y}$ and $\boldsymbol{y}^{\mathrm{c}}$ are also members of this two dimensional linear space, they can form a set of basis vectors of the space. Therefore $\boldsymbol{y}^{\mathrm{s}}$ can be expressed as a linear combination of $\boldsymbol{y}$ and $\boldsymbol{y}^{\mathrm{c}}$. This means that the two linear spaces L^{c} and L^{s} are identical to each other. We thus understand that $\mu = 0$ if and only if we find degenerate eigenmodes.

2.5. *Alternative Choice of the Dependent Variables*

Once we understand the properties of the problem as a linear Hamiltonian system with two degrees of freedom, it is convenient to change the dependent variables from Y to vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, which are defined by

$$\boldsymbol{u} = \frac{1}{2}\begin{pmatrix} p_1 \wedge p_2 + q_1 \wedge q_2 \\ p_2 \wedge q_1 + p_1 \wedge q_2 \\ p_1 \wedge q_1 - p_2 \wedge q_2 \end{pmatrix} \tag{41}$$

and

$$\boldsymbol{v} = \frac{1}{2}\begin{pmatrix} p_1 \wedge p_2 - q_1 \wedge q_2 \\ p_2 \wedge q_1 - p_1 \wedge q_2 \end{pmatrix} , \tag{42}$$

respectively. We can translate equation (14) into the system of ordinary differential equations,

$$\frac{d\boldsymbol{m}}{dx} = \mathsf{B}\boldsymbol{m} , \tag{43}$$

where $\boldsymbol{m}$ is the five component vector defined by

$$\boldsymbol{m} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} , \tag{44}$$

and B is a five by five matrix whose explicit expression is given in appendix 2.2. If we introduce a five by five diagonal matrix,

$$\mathsf{K} = \mathrm{diag}\,(1, 1, 1, -1, -1) , \tag{45}$$

we can show

$$\mathsf{KB} = -\mathsf{B}^T\mathsf{K} , \tag{46}$$

which means KB is antisymmetric [cf. equation (A39)]. Equation (46) ensures that

$$\left[\boldsymbol{m}^{(1)}(x)\right]^T \mathsf{K}\boldsymbol{m}^{(2)}(x) = \boldsymbol{u}^{(1)}(x)\cdot\boldsymbol{u}^{(2)}(x) - \boldsymbol{v}^{(1)}(x)\cdot\boldsymbol{v}^{(2)}(x) \quad (47)$$

is independent of x for any

$$\boldsymbol{m}^{(i)}(x) = \begin{pmatrix} \boldsymbol{u}^{(i)}(x) \\ \boldsymbol{v}^{(i)}(x) \end{pmatrix} \quad (48)$$

($i = 1, 2$) that satisfy equation (43). Let us put the superscript b (b = c, s) to $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{m}$ to indicate that they are constructed from the elements of $\mathsf{Y}^{\rm b}$. As we find for Y, the condition $\boldsymbol{m}^{\rm c} = a\boldsymbol{m}^{\rm s}$ holds at $x = x_{\rm f}$ for a non-zero constant a, if and only if we encounter the case of degeneracy.

We present three crucial relations for the mode classification. The first one is

$$D(\lambda) = 2\left(\boldsymbol{u}^{\rm c}\cdot\boldsymbol{u}^{\rm s} - \boldsymbol{v}^{\rm c}\cdot\boldsymbol{v}^{\rm s}\right), \quad (49)$$

which replaces equation (4) with the help of equation (28). The constancy of D with respect to $x_{\rm f}$ is then confirmed by that of equation (47). The second relation is provided by

$$\left|\boldsymbol{u}^{\rm b}(x)\right|^2 = \left|\boldsymbol{v}^{\rm b}(x)\right|^2 \quad \text{for } 0 \le x \le 1 \text{ and b} = \text{c, s}, \quad (50)$$

which represents equation (11) because of equation (28). Equation (50) can be regarded as a particular case of equation (47). The last one is obtained by taking the trace of the both sides of equation (39) as

$$\mu|\boldsymbol{y}|^2 = -4v_1 \wedge v_2(\boldsymbol{v}^{\rm c}, \boldsymbol{v}^{\rm s}) = 4\left(v_2^{\rm c}v_1^{\rm s} - v_1^{\rm c}v_2^{\rm s}\right). \quad (51)$$

With suitable expansions near the boundaries [cf. equations (A67) and (A90)], we may integrate equation (43) instead of equation (14) to calculate the discriminant given by equation (49). Every time we find a root of D (an eigenvalue) for a non-degenerate eigenmode, we can determine the corresponding eigenfunctions based on equation (39), the right-hand side of which can be evaluated by $\boldsymbol{m}^{\rm b}$ (b = c, s) using the relation

$$\mathsf{Y}^{\rm b} = \begin{pmatrix} 0 & u_1^{\rm b} + v_1^{\rm b} & u_3^{\rm b} & u_2^{\rm b} - v_2^{\rm b} \\ -u_1^{\rm b} - v_1^{\rm b} & 0 & u_2^{\rm b} + v_2^{\rm b} & -u_3^{\rm b} \\ -u_3^{\rm b} & -u_2^{\rm b} - v_2^{\rm b} & 0 & u_1^{\rm b} - v_1^{\rm b} \\ -u_2^{\rm b} + v_2^{\rm b} & u_3^{\rm b} & -u_1^{\rm b} + v_1^{\rm b} & 0 \end{pmatrix}. \quad (52)$$

3. Mode Classification

3.1. A Basic Idea

In the case of radial oscillations, we may interpret that the eigenmodes are classified based on the surface values of the phase angle, which is defined as the polar angle of the phase point $(\xi_r, \delta p)$, where δp is the Lagrangian perturbation to the pressure. Similarly, in the case of dipolar oscillations, a different pair of the eigenfunctions is used to define the appropriate phase angle, which is then used to classify the eigenmodes. Following these examples, we look for an appropriate definition of the phase angle for the eigenmodes with any spherical degree.

For each λ, we can calculate D and its derivative (with respect to λ) D_λ. We assume for a while $(D_\lambda, D) \neq (0, 0)$. If we follow the position of the point (D_λ, D) as a continuous function of λ on the two dimensional plane, an eigenmode is found every time the point comes on the abscissa axis. If we then increase λ, the ordinate of the point increases or decreases, if the abscissa is positive or negative, respectively. Thus, the point rotates about the origin in the counterclockwise direction, when $D = 0$. Once the point moves from the positive (negative) part of the abscissa axis to the first (third) quadrant, it never comes back to the abscissa axis from the same quadrant, because the ordinate must increase (decrease) in the first (third) quadrant. The only possibility for the point to get to the abscissa axis again is to move in advance to the second (fourth) quadrant, where the ordinate decreases (increases). Therefore, next time the point comes back on the abscissa axis, it reaches the part of the opposite sign. In other words, if we compare the polar angles of the point that corresponds to two successive eigenmodes in the frequency spectrum, that for the larger eigenvalue is larger by π than that for the smaller eigenvalue. This consideration suggests an idea that each eigenmode is uniquely characterized by the polar angle of the corresponding point (D_λ, D), which is equal to an integral multiple of π. In order to establish a scheme of the mode classification based on this idea, all we need is to evaluate the sign of D, and that of D_λ for $D = 0$, which will be discussed in subsections 3.2 and 3.3, respectively.

The case of $(D_\lambda, D) = (0, 0)$ requires a special care, because it could make the proposed idea void. It will turn out that this corresponds to the case of degeneracy in subsection 3.3. We study this case in detail in appendix 4.

3.2. The Sign of the Discriminant

Let us introduce the amplitude $A^{\rm b}$ and the polar angle $\theta^{\rm b}$ of $\boldsymbol{v}^{\rm b}$ (for b = c, s) by

$$v_1^{\rm b} = A^{\rm b}\cos\theta^{\rm b} \quad (53)$$

and

$$v_2^{\rm b} = A^{\rm b}\sin\theta^{\rm b}. \quad (54)$$

Equation (50) guarantees $\left|\boldsymbol{u}^{\rm b}\right| = A^{\rm b}$. It is important that $A^{\rm b} \neq 0$ for $0 < x < 1$. In fact, if $A^{\rm b} = 0$ at a certain radius in $(0, 1)$, then both of $\boldsymbol{u}^{\rm b}$ and $\boldsymbol{v}^{\rm b}$ are null vectors there. Since these vectors form the dependent variables of the linear and homogeneous system of ordinary differential equations, which is given by equation (43), the dependent variables are totally zero for $0 \le x \le 1$, once they are all zero at a radius except for the boundaries. This clearly contradicts the boundary conditions that constrain the dependent variables to be non-trivial. We thus understand $A^{\rm b} > 0$ for $0 < x < 1$.

On the other hand, we determine $\theta^{\rm b}$ at each radius by following the profiles of $v_1^{\rm b}$ and $v_2^{\rm b}$ from the corresponding boundary, which are in turn obtained by integrating equation (43). Consequently, we can define $\theta^{\rm b}$ as a continuous function of x. (If $A^{\rm b} = 0$ at an intermediate radius, $\theta^{\rm b}$ generally changes discontinuously at the radius.) If we set the initial value of $\theta^{\rm b}$ ($\theta^{\rm c}$ at the center and $\theta^{\rm s}$ at the surface) to be a continuous function of λ and the spherical degree, it also depends on these parameters continuously. Moreover, if the initial value changes continuously against any continuous change in the equilibrium structure, $\theta^{\rm b}$ varies continuously against the change. These properties of $\theta^{\rm b}$ are essential for the mode classification.

If we introduce

$$\Delta\theta = \theta^{\rm c} - \theta^{\rm s}, \quad (55)$$

and define $\Delta\phi$ as the angle between $\boldsymbol{u}^{\rm c}$ and $\boldsymbol{u}^{\rm s}$, equation (49) is rewritten as

$$D(\lambda) = 2A^{\rm c}A^{\rm s}\left[\cos(\Delta\phi) - \cos(\Delta\theta)\right] . \tag{56}$$

Note that $\Delta\theta$ and $\Delta\phi$ have to be calculated from $\boldsymbol{v}^{\rm b}$ and $\boldsymbol{u}^{\rm b}$, respectively, at the same radius $x = x_{\rm f}$. From equation (56), we find

$$\mathrm{sign}\left[D(\lambda)\right] = \mathrm{sign}\left[\cos(\Delta\phi) - \cos(\Delta\theta)\right] \quad \text{for } 0 < x_{\rm f} < 1 . \tag{57}$$

The domain of $\Delta\theta$ is $(-\infty, +\infty)$, because $\theta^{\rm c}$ and $\theta^{\rm s}$ are calculated separately as the polar angle of $\boldsymbol{v}^{\rm c}$ and $\boldsymbol{v}^{\rm s}$, respectively. On the other hand, since $\Delta\phi$ is defined as the angle between the two three dimensional vectors, $\boldsymbol{u}^{\rm c}$ and $\boldsymbol{u}^{\rm s}$, its domain is restricted to $[0, \pi]$. It is of particular importance to examine in which case we have $\sin(\Delta\phi) = 0$ for $D = 0$. Since $\cos(\Delta\phi) = \pm 1$, $D = 0$ yields $\cos(\Delta\theta) = \pm 1$. We therefore obtain $(\Delta\theta, \Delta\phi) = (0, 2n\pi)$, $(\pi, (2n+1)\pi)$ for some integer n, which means that $\boldsymbol{m}^{\rm c}$ and $\boldsymbol{m}^{\rm s}$ are parallel or antiparallel to each other. This corresponds to the case of degeneracy. We thus find

$$\begin{cases} \sin(\Delta\phi) > 0 & \text{for non-degenerate eigenmodes} , \\ \sin(\Delta\phi) = 0 & \text{for degenerate eigenmodes} . \end{cases} \tag{58}$$

3.3. The Sign of the Derivative of the Discriminant with Respect to λ

We evaluate D_λ for $D = 0$ based on equation (29). If we substitute equations (34)–(37) after differentiating equation (29) with respect to λ, we obtain

$$D_\lambda = \mu\mathcal{S}_\lambda\left(\boldsymbol{y}^{\rm c,(1)}, \boldsymbol{y}^{\rm s,(1)}\right) \quad \text{for } D = 0 , \tag{59}$$

because of equation (38). Since the expression of $\mathcal{S}_\lambda$ is given by equation (A125) with $\alpha = \lambda$, we have

$$D_\lambda = \mu\mathcal{I}(\boldsymbol{y}, \boldsymbol{y}) , \tag{60}$$

where

$$\mathcal{I}(\boldsymbol{y}, \boldsymbol{y}) = \int_0^1 \boldsymbol{y}^T(-\mathsf{J}\mathsf{A}_\lambda)\,\boldsymbol{y}\,\mathrm{d}x > 0 \tag{61}$$

[cf. equation (A51)]. Thus, the sign of D_λ is equal to that of μ for $D = 0$. Therefore, the case of $(D_\lambda, D) = (0, 0)$ is equivalent to that of degeneracy. On the other hand, because equation (51) can be rewritten as

$$\mu|\boldsymbol{y}|^2 = 4A^{\rm c}A^{\rm s}\sin(\Delta\theta) \tag{62}$$

[cf. equations (53), (54), and (55)], μ has the same sign as $\sin(\Delta\theta)$ for $0 < x_{\rm f} < 1$. Considering equation (58), we find

$$\mathrm{sign}\left[D_\lambda\right] = \mathrm{sign}\left[\sin(\Delta\theta)\sin(\Delta\phi)\right] \quad \text{for } D = 0 \text{ and } 0 < x_{\rm f} < 1 . \tag{63}$$

We may evaluate in passing the second derivative of D in the degenerate case based on equation (29). If we denote the two linearly independent eigenfunction vectors that are associated with the degenerate eigenvalue λ by $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$, we may set in line with equations (34)–(37) $\boldsymbol{y}^{(3)} = \boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(4)} = \boldsymbol{y}^{(2)}$. Then, it is straightforward to show

$$D_{\lambda\lambda} = \mathcal{I}\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)}\right)\mathcal{I}\left(\boldsymbol{y}^{(2)}, \boldsymbol{y}^{(2)}\right) > 0 \quad \text{for } D = 0 \text{ and } D_\lambda = 0 \tag{64}$$

[cf. equation (61)].

3.4. A Scheme of the Mode Classification

Based on equations (57) and (63), we intend to define the angle Θ, which satisfies

$$\tan\Theta = \frac{\cos(\Delta\phi) - \cos(\Delta\theta)}{\sin(\Delta\theta)\sin(\Delta\phi)} , \tag{65}$$

as a continuous function of $\Delta\theta$ and $\Delta\phi$. Note that we concentrate on non-degenerate modes for a while because both of the denominator and the numerator of the right-hand side of equation (65) are equal to zero for the degenerate case. Standard formulas of trigonometry allow us to rewrite equation (65) as

$$\tan\Theta = \tan\left(2\Phi + \frac{\pi}{2}\right) , \tag{66}$$

where we require that Φ satisfies

$$\tan\Phi = \frac{\tan\frac{\Delta\theta}{2}}{\tan\frac{\Delta\phi}{2}} . \tag{67}$$

If Φ can be defined as a continuous function of $\Delta\theta$ and $\Delta\phi$, we may set

$$\Theta = 2\Phi + \frac{\pi}{2} . \tag{68}$$

We need to treat the right-hand side of equation (67) carefully. The denominator can be zero or divergent for $\Delta\phi = 0$ or π, respectively, while the numerator is so for $\Delta\theta = 2n\pi$ or $(2n+1)\pi$, respectively, for any integer n. When the denominator approaches zero, it always does so from a positive side, because $0 \le \Delta\phi \le \pi$. Therefore, as far as the numerator is different from zero, which is true in the absence of degenerate modes, we find $\tan\Phi$ goes to $\pm\infty$, where the double sign corresponds to the sign of the numerator. When the denominator diverges, we can safely set $\tan\Phi = 0$, since the numerator is finite if there is no degenerate modes. Similarly, we find $\tan\Phi = 0$ when the numerator is equal to zero, because the denominator is different from zero for a non-degenerate mode. The most delicate case is found when the numerator is divergent, which happens when $\Delta\theta$ approaches $(2n+1)\pi$ for some integer n. We may safely regard that the denominator is positive or zero, and not divergent, since $0 \le \Delta\phi < \pi$ if there exist only non-degenerate modes. What we should pay attention to is the fact that the sign of $\tan\Phi$ depends on the direction of the approach of $\Delta\theta$ to $(2n+1)\pi$. In fact, we find $\tan\Phi \to \pm\infty$ as $\Delta\theta \to (2n+1)\pi\mp$, where the double signs correspond to each other. This means that, when $\Delta\theta$ passes through $(2n+1)\pi$ in the increasing direction, $\tan\Phi$ changes discontinuously from $+\infty$ to $-\infty$.

When we encounter the case of degeneracy, both of the denominator and the numerator of equation (67) are divergent or equal to zero, which means that $\tan\Phi$ is indefinite from this equation. We should rather consider how the trajectory of the point (D_λ, D) behaves geometrically near the degenerate case. In figure 1, the case (2) corresponds to the equilibrium structure with the degenerate eigenvalue $\lambda_{\rm d}$ at the origin, while the cases (1) and (3) are the profiles of the perturbed structures. Based on the considerations in appendix 4, we find that the curve in the figure can transit from the case (2) to only the case (3), and to the case (2) from only the case (3), whatever perturbation we put to the equilibrium structure. The case (2) can thus be

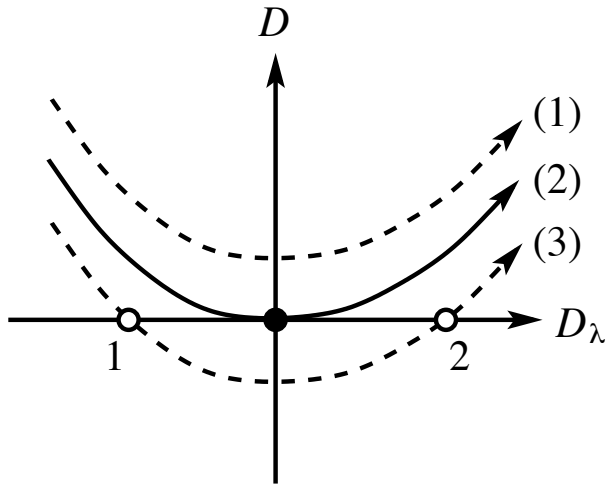

Fig. 1. Schematic behavior of the curves on the (D_λ, D) plane. Different curves correspond to different equilibrium structures. The arrow of each curve indicates the direction in which λ increases. In the case (2), the equilibrium structure has a degenerate eigenvalue, which is found at the origin. The labels (1), (2), and (3) of the curves correspond to those in figure 4.

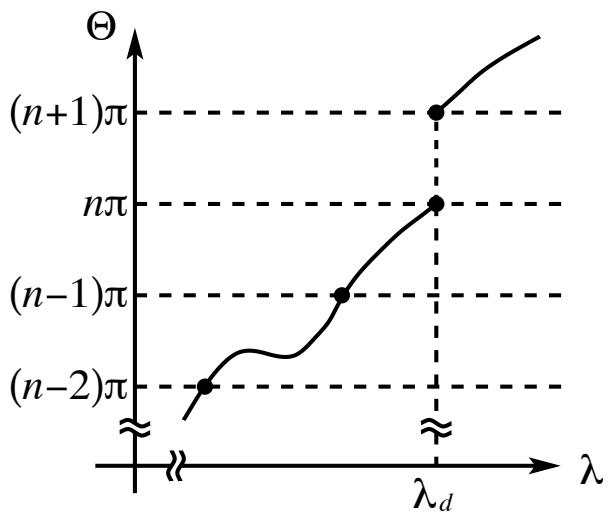

Fig. 2. Schematic behavior of Θ as a function of λ. The points with dots correspond to eigenmodes. When $\lambda = \lambda_d$, Θ takes the two values, $n\pi$ and $(n+1)\pi$, which correspond to the two degenerate eigenmodes.

regarded as the limiting case of the case (3). We may therefore consider that, if the polar angle of the point on the curve (2), hence Θ, approaches $n_d\pi$ for some integer n_d as $\lambda \to \lambda_{\rm d}-$, then it does $(n_d+1)\pi$ as $\lambda \to \lambda_{\rm d}+$. The value of n_d should be the same as that adopted for the point 1 on the curve (3). We should thus allocate the two values of $n_d\pi$ and $(n_d+1)\pi$ to Θ for the degenerate case. The value of Φ should be arranged so as to satisfy this requirement.

From all of the discussions given above, we relate $\Delta\theta$ and $\Delta\phi$ to Φ as

$$\Phi = \Phi_0 + \left\lfloor \frac{\Delta\theta}{2\pi} + \frac{1}{2} \right\rfloor \pi \,, \tag{69}$$

in which $\lfloor X \rfloor$ means the floor function of X that returns the maximum integer not larger than X. The angle Φ_0 is defined by

$$\Phi_0 = \begin{cases} \arctan\left(\frac{\tan\frac{\Delta\theta}{2}}{\tan\frac{\Delta\phi}{2}}\right) & \text{if } \Delta\theta \neq (2n+1)\pi \text{ and} \\ & \quad 0 < \Delta\phi < \pi \,, \\ \mathrm{sign}\left[\tan\frac{\Delta\theta}{2}\right]\frac{\pi}{2} & \text{if } \Delta\theta \neq n\pi \text{ and } \Delta\phi = 0 \,, \\ 0 & \text{if } \Delta\theta \neq (2n+1)\pi \text{ and } \Delta\phi = \pi \,, \\ -\frac{\pi}{2} & \text{if } \Delta\theta = (2n+1)\pi \text{ and } \Delta\phi \neq \pi \,, \\ \pm\frac{\pi}{4} & \text{if } (\Delta\theta, \Delta\phi) = (2n\pi, 0) \text{ or} \\ & \quad ((2n+1)\pi, \pi) \,, \end{cases} \tag{70}$$

where n is any integer, and the domain of the arctangent function (in the first case) is restricted to $(-\pi/2, \pi/2)$, so that $-\pi/2 \le \Phi \le \pi/2$. Note that, although Φ_0 is discontinuous at $\Delta\theta = (2n+1)\pi$ for any integer n, the discontinuity is compensated by that of the floor function in equation (69). The last case on the right-hand side of equation (70) corresponds to the case of degeneracy. From equations (68) and (69), we finally give the definition of Θ as

$$\Theta = 2\Phi_0 + \left(2\left\lfloor \frac{\Delta\theta}{2\pi} + \frac{1}{2} \right\rfloor + \frac{1}{2}\right)\pi \,. \tag{71}$$

The angle Θ changes discontinuously as a function of λ only when λ is equal to a degenerate eigenvalue, for which the two values of $n_d\pi$ and $(n_d+1)\pi$ are returned with

$$n_d = 2\left\lfloor \frac{\Delta\theta}{2\pi} + \frac{1}{2} \right\rfloor . \tag{72}$$

When Θ is equal to a multiple of π, its derivative with respect to λ must be positive for non-degenerate eigenmodes. On the other hand, if Θ is not equal to a multiple of π, it can increase or decrease as a function of λ. These behaviors are schematically shown in figure 2.

The angle Θ defined by equation (71) provides the fundamental relation for the mode classification. The condition for an eigenmode is clearly given by $\Theta = N\pi$ for some integer N. The integer N should be regarded as the radial order of the eigenmode. Note that Θ generally depends on $x_{\rm f}$, because $\Delta\theta$ and $\Delta\phi$ do so. However, since equations (57) and (63) hold irrespectively of the value of $x_{\rm f}$, N is independent of $x_{\rm f}$. Two adjacent eigenmodes in the frequency spectrum with different eigenvalues have two successive values of the index N, which is also true for any two (degenerate) eigenmodes with the same eigenvalue. The index N is therefore unique and continuous. It appears that we may add any integral multiples of π to the right-hand side of equation (71). Actually, we have already used this freedom to fix the origin of the index N so that $N = 1$ is allocated to the radial fundamental mode as we show in appendix 5. For this purpose, the initial values of $\theta^{\rm c}$ and $\theta^{\rm s}$ have also been chosen properly at the center and the surface, respectively [cf. equations (A69) and (A93)]. Note that these adjustments generally depend on the definitions of the canonical variables, which are given by equations (A12)–(A15) in the present study. Once we fix the fiducial value of the index N for the radial fundamental mode, we may determine the index of any eigenmodes of any models. The classification of non-degenerate eigenmodes is established as follows: we regard the eigenmodes with positive N as the N-th p mode (the p_N mode); we interpret the eigenmode with $l \neq 1$ and $N < 0$

as the $|N|$-th g mode (the $g_{|N|}$ mode), while that with $l \geq 2$ and $N = 0$ is considered as the f mode; the case of $l = 1$ needs a special care, because it is known by physical discussion that the frequency of the f mode goes to zero; the dipolar eigenmode with $N \leq 0$ should therefore be classified as the $(|N| + 1)$-th g mode (the $g_{|N|+1}$ mode). This is consistent with the numerical experiments in which it is claimed that the quadrupolar f and g_n modes turn into the dipolar g_1 and g_{n+1} modes, respectively, by changing the spherical degree continuously from 2 to 1 (Aizenman et al. 1977; Christensen-Dalsgaard & Gough 2001). Note that, in the scheme of the dipolar mode classification proposed by Takata (2006a), the mode index is adjusted so that the dipolar g_n mode has the index of $-n$, which is smaller by one than the value in the present study. A pair of degenerate eigenmodes with the same eigenvalue has two successive values of N. We naturally regard each of these eigenmodes as a mixture of the two modes to which the two indexes are allocated in the case of non-degenerate modes.

4. Discussion

We have developed a scheme to define the radial order N of each eigenmode, which can be used to classify p modes, g modes, and f modes. The scheme is directly based on Θ, which is defined by equation (71). The angle Θ does not diverge for finite and positive values of λ. This is because the derivative of $\theta^{\rm b}$ (for b = c, s) with respect to x is finite for $0 < x < 1$ [cf. equation (A46)], and $\theta^{\rm b}$ behaves regularly near the boundaries [cf. equations (A68) and (A92)]. This fact implies that the eigenvalue spectrum is discrete (for $\lambda > 0$). In fact, we can evaluate the number of eigenmodes in a given frequency interval $[\lambda_1, \lambda_2]$ for $0 < \lambda_1 < \lambda_2$. If we introduce

$$N_1 = \left\lceil \frac{\Theta(\lambda_1)}{\pi} \right\rceil \tag{73}$$

and

$$N_2 = \left\lfloor \frac{\Theta(\lambda_2)}{\pi} \right\rfloor , \tag{74}$$

where $\lceil X \rceil$ is the ceiling function of X that returns the minimum integer not smaller than X, we should find all of the eigenmodes of the index N that satisfies $N_1 \leq N \leq N_2$ in the frequency interval. We therefore understand the number of eigenmodes in $[\lambda_1, \lambda_2]$ is given by

$$\mathcal{N}(\lambda_1, \lambda_2) = N_2 - N_1 + 1 = \left\lfloor \frac{\Theta(\lambda_2)}{\pi} \right\rfloor - \left\lceil \frac{\Theta(\lambda_1)}{\pi} \right\rceil + 1 . \tag{75}$$

Equations (73)–(75) are particularly useful when we numerically compute all of the eigenmodes of a given stellar model in a given range of the frequency and the spherical degree systematically.

Moreover, if we need to find a particular eigenmode (with the index N) of a given equilibrium model or a given series of equilibrium models, we may use the discriminant for the eigenmode which is defined by

$$\mathcal{D}_N(\lambda) = \Theta - N\pi . \tag{76}$$

The eigenfrequency of the eigenmode is identified as the root of this discriminant.

When we implement the thus constructed scheme in numerical programs, we have to remember that it is not sufficient to have the eigenvalue λ and the profile of the corresponding eigenfunction $\boldsymbol{y}$ in order to calculate the index N, because we also need the profiles of $\boldsymbol{y}^{\rm c}$ and $\boldsymbol{y}^{\rm s}$ [cf. equations (34)–(37)] to compute $\Delta\theta$ and $\Delta\phi$.

We may compare the scheme of the mode classification presented in section 3 with the Eckart-Scuflaire-Osaki (ESO) scheme that is justified only with the Cowling approximation (Eckart 1960; Scuflaire 1974; Osaki 1975). Although the ESO scheme has troubles with dipolar modes of stellar models with high central mass concentration (cf. Lee 1985; Guenther 1991; Christensen-Dalsgaard & Mullan 1994), it usually provides satisfactory results for $l \geq 2$. For dipolar modes, another particular scheme has been developed (Takata 2006a). Therefore, there is so far little practical need for the alternative scheme at least for $l \geq 2$. However, there still exists an advantage of the new scheme. When we compute eigenmodes numerically, we fully take the perturbation to the gravitational potential into account, whereas we totally ignore it when we classify the computed modes based on the ESO scheme. Such clear conceptual inconsistency is removed, if we adopt the mode classification scheme developed in section 3. More importantly, we believe that the analyses invoked to construct the new scheme help us have a deeper insight into the mathematical structure of adiabatic stellar oscillations, which would in turn serve to establish a more profound physical picture of the problem and make a better interpretation of observational data of stellar oscillations.

5. Conclusion

A scheme have been presented to classify the eigenmodes of adiabatic oscillations of spherically symmetric stars. The scheme provides the definition of the radial order of eigenmodes, which can in turn be used to define p modes, g modes, and f modes. The analysis is based on the framework developed in Woodhouse (1988), which is extended in two respects. We firstly argue that degenerate modes do not affect the classification of the other modes because the number of modes is conserved during continuous change in the equilibrium structure. The pair of degenerate modes should simply have two successive values of the radial order N. In the other respect, we have devised a scheme of the mode classification that is specific to Hamiltonian systems with two degrees of freedom.

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Appendix 1. The Proofs of the Algebraic Identities

A.1.1. The Proof of Equation (29)

Applying the Laplace expansion, we obtain

$$\begin{aligned} &\left| \boldsymbol{y}^{(1)} \quad \boldsymbol{y}^{(2)} \quad \boldsymbol{y}^{(3)} \quad \boldsymbol{y}^{(4)} \right| \\ &= -(y_1 \wedge y_3)^{(1,2)} (y_2 \wedge y_4)^{(3,4)} + (y_1 \wedge y_3)^{(1,3)} (y_2 \wedge y_4)^{(2,4)} \\ &\quad - (y_1 \wedge y_3)^{(1,4)} (y_2 \wedge y_4)^{(2,3)} - (y_1 \wedge y_3)^{(2,3)} (y_2 \wedge y_4)^{(1,4)} \end{aligned}$$

$$+ (y_1 \wedge y_3)^{(2,4)} (y_2 \wedge y_4)^{(1,3)} - (y_1 \wedge y_3)^{(3,4)} (y_2 \wedge y_4)^{(1,2)} , \tag{A1}$$

in which we have used the notation,

$$\left(y_i \wedge y_j\right)^{(m,n)} = y_i \wedge y_j \left(\boldsymbol{y}^{(m)}, \boldsymbol{y}^{(n)}\right) . \tag{A2}$$

If we replace $y_2^{(i)}$ and $y_4^{(i)}$ by $y_1^{(i)}$ and $y_3^{(i)}$, respectively, in equation (A1), we get

$$0 = -(y_1 \wedge y_3)^{(1,2)} (y_1 \wedge y_3)^{(3,4)} + (y_1 \wedge y_3)^{(1,3)} (y_1 \wedge y_3)^{(2,4)} - (y_1 \wedge y_3)^{(1,4)} (y_1 \wedge y_3)^{(2,3)} . \tag{A3}$$

Similarly, if $y_1^{(i)}$ and $y_3^{(i)}$ are replaced by $y_2^{(i)}$ and $y_4^{(i)}$, respectively, in equation (A1), we have

$$0 = -(y_2 \wedge y_4)^{(1,2)} (y_2 \wedge y_4)^{(3,4)} + (y_2 \wedge y_4)^{(1,3)} (y_2 \wedge y_4)^{(2,4)} - (y_2 \wedge y_4)^{(1,4)} (y_2 \wedge y_4)^{(2,3)} . \tag{A4}$$

If we add equations (A1), (A3), and (A4), we obtain equation (29) [cf. equation (18)].

A.1.2. The Proof of Equation (31)

We first note that the (i, j) element of J is given by

$$\mathsf{J}_{i,j} = \delta_{i+2,j} - \delta_{i,j+2} , \tag{A5}$$

where $\delta_{i,j}$ is the Kronecker symbol, and that the symplectic form can be expressed as

$$\mathcal{S}^{(i,j)} = \sum_{k=1}^{2} \left[y_k^{(i)} y_{k+2}^{(j)} - y_{k+2}^{(i)} y_k^{(j)} \right] \tag{A6}$$

[cf. equations (18) and (23)]. The first term on the right-hand side of equation (31) is then evaluated as

$$\begin{aligned} &\left(\mathsf{Y}^{(3,4)} \mathsf{J} \mathsf{Y}^{(1,2)}\right)_{i,j} \\ &= \sum_{k=1}^{2} \Big[(y_i \wedge y_k)^{(3,4)} \left(y_{k+2} \wedge y_j\right)^{(1,2)} \\ &\quad - (y_i \wedge y_{k+2})^{(3,4)} \left(y_k \wedge y_j\right)^{(1,2)} \Big] \\ &= \mathcal{S}^{(2,4)} y_i^{(3)} y_j^{(1)} - \mathcal{S}^{(1,4)} y_i^{(3)} y_j^{(2)} - \mathcal{S}^{(2,3)} y_i^{(4)} y_j^{(1)} \\ &\quad + \mathcal{S}^{(1,3)} y_i^{(4)} y_j^{(2)} . \end{aligned} \tag{A7}$$

From equation (A7) and the fact that the second term on the right-hand side of equation (31) is equal to the transpose of the first term (because $\mathsf{Y}^{(i,j)}$ and J are symmetric), we obtain equation (31).

Appendix 2. Hamiltonian Formulation of Linear Adiabatic Stellar Oscillations

A.2.1. Hamiltonian System

We summarize the formulation of the problem of linear adiabatic oscillations of stars as a Hamiltonian system (Takata 2006b). The equilibrium structure that concerns the problem is specified by the following four dimensionless quantities:

$$c_1 = \frac{x^3}{M_r/M} , \tag{A8}$$

$$U = \frac{d\ln M_r}{d\ln r} = \frac{4\pi r^3 \rho}{M_r} \tag{A9}$$

$$V_g = -\frac{1}{\Gamma_1} \frac{d\ln p}{d\ln r} = \frac{gr}{c^2} , \tag{A10}$$

and

$$A^* = \frac{1}{\Gamma} \frac{d\ln p}{d\ln r} - \frac{d\ln \rho}{d\ln r} , \tag{A11}$$

where M_r is the concentric mass, M the total mass, r the radius, ρ the density, Γ_1 the first adiabatic index, p the pressure, g the gravitational acceleration, and c the sound speed. A set of the canonical variables, which is convenient for the present analysis, is introduced by

$$p_1 = -\frac{\sqrt{x^5} U}{c_1} \left(y_1^{\mathrm{D}} - y_2^{\mathrm{D}} + y_3^{\mathrm{D}}\right) , \tag{A12}$$

$$p_2 = -\frac{\sqrt{x^5}}{c_1} \left[U y_1^{\mathrm{D}} + (l+1) y_3^{\mathrm{D}} + y_4^{\mathrm{D}} \right] , \tag{A13}$$

$$q_1 = \frac{\sqrt{x^5}}{c_1} y_1^{\mathrm{D}} , \tag{A14}$$

and

$$q_2 = \frac{\sqrt{x^5}}{c_1} y_3^{\mathrm{D}} , \tag{A15}$$

in which y_i^{D} $(i = 1, 2, 3, 4)$ are defined by

$$y_1^{\mathrm{D}} = \frac{\xi_r}{r} , \tag{A16}$$

$$y_2^{\mathrm{D}} = \frac{1}{gr} \left(\frac{p'}{\rho} + \Phi' \right) = \lambda c_1 \frac{\xi_h}{r} , \tag{A17}$$

$$y_3^{\mathrm{D}} = \frac{\Phi'}{gr} , \tag{A18}$$

and

$$y_4^{\mathrm{D}} = \frac{1}{g} \frac{d\Phi'}{dr} , \tag{A19}$$

respectively (Dziembowski 1971). Here the prime ($'$) denotes the Eulerian perturbation, Φ is the gravitational potential, and ξ_h is (the radial part of) the horizontal component of the displacement vector. The system of ordinary differential equations that y_i^{D} $(i = 1, 2, 3, 4)$ satisfy is given by

$$\frac{d\boldsymbol{y}^{\mathrm{D}}}{dx} = \mathsf{A}^{\mathrm{D}} \boldsymbol{y}^{\mathrm{D}} , \tag{A20}$$

in which

$$\mathsf{A}^{\mathrm{D}} = \frac{1}{x} \begin{pmatrix} V_g - 3 & \frac{\Lambda}{\lambda c_1} - V_g & V_g & 0 \\ \lambda c_1 - A^* & A^* - U + 1 & -A^* & 0 \\ 0 & 0 & 1 - U & 1 \\ U A^* & U V_g & \Lambda - U V_g & -U \end{pmatrix} . \tag{A21}$$

Here we have defined

$$\Lambda = l(l+1) . \tag{A22}$$

The central boundary conditions collateral to equation (A20) are

$$\lambda c_1 y_1^{\mathrm{D}} - l y_2^{\mathrm{D}} = 0 \tag{A23}$$

and

$$l y_3^{\mathrm{D}} - y_4^{\mathrm{D}} = 0\,, \tag{A24}$$

while those at the surface are

$$y_1^{\mathrm{D}} - y_2^{\mathrm{D}} + y_3^{\mathrm{D}} = 0 \tag{A25}$$

and

$$(l+1) y_3^{\mathrm{D}} + y_4^{\mathrm{D}} = 0\,, \tag{A26}$$

in which we have assumed that the density vanishes at the surface. It can be shown that the system given by equation (A20) is equivalent to a Hamiltonian system whose Hamiltonian is provided by

$$\begin{aligned} H = \frac{1}{2x}\Big\{ & \frac{\Lambda}{\lambda c_1 U}\left[p_1 + U(q_1+q_2)\right]^2 - (\lambda c_1 + 4) U q_1^2 \\ & - (2l+1) p_2 q_2 - 2(l+1) U q_1 q_2 - \frac{V_g}{U} p_1^2 - p_2^2 \\ & + (2U-7) p_1 q_1 - 2U p_2 q_1 \Big\}\,. \end{aligned} \tag{A27}$$

Accordingly, the matrixes P, Q, and R [cf. equations (20)–(22)] are provided by

$$\mathsf{P} = \frac{1}{x}\begin{pmatrix} \frac{1}{U}\left(\frac{\Lambda}{\lambda c_1} - V_g\right) & 0 \\ 0 & -1 \end{pmatrix}, \tag{A28}$$

$$\mathsf{Q} = \frac{U}{x}\begin{pmatrix} \frac{\Lambda}{\lambda c_1} - \lambda c_1 - 4 & \frac{\Lambda}{\lambda c_1} - l - 1 \\ \frac{\Lambda}{\lambda c_1} - l - 1 & \frac{\Lambda}{\lambda c_1} \end{pmatrix}, \tag{A29}$$

and

$$\mathsf{R} = \frac{1}{x}\begin{pmatrix} \frac{\Lambda}{\lambda c_1} + U - \frac{7}{2} & \frac{\Lambda}{\lambda c_1} \\ -U & -l - \frac{1}{2} \end{pmatrix}, \tag{A30}$$

respectively.

A.2.2. The Expression of Matrix B

The matrix B, which is introduced by equation (43), is given by

$$\mathsf{B} = \begin{pmatrix} \mathsf{B}_1 & \mathsf{B}_3 \\ \mathsf{B}_3^T & \mathsf{B}_2 \end{pmatrix}, \tag{A31}$$

in which

$$\mathsf{B}_1 = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \tag{A32}$$

$$\mathsf{B}_2 = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}, \tag{A33}$$

and

$$\mathsf{B}_3 = \begin{pmatrix} \boldsymbol{k}^{(1)} & \boldsymbol{k}^{(2)} \end{pmatrix}. \tag{A34}$$

Here $\boldsymbol{w}$, Ω, $\boldsymbol{k}^{(1)}$, and $\boldsymbol{k}^{(2)}$ are defined by

$$\begin{aligned} \boldsymbol{w} &= \begin{pmatrix} \mathsf{R}_{12} - \mathsf{R}_{21} \\ -\mathsf{P}_{12} - \mathsf{Q}_{12} \\ -\frac{1}{2}(\mathsf{P}_{11} - \mathsf{P}_{22} + \mathsf{Q}_{11} - \mathsf{Q}_{22}) \end{pmatrix} \\ &= \frac{1}{x}\begin{pmatrix} \frac{\Lambda}{\lambda c_1} + U \\ U\left(-\frac{\Lambda}{\lambda c_1} + l + 1\right) \\ \frac{\lambda c_1 U}{2} - \frac{\Lambda}{2\lambda c_1 U} + \frac{V_g}{2U} + 2U - \frac{1}{2} \end{pmatrix}, \end{aligned} \tag{A35}$$

$$\begin{aligned} \Omega &= -\frac{1}{2}(\mathsf{P}_{11} + \mathsf{P}_{22} + \mathsf{Q}_{11} + \mathsf{Q}_{22}) \\ &= \frac{1}{2x}\left[-\frac{\Lambda}{\lambda c_1}\left(2U + \frac{1}{U}\right) + U(\lambda c_1 + 4) + \frac{V_g}{U} + 1\right], \end{aligned} \tag{A36}$$

$$\begin{aligned} \boldsymbol{k}^{(1)} &= \begin{pmatrix} -\mathsf{R}_{11} - \mathsf{R}_{22} \\ -\frac{1}{2}(\mathsf{P}_{11} - \mathsf{P}_{22} - \mathsf{Q}_{11} + \mathsf{Q}_{22}) \\ \mathsf{P}_{12} - \mathsf{Q}_{12} \end{pmatrix} \\ &= \frac{1}{x}\begin{pmatrix} -\frac{\Lambda}{\lambda c_1} - U + l + 4 \\ -\frac{\Lambda}{2\lambda c_1 U} - \frac{1}{2}\lambda c_1 U + \frac{V_g}{2U} - 2U - \frac{1}{2} \\ U\left(-\frac{\Lambda}{\lambda c_1} + l + 1\right) \end{pmatrix}, \end{aligned} \tag{A37}$$

and

$$\begin{aligned} \boldsymbol{k}^{(2)} &= \begin{pmatrix} -\frac{1}{2}(\mathsf{P}_{11} + \mathsf{P}_{22} - \mathsf{Q}_{11} - \mathsf{Q}_{22}) \\ \mathsf{R}_{11} - \mathsf{R}_{22} \\ -\mathsf{R}_{12} - \mathsf{R}_{21} \end{pmatrix} \\ &= \frac{1}{x}\begin{pmatrix} \frac{\Lambda}{\lambda c_1}\left(U - \frac{1}{2U}\right) - \frac{1}{2}\lambda c_1 U - 2U + \frac{V_g}{2U} + \frac{1}{2} \\ \frac{\Lambda}{\lambda c_1} + U + l - 3 \\ -\frac{\Lambda}{\lambda c_1} + U \end{pmatrix}, \end{aligned} \tag{A38}$$

respectively. We can confirm equation (46) from equations (45) and (A31), because we have

$$\mathsf{KB} = \begin{pmatrix} \mathsf{B}_1 & \mathsf{B}_3 \\ -\mathsf{B}_3^T & -\mathsf{B}_2 \end{pmatrix}, \tag{A39}$$

which demonstrates that KB is antisymmetric due to the antisymmetry of B_1 and B_2.

A.2.3. Differential Equations of the Amplitude and the Phases

In this subsection, we use, for simplicity, the symbols A, $\boldsymbol{u}$, $\boldsymbol{v}$, and θ for those with the superscript b (b = c, s). Because of equation (50), we may separate the amplitude A of $\boldsymbol{u}$ and $\boldsymbol{v}$ from

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{A} \tag{A40}$$

and

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{A} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{A41}$$

We provide the differential equations that A, $\hat{\boldsymbol{u}}$, and θ satisfy. If we differentiate $A^2 = \boldsymbol{u}\cdot\boldsymbol{u}$ and use equation (43), we get

$$\begin{aligned} \frac{dA^2}{dx} &= 2\boldsymbol{u}\cdot\frac{d\boldsymbol{u}}{dx} = 2\boldsymbol{u}\cdot(\mathsf{B}_1\boldsymbol{u} + \mathsf{B}_3\boldsymbol{v}) \\ &= 2A^2\hat{\boldsymbol{u}}\cdot\left[\cos\theta\boldsymbol{k}^{(1)} + \sin\theta\boldsymbol{k}^{(2)}\right], \end{aligned} \tag{A42}$$

where we have used $\mathsf{B}_1\boldsymbol{u} = \boldsymbol{w}\times\boldsymbol{u}$. We thus obtain

$$\frac{d\ln A}{dx} = \hat{\boldsymbol{u}}\cdot\left[\cos\theta\boldsymbol{k}^{(1)} + \sin\theta\boldsymbol{k}^{(2)}\right]. \tag{A43}$$

If we substitute $\boldsymbol{u} = A\hat{\boldsymbol{u}}$ and $\boldsymbol{v} = A\hat{\boldsymbol{v}}$ into equation (43) and utilize equation (A43), we get

$$\frac{d\hat{\boldsymbol{u}}}{dx} = \left\{\boldsymbol{w} + \hat{\boldsymbol{u}}\times\left[\cos\theta\boldsymbol{k}^{(1)} + \sin\theta\boldsymbol{k}^{(2)}\right]\right\}\times\hat{\boldsymbol{u}} \tag{A44}$$

and

$$\frac{d\theta}{dx} = \Omega + \hat{\boldsymbol{u}}\cdot\left[\cos\theta\boldsymbol{k}^{(2)} - \sin\theta\boldsymbol{k}^{(1)}\right]. \tag{A45}$$

Equation (A45) tells us

$$\left|\frac{d\theta}{dx}\right| \le |\Omega| + \left|\boldsymbol{k}^{(1)}\right| + \left|\boldsymbol{k}^{(2)}\right| , \tag{A46}$$

which means that the derivative of θ with respect to x is finite for $0 < x < 1$ and any finite positive values of λ.

A.2.4. Derivatives with Respect to λ

Since the derivatives of P, Q, and R with respect to λ are calculated as

$$\mathsf{P}_\lambda = -\frac{\Lambda}{\lambda^2 x c_1 U}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A47}$$

$$\mathsf{Q}_\lambda = -\frac{\Lambda U}{\lambda^2 x c_1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{c_1 U}{x}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A48}$$

and

$$\mathsf{R}_\lambda = -\frac{\Lambda}{\lambda^2 x c_1}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \tag{A49}$$

respectively, the derivative of JA is given by

$$\begin{aligned}\mathsf{JA}_\lambda &= \begin{pmatrix} \mathsf{P}_\lambda & \mathsf{R}_\lambda \\ \mathsf{R}_\lambda^T & \mathsf{Q}_\lambda \end{pmatrix} \\ &= -\frac{\Lambda}{\lambda^2 x c_1}\begin{pmatrix} \frac{1}{U} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & U & U \\ 1 & 0 & U & U \end{pmatrix} \\ &\quad - \frac{c_1 U}{x}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{A50}$$

We then obtain

$$\begin{aligned}\mathcal{I} &= \int_0^1 \boldsymbol{y}^T (-\mathsf{JA}_\lambda)\, \boldsymbol{y}\, \mathrm{d}x \\ &= \int_0^1 \frac{1}{x}\left\{\frac{\Lambda}{\lambda^2 c_1 U}\left[p_1 + U(q_1+q_2)\right]^2 + c_1 U q_1^2\right\} \mathrm{d}x \\ &= \frac{4\pi}{MR^2}\int_0^R \left(\xi_r^2 + \Lambda \xi_h^2\right)\rho r^2\, \mathrm{d}r , \end{aligned} \tag{A51}$$

which is essentially equal to the mode inertia. It is possible to show that $\mathcal{I} > 0$ for any non-trivial solutions of the Hamiltonian system or, equivalently, equation (A20). In fact, if we assume $\mathcal{I} = 0$ for $l > 0$, we have $y_1^{\mathrm{D}} = 0$ and $y_2^{\mathrm{D}} = 0$ for $0 \le x \le 1$. If we substitute these conditions in equation (A20), the first equation tells us that $y_3^{\mathrm{D}} = 0$ for $0 \le x \le 1$ since $V_g > 0$ for $x > 0$. Then, the third equation yields $y_4^{\mathrm{D}} = 0$ for $0 \le x \le 1$. We thus find that $\boldsymbol{y}^{\mathrm{D}} = \boldsymbol{0}$ for $0 \le x \le 1$ if $\mathcal{I} = 0$ and $l > 0$. When $l = 0$, the assumption of $\mathcal{I} = 0$ yields $y_1^{\mathrm{D}} = 0$ for $0 \le x \le 1$. We obtain from the first equation of equation (A20) that $y_2^{\mathrm{D}} = y_3^{\mathrm{D}}$ for $0 \le x \le 1$. The fourth equation of equation (A20) is accordingly reduced to

$$\frac{dy_4^{\mathrm{D}}}{dx} = -\frac{U}{x} y_4^{\mathrm{D}} , \tag{A52}$$

which is integrated to generate

$$y_4^{\mathrm{D}} \propto \frac{1}{M_r} . \tag{A53}$$

For the regular behavior of y_4^{D} near the center [cf. equation (A24)], we must have $y_4^{\mathrm{D}} = 0$ for $0 \le x \le 1$. The third equation of equation (A20) is therefore simplified to

$$\frac{dy_3^{\mathrm{D}}}{dx} = \frac{1-U}{x} y_3^{\mathrm{D}} , \tag{A54}$$

whose solution is given by

$$y_3^{\mathrm{D}} \propto \frac{x}{M_r} . \tag{A55}$$

Then the surface boundary condition given by equation (A26) requires that $y_3^{\mathrm{D}} = 0$ for $0 \le x \le 1$. We have thus shown that $\boldsymbol{y}^{\mathrm{D}} = \boldsymbol{0}$ if $\mathcal{I} = 0$ and $l = 0$.

A.2.5. Expansions Near the Boundaries

A.2.5.1. Expansions Near the Center

In terms of y_i^{D} ($i = 1, 2, 3, 4$), a pair of linearly independent solution vectors that satisfy the central boundary conditions [cf. equations (A23) and (A24)] are expanded near $x = 0$ as

$$\boldsymbol{y}^{\mathrm{D,c,(1)}} = x^{l-2}\begin{pmatrix} l \\ \lambda c_1^{\mathrm{c}} \\ 0 \\ 0 \end{pmatrix} + O\left(x^l\right) \tag{A56}$$

and

$$\boldsymbol{y}^{\mathrm{D,c,(2)}} = x^{l-2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ l \end{pmatrix} + O\left(x^l\right) , \tag{A57}$$

where c_1^{c} is the central value of c_1 (cf. Dziembowski 1971). By substituting equations (A56) and (A57) into equations (A12)–(A15), we may construct the expansions of $\boldsymbol{y}^{\mathrm{c},(1)}$ and $\boldsymbol{y}^{\mathrm{c},(2)}$ near $x = 0$ as

$$\boldsymbol{y}^{\mathrm{c},(1)} = \frac{x^{l+1/2}}{c_1^{\mathrm{c}}}\begin{pmatrix} 3\left(\lambda c_1^{\mathrm{c}} - l\right) \\ -3l \\ l \\ 0 \end{pmatrix} + O\left(x^{l+5/2}\right) \tag{A58}$$

and

$$\boldsymbol{y}^{\mathrm{c},(2)} = \frac{x^{l+1/2}}{c_1^{\mathrm{c}}}\begin{pmatrix} -3 \\ -2l-1 \\ 0 \\ 1 \end{pmatrix} + O\left(x^{l+5/2}\right) , \tag{A59}$$

respectively. The expansions of the exterior products near $x = 0$ are then provided by

$$\begin{aligned}(p_1 \wedge p_2)^{\mathrm{c}} &= \frac{3}{\left(c_1^{\mathrm{c}}\right)^2}\left[2l(l-1) - (2l+1)\lambda c_1^{\mathrm{c}}\right] x^{2l+1} \\ &\quad + O\left(x^{2l+3}\right) , \end{aligned} \tag{A60}$$

$$(p_1 \wedge q_1)^{\mathrm{c}} = \frac{3l}{\left(c_1^{\mathrm{c}}\right)^2} x^{2l+1} + O\left(x^{2l+3}\right) , \tag{A61}$$

$$(p_1 \wedge q_2)^{\mathrm{c}} = \frac{3\left(\lambda c_1^{\mathrm{c}} - l\right)}{\left(c_1^{\mathrm{c}}\right)^2} x^{2l+1} + O\left(x^{2l+3}\right) , \tag{A62}$$

$$(p_2 \wedge q_1)^{\rm c} = \frac{l(2l+1)}{\left(c_1^{\rm c}\right)^2} x^{2l+1} + O\left(x^{2l+3}\right) , \tag{A63}$$

$$(p_2 \wedge q_2)^{\rm c} = -\frac{3l}{\left(c_1^{\rm c}\right)^2} x^{2l+1} + O\left(x^{2l+3}\right) , \tag{A64}$$

and

$$(q_1 \wedge q_2)^{\rm c} = \frac{l}{\left(c_1^{\rm c}\right)^2} x^{2l+1} + O\left(x^{2l+3}\right) . \tag{A65}$$

Equations (A61) and (A64) tell us that $\mathcal{S}\left(\boldsymbol{y}^{\rm c,(1)}, \boldsymbol{y}^{\rm c,(2)}\right)$ goes to zero on the order of x^{2l+3} at the limit of $x \to 0$. Since it is constant for all x, we obtain

$$\mathcal{S}\left(\boldsymbol{y}^{\rm c,(1)}, \boldsymbol{y}^{\rm c,(2)}\right) = 0 \quad \text{for } 0 \le x \le 1 . \tag{A66}$$

We find from equations (A60)–(A65) that $\boldsymbol{m}^{\rm c}$ [cf. equation (44)] is expanded near the center as

$$\boldsymbol{m}^{\rm c} = -\frac{x^{2l+1}}{2\left(c_1^{\rm c}\right)^2} \begin{pmatrix} 3(2l+1)\lambda c_1^{\rm c} - l(6l-5) \\ -3\lambda c_1^{\rm c} - 2l(l-1) \\ -6l \\ 3(2l+1)\lambda c_1^{\rm c} - l(6l-7) \\ 3\lambda c_1^{\rm c} - 2l(l+2) \end{pmatrix} + O\left(x^{2l+3}\right) . \tag{A67}$$

Since $\boldsymbol{m}^{\rm c}$ is a solution of the linear homogeneous system of ordinary differential equations, we may multiply it by any nonzero constant as the normalization factor. If we adopt a negative constant as the normalization factor, $\theta^{\rm c}$ [cf. equations (53) and (54)] behaves near the center as

$$\theta^{\rm c} = \theta_0^{\rm c} + O\left(x^2\right) , \tag{A68}$$

in which we have defined

$$\theta_0^{\rm c} = \mathrm{atan2}\left[3\lambda c_1^{\rm c} - 2l(l+2) , 3(2l+1)\lambda c_1^{\rm c} - l(6l-7)\right] . \tag{A69}$$

We can show that, for $l \ge 0$ and $\lambda > 0$, the domain of $\theta^{\rm c}$ is given by

$$\mathrm{atan2}(-1,-3) < \theta_0^{\rm c} \le \frac{\pi}{4} , \tag{A70}$$

so that $\theta_0^{\rm c}$ changes continuously as a function of λ, $c_1^{\rm c}$, and l.

The other two solutions of equation (A20) near the center than those in equations (A56) and (A57) are given by

$$\boldsymbol{y}^{\rm D,c,(3)} = x^{-l-3} \begin{pmatrix} l+1 \\ -\lambda c_1^{\rm c} \\ 0 \\ 0 \end{pmatrix} + O\left(x^{-l-1}\right) \tag{A71}$$

and

$$\boldsymbol{y}^{\rm D,c,(4)} = x^{-l-3} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -l-1 \end{pmatrix} + O\left(x^{-l-1}\right) . \tag{A72}$$

Note that these do not satisfy the central boundary conditions. The corresponding solution vectors of the canonical variables then satisfy

$$\boldsymbol{y}^{{\rm c},(i)} = O\left(x^{-l-1/2}\right) \quad \text{for } i = 3, 4 . \tag{A73}$$

A.2.5.2. Expansions Near the Surface

In order to determine the expansions of $y_i^{\rm D}$ $(i = 1, 2, 3, 4)$ that satisfy the surface boundary conditions, we assume the behavior of the equilibrium quantities near the surface as

$$U = s^\beta [u_0 + O(s)] , \tag{A74}$$

$$V_g = s^{-1} v_{-1} + O\left(s^0\right) , \tag{A75}$$

$$A^* = s^{-1} a_{-1} + O\left(s^0\right) , \tag{A76}$$

and

$$c_1 = 1 + O(s) , \tag{A77}$$

in which we have introduced $s = 1 - x$, and β, u_0, v_{-1}, and a_{-1} are all constant. These constants satisfy $\beta > 0$, $u_0 > 0$, and $v_{-1} > 0$. Note that the relation

$$\frac{d\ln U}{d\ln x} = 3 - U - V_g - A^* , \tag{A78}$$

which comes from the mass conservation, yields the condition,

$$\beta = v_{-1} + a_{-1} . \tag{A79}$$

According to a detailed analysis, equation (A20) is formally satisfied by the expansions,

$$\boldsymbol{y}^{\rm D,s,(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -U \end{pmatrix} + \begin{pmatrix} O(s) \\ O(s) \\ O\left(s^{1+\beta}\right) \\ O\left(s^{1+\beta}\right) \end{pmatrix} \tag{A80}$$

and

$$\boldsymbol{y}^{\rm D,s,(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -l-1 \end{pmatrix} + O(s) , \tag{A81}$$

which are consistent with the surface boundary conditions [cf. equations (A25) and (A26)] as $s \to 0$. We correspondingly obtain the expansions of $\boldsymbol{y}^{\rm s,(1)}$ and $\boldsymbol{y}^{\rm s,(2)}$ near $x = 1$ as

$$\boldsymbol{y}^{\rm s,(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} O\left(s^{1+\beta}\right) \\ O\left(s^{1+\beta}\right) \\ O(s) \\ O\left(s^{1+\beta}\right) \end{pmatrix} \tag{A82}$$

and

$$\boldsymbol{y}^{\rm s,(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} O\left(s^{1+\beta}\right) \\ O(s) \\ O(s) \\ O(s) \end{pmatrix} , \tag{A83}$$

respectively. The expansions of the exterior products are accordingly given by

$$(p_1 \wedge p_2)^{\rm s} = O\left(s^{2+\beta}\right) , \tag{A84}$$

$$(p_1 \wedge q_1)^{\rm s} = O\left(s^{1+\beta}\right) , \tag{A85}$$

$$(p_1 \wedge q_2)^{\rm s} = O\left(s^{1+\beta}\right) , \tag{A86}$$

$$(p_2 \wedge q_1)^{\rm s} = O(s) , \tag{A87}$$

$$(p_2 \wedge q_2)^{\rm s} = O\left(s^{1+\beta}\right) , \tag{A88}$$

and

$$(q_1 \wedge q_2)^{\rm s} = 1 + O(s) \ , \tag{A89}$$

from which we obtain the formal expansion of $\boldsymbol{m}^{\rm s}$ [cf. equation (44)] as

$$\boldsymbol{m}^{\rm s} = -\frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} O(s) \\ O(s) \\ O\left(s^{1+\beta}\right) \\ O(s) \\ O(s) \end{pmatrix} . \tag{A90}$$

We find from equations (A85) and (A88) that $\mathcal{S}\left(\boldsymbol{y}^{\rm s,(1)}, \boldsymbol{y}^{\rm s,(2)}\right) \to 0$ as $x \to 1$, so that

$$\mathcal{S}\left(\boldsymbol{y}^{\rm s,(1)}, \boldsymbol{y}^{\rm s,(2)}\right) = 0 \quad \text{for } 0 \le x \le 1 \ . \tag{A91}$$

If we multiply $\boldsymbol{m}^{\rm s}$ by a negative normalization factor, the expression of $\theta^{\rm s}$ [cf. equations (53) and (54)] that corresponds to equation (A90) is provided by

$$\theta^{\rm s} = \theta^{\rm s}_0 + O(s) \ , \tag{A92}$$

in which we require

$$\theta^{\rm s}_0 = 0 \ . \tag{A93}$$

The formal expansions of the other solutions of equation (A20) than those given by equations (A80) and (A81) near the surface are provided by

$$\boldsymbol{y}^{\rm D,s,(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} O(s) \\ O(s) \\ O(s) \\ O\left(s^2\right) \end{pmatrix} \tag{A94}$$

and

$$\boldsymbol{y}^{\rm D,s,(4)} = s^{-\beta}\begin{pmatrix} v_{-1} \\ -a_{-1} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} O\left(s^{1-\beta}\right) \\ O\left(s^{1-\beta}\right) \\ O\left(s^2\right) \\ O(s) \end{pmatrix} . \tag{A95}$$

Note that equation (A95) should be replaced by

$$\boldsymbol{y}^{\rm D,s,(4)} = s^{-\beta}\begin{pmatrix} v_{-1} \\ -a_{-1} \\ 0 \\ 0 \end{pmatrix} + \mathfrak{A}(\ln s)\,\boldsymbol{y}^{\rm D,s,(1)} + \begin{pmatrix} O\left(s^{1-\beta}\right) \\ O\left(s^{1-\beta}\right) \\ O\left(s^2\right) \\ O(s) \end{pmatrix} , \tag{A96}$$

in which $\mathfrak{A}$ is a constant, if β is equal to a positive integer. If we regard the vectors of the canonical variables that correspond to equation (A94) and equation (A95) [or (A96) for $\beta = 1, 2, \ldots$] as $\boldsymbol{y}^{\rm s,(3)}$ and $\boldsymbol{y}^{\rm s,(4)}$, respectively, we obtain

$$\boldsymbol{y}^{\rm s,(3)} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} O\left(s^{1+\beta}\right) \\ O(s) \\ O(s) \\ O(s) \end{pmatrix} \tag{A97}$$

and

$$\boldsymbol{y}^{\rm s,(4)} = \begin{cases} \boldsymbol{y}^{\rm s,(4)}_a & \text{for } \beta = 1, 2, \ldots , \\ \boldsymbol{y}^{\rm s,(4)}_b & \text{otherwise } (\beta > 0) , \end{cases} \tag{A98}$$

in which we have defined

$$\boldsymbol{y}^{\rm s,(4)}_a = v_{-1}\, s^{-\beta}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mathfrak{A}(\ln s)\,\boldsymbol{y}^{\rm s,(1)} - u_0 \begin{pmatrix} \beta \\ v_{-1} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} O(s) \\ O(s) \\ O\left(s^{1-\beta}\right) \\ O\left(s^2\right) \end{pmatrix} , \tag{A99}$$

and

$$\boldsymbol{y}^{\rm s,(4)}_b = v_{-1}\, s^{-\beta}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - u_0 \begin{pmatrix} \beta \\ v_{-1} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} O(s) \\ O(s) \\ O\left(s^{1-\beta}\right) \\ O\left(s^2\right) \end{pmatrix} . \tag{A100}$$

An important property of $\boldsymbol{y}^{\rm s,(4)}$ is that, by suitably choosing a function $\mathfrak{F}(s)$, we can show

$$\boldsymbol{y}^{\rm s,(4)} + \mathfrak{F}(s)\,\boldsymbol{y}^{\rm s,(1)} = -u_0 \begin{pmatrix} \beta \\ v_{-1} \\ 0 \\ 0 \end{pmatrix} + O(s) \ . \tag{A101}$$

Appendix 3. Derivatives of the Symplectic Form

We evaluate the derivatives of the symplectic form defined by equation (23) with respect to a parameter α, on which the coefficient matrix A depends. We do not first specify the definition of α to develop the general analysis. We will later consider the cases of $\alpha = \lambda$ and $\alpha = \epsilon$ (structure perturbation).

A.3.1. General Expressions

If the partial derivative of $\boldsymbol{y}^{(i)}$ with respect to α is denoted by $\boldsymbol{y}^{(i)}_\alpha$, we can write

$$\begin{aligned} S_\alpha\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) &= \frac{\partial}{\partial\alpha}\left(\boldsymbol{y}^{(1)T}\mathsf{J}\boldsymbol{y}^{(2)}\right) \\ &= \boldsymbol{y}^{(1)T}_\alpha \mathsf{J}\boldsymbol{y}^{(2)} + \boldsymbol{y}^{(1)T}\mathsf{J}\boldsymbol{y}^{(2)}_\alpha \ . \end{aligned} \tag{A102}$$

In order to evaluate $\boldsymbol{y}_\alpha$, we introduce the fundamental matrix F of equation (1). It is a four by four regular matrix whose columns consist of four linearly independent solution vectors of equation (1). By definition, F satisfies

$$\frac{d\mathsf{F}}{dx} = \mathsf{A}\mathsf{F} \ . \tag{A103}$$

Since $\boldsymbol{y}^{(i)}$ can be expressed as a linear combination of the four solution vectors that compose F, there exists a constant vector $\boldsymbol{a}^{(i)}$ for each of $i = 1, 2$ that satisfies

$$\boldsymbol{y}^{(i)}(x) = \mathsf{F}(x)\boldsymbol{a}^{(i)} \ . \tag{A104}$$

Similarly, since F is a regular matrix, $\boldsymbol{y}^{(i)}_\alpha$ can also be expressible in terms of a linear combination of the four solution vectors as

$$\boldsymbol{y}^{(i)}_\alpha(x) = \mathsf{F}(x)\boldsymbol{b}^{(i)}(x) \ , \tag{A105}$$

where $\boldsymbol{b}^{(i)}$ generally depends on x. Partially differentiating equation (1), we obtain

$$\frac{d\boldsymbol{y}^{(i)}_\alpha}{dx} = \mathsf{A}_\alpha \boldsymbol{y}^{(i)} + \mathsf{A}\mathsf{F}\boldsymbol{b}^{(i)} \ , \tag{A106}$$

where we have used equation (A105), and A_α is the partial derivative of A with respect to α. On the other hand, if we (partially) differentiate equation (A105) with respect to x, we get

$$\frac{d\boldsymbol{y}_\alpha^{(i)}}{dx} = \mathsf{AF}\boldsymbol{b}^{(i)} + \mathsf{F}\frac{d\boldsymbol{b}^{(i)}}{dx} , \tag{A107}$$

in which we have utilized equation (A103). From equations (A106) and (A107), we are led to

$$\frac{d\boldsymbol{b}^{(i)}}{dx} = \mathsf{F}^{-1}\mathsf{A}_\alpha \boldsymbol{y}^{(i)} , \tag{A108}$$

from which we find

$$\boldsymbol{y}_\alpha^{(i)}(x) = \mathsf{F}(x)\int_{x_0^{(i)}}^{x} \mathsf{F}^{-1}(x_1)\mathsf{A}_\alpha(x_1)\boldsymbol{y}^{(i)}(x_1)\,\mathrm{d}x_1 + \mathsf{F}(x)\,\boldsymbol{b}\left(x_0^{(i)}\right) . \tag{A109}$$

Here we take the initial positions of integration $x_0^{(i)}$ as $x_0^{(1)} = 0$ and $x_0^{(2)} = 1$. Then, substitution of equation (A109) into equation (A102) yields

$$S_\alpha\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = \int_0^1 \boldsymbol{y}^{(1)T}(-\mathsf{JA}_\alpha)\boldsymbol{y}^{(2)}\,\mathrm{d}x + \mathcal{B} , \tag{A110}$$

in which

$$\mathcal{B} = \mathsf{C}^{(1)}\boldsymbol{b}^{(2)}(1) - \left[\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(0)\right]^T . \tag{A111}$$

We have used the property,

$$\mathsf{JA}_\alpha = -\mathsf{A}_\alpha^T\mathsf{J} , \tag{A112}$$

which comes from equation (27), and the fact that the one by four matrix,

$$\mathsf{C}^{(i)} = \boldsymbol{y}^{(i)}(x)^T\mathsf{JF}(x) , \tag{A113}$$

does not depend on x, because each column is a symplectic form.

A.3.2. Conditions for the Boundary Terms to Vanish

We discuss the conditions for each term on the right-hand side of equation (A111) to vanish, when $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ are eigenfunction vectors that can be either linearly dependent on, or linearly independent of each other. Since $\boldsymbol{b}^{(1)}(x)$ and $\boldsymbol{b}^{(2)}(x)$ can be singular at $x = 0$ and $x = 1$, respectively, we evaluate them by their limiting values, so that we regard

$$\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(0) = \lim_{x_0\to 0+}\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(x_0) \tag{A114}$$

and

$$\mathsf{C}^{(1)}\boldsymbol{b}^{(2)}(1) = \lim_{x_0\to 1-}\mathsf{C}^{(1)}\boldsymbol{b}^{(2)}(x_0) . \tag{A115}$$

As for

$$\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(x_0) = \boldsymbol{y}^{(2)}(x)^T\mathsf{JF}(x)\boldsymbol{b}^{(1)}(x_0) , \tag{A116}$$

we may construct $\mathsf{F}(x)$ from the four linearly independent solutions $\boldsymbol{y}^{\mathrm{c},(i)}(x)$ $(i = 1, 2, 3, 4)$ as

$$\mathsf{F}(x) = \begin{pmatrix} \boldsymbol{y}^{\mathrm{c},(1)} & \boldsymbol{y}^{\mathrm{c},(2)} & \boldsymbol{y}^{\mathrm{c},(3)} & \boldsymbol{y}^{\mathrm{c},(4)} \end{pmatrix} , \tag{A117}$$

in which only $\boldsymbol{y}^{\mathrm{c},(1)}$ and $\boldsymbol{y}^{\mathrm{c},(2)}$ satisfy the central boundary conditions. We then have

$$\mathsf{F}(x)\,\boldsymbol{b}^{(1)}(x_0) = \sum_{i=1}^{4} b_i^{(1)}(x_0)\,\boldsymbol{y}^{\mathrm{c},(i)}(x) \tag{A118}$$

If $\boldsymbol{y}^{(2)}(x)$ is an eigenfunction vector, it can be expressed as a linear combination of $\boldsymbol{y}^{\mathrm{c},(1)}(x)$ and $\boldsymbol{y}^{\mathrm{c},(2)}(x)$, so that we find

$$\boldsymbol{y}^{(2)}(x)^T\mathsf{J}\boldsymbol{y}^{\mathrm{c},(i)}(x) = 0 \quad \text{for } i = 1, 2 \tag{A119}$$

[cf. equation (28)]. We therefore have

$$\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(x_0) = \sum_{i=3}^{4} b_i^{(1)}(x_0)\,\mathfrak{C}_i , \tag{A120}$$

in which we have introduced a constant,

$$\mathfrak{C}_i = \boldsymbol{y}^{(2)}(x)^T\mathsf{J}\boldsymbol{y}^{\mathrm{c},(i)}(x) \tag{A121}$$

[cf. equation (23)]. We thus find $\mathsf{C}^{(2)}\boldsymbol{b}^{(1)}(0) = 0$, if

$$\lim_{x_0\to 0+} b_i^{(1)}(x_0) = 0 \quad \text{for } i = 3, 4 . \tag{A122}$$

By a completely parallel analysis, we understand that $\mathsf{C}^{(1)}\boldsymbol{b}^{(2)}(1) = 0$, if

$$\lim_{x_0\to 1-} b_i^{(2)}(x_0) = 0 \quad \text{for } i = 3, 4 , \tag{A123}$$

provided that $\boldsymbol{y}^{(1)}(x)$ is an eigenfunction vector, and that $\mathsf{F}(x)$ is chosen as

$$\mathsf{F}(x) = \begin{pmatrix} \boldsymbol{y}^{\mathrm{s},(1)} & \boldsymbol{y}^{\mathrm{s},(2)} & \boldsymbol{y}^{\mathrm{s},(3)} & \boldsymbol{y}^{\mathrm{s},(4)} \end{pmatrix} , \tag{A124}$$

where only $\boldsymbol{y}^{\mathrm{s},(1)}$ and $\boldsymbol{y}^{\mathrm{s},(2)}$ satisfy the surface boundary conditions. Equations (A122) and (A123) provide the sufficient conditions for $\mathcal{B} = 0$. If these conditions are satisfied, equation (A110) is reduced to

$$S_\alpha\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = \int_0^1 \boldsymbol{y}^{(1)T}(-\mathsf{JA}_\alpha)\boldsymbol{y}^{(2)}\,\mathrm{d}x , \tag{A125}$$

which implies

$$S_\alpha\left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}\right) = S_\alpha\left(\boldsymbol{y}^{(2)}, \boldsymbol{y}^{(1)}\right) . \tag{A126}$$

We confirm below that we actually have $\mathcal{B} = 0$ for $\alpha = \lambda$ and $\alpha = \epsilon$.

A.3.3. The Case of $\alpha = \lambda$

By partially differentiating equations (A58) and (A59) with respect to λ, we obtain near the center

$$\boldsymbol{y}_\lambda^{\mathrm{c},(1)}(x) = O\left(x^{l+1/2}\right) \tag{A127}$$

and

$$\boldsymbol{y}_\lambda^{\mathrm{c},(2)}(x) = O\left(x^{l+5/2}\right) , \tag{A128}$$

respectively, from which we find

$$\boldsymbol{y}_\lambda^{(1)}(x) = O\left(x^{l+1/2}\right) , \tag{A129}$$

because $\boldsymbol{y}_\lambda^{(1)}$ can be expressed as linear combinations of $\boldsymbol{y}_\lambda^{\mathrm{c},(1)}$ and $\boldsymbol{y}_\lambda^{\mathrm{c},(2)}$. Then, because of equation (A73), we find

$$b_i^{(1)}(x) = O\left(x^{2l+1}\right) \quad \text{for } i = 3, 4 . \tag{A130}$$

We thus understand that the condition in equation (A122) is satisfied.

If we differentiate equations (A82) and (A83) with respect to λ, we find near the surface

$$\boldsymbol{y}_{\lambda}^{\mathrm{s},(i)}(x) = O(s) \quad \text{for } i = 1, 2\,, \tag{A131}$$

so that we obtain

$$\boldsymbol{y}_{\lambda}^{(2)}(x) = O(s)\,. \tag{A132}$$

Therefore, from equations (A97) and (A101) we get

$$b_i^{(2)}(x) = O(s) \quad \text{for } i = 3, 4\,, \tag{A133}$$

which ensure that the conditions in equation (A123) are satisfied.

A.3.4. The Case of the perturbation to the equilibrium structure

Here we consider the situation in which the equilibrium structure is perturbed. The perturbation is specified by those to the profiles of c_1, U, and V_g. We introduce a parameter ϵ to describe these perturbations, so that these quantities are perturbed as

$$c_1 \to c_1 + \epsilon \Delta c_1 \tag{A134}$$

and

$$V_g \to V_g + \epsilon \Delta V_g\,. \tag{A135}$$

Note that the perturbation to U is not independent of that to c_1 because of the relation

$$U = 3 - \frac{d\ln c_1}{d\ln x}\,, \tag{A136}$$

which comes from the mass conservation. Differentiation of equations (A58) and (A59) with respect to ϵ yields

$$\boldsymbol{y}_{\epsilon}^{\mathrm{c},(1)} = \frac{\Delta c_1^{\mathrm{c}}}{\left(c_1^{\mathrm{c}}\right)^2} x^{l+1/2} \begin{pmatrix} 3l \\ 3l \\ -l \\ 0 \end{pmatrix} + O\left(x^{l+5/2}\right) \tag{A137}$$

and

$$\boldsymbol{y}_{\epsilon}^{\mathrm{c},(2)} = \frac{\Delta c_1^{\mathrm{c}}}{\left(c_1^{\mathrm{c}}\right)^2} x^{l+1/2} \begin{pmatrix} 3 \\ 2l+1 \\ 0 \\ -1 \end{pmatrix} + O\left(x^{l+5/2}\right)\,, \tag{A138}$$

respectively. We therefore find

$$\boldsymbol{y}_{\epsilon}^{(1)} = O\left(x^{l+1/2}\right)\,. \tag{A139}$$

Then, by the same argument after equation (A129), we understand that

$$\lim_{x_0 \to 0+} b_i^{(1)}(x_0) = 0 \quad \text{for } i = 3, 4\,. \tag{A140}$$

As for the expansion near the surface, we obtain from equations (A82) and (A83)

$$\boldsymbol{y}_{\epsilon}^{\mathrm{s},(i)} = O(s) \quad \text{for } i = 1, 2\,, \tag{A141}$$

hence

$$\boldsymbol{y}_{\epsilon}^{(2)} = O(s)\,. \tag{A142}$$

Again, the same argument after equation (A132) can be applied to confirm that

$$\lim_{x_0 \to 1-} b_i^{(2)}(x_0) = 0 \quad \text{for } i = 3, 4\,. \tag{A143}$$

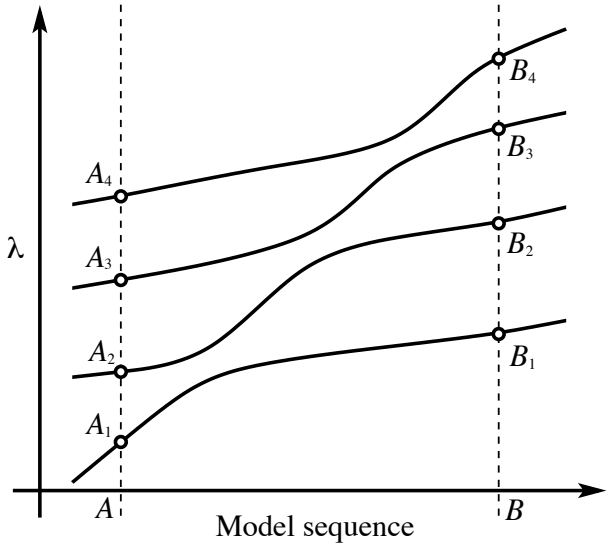

Fig. 3. Typical evolution of the eigenvalue spectrum for a model sequence without degeneracy. The points with open dots correspond to eigenmodes. The eigenmodes of the models A and B are expressed by A_i and B_i (i = 1–4), respectively. We observe that the eigenmodes undergo avoided crossing.

Appendix 4. Properties of the Degeneracy

We explain in this section that the possibility of degenerate eigenmodes is very important for the mode classification. We do not ask whether the degenerate case can actually be found in the problem of adiabatic stellar oscillations or not. Instead, we discuss what kind of influence the degeneracy can have on the mode classification if it exists.

A.4.1. Standard evolution of the eigenvalue spectrum in the absence of degeneracy

For comparison with the case of degenerate eigenmodes, we first depict typical evolution of the eigenvalue spectrum of stellar oscillations for a model sequence without degeneracy in figure 3. Since there is no degeneracy, any two eigenvalues of a given model are never exactly equal to each other. We actually observe that they once come close to each other, but go apart eventually. Curiously, the structure of the eigenfunctions of the two modes is exchanged during the close encounter. This is called the avoided crossing phenomenon (cf. Osaki 1975). As for the mode classification, we have in principle no trouble in following each eigenmode. It is clear that the mode A_i (i = 1–4) of model A evolves into the mode B_i of model B in figure 3, though the structure of the eigenfunctions of the mode A_i (i = 2–4) is closer to that of the mode B_{i-1} than that of the mode B_i.

A.4.2. Possible Influences of the Degenerate Case on the Mode Classification

We next illustrate how the degenerate eigenmodes can affect the mode classification by considering the continuous variation in the equilibrium structure near the degenerate case. As we have shown in subsection 3.3, the degenerate case is characterized by $D = 0$ and $D_\lambda = 0$. Moreover, we find $D_{\lambda\lambda} > 0$ in

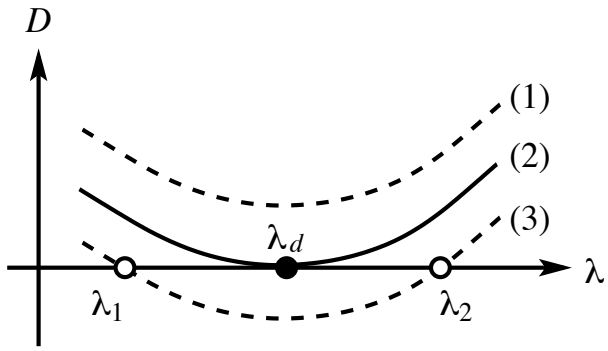

Fig. 4. Schematic profiles of the discriminant $D(\lambda)$ as a function of λ near the degenerate case. The solid curve (2) is the profile observed in the equilibrium structure that has the degenerate eigenvalue λ_d, while the dashed curves (1) and (3) indicate the possible profiles into which the curve (2) is changed by the infinitesimal variation in the equilibrium structure. The filled dot corresponds to the degenerate eigenvalue λ_d, while the open dots depict the positions of the two non-degenerate eigenvalues λ_1 and λ_2.

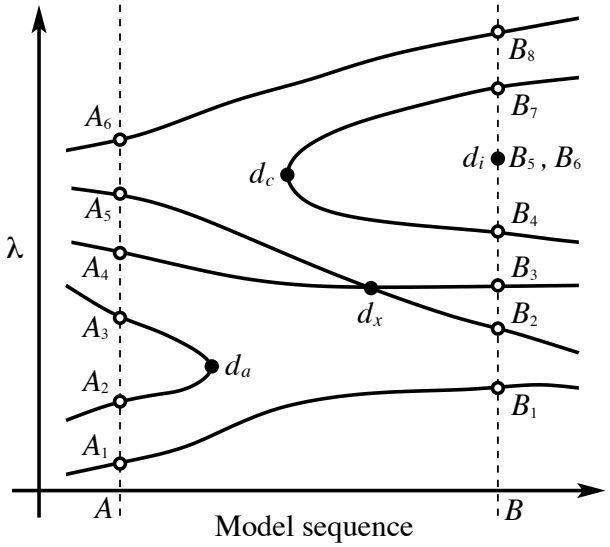

Fig. 5. A schematic example of peculiar evolution of the eigenvalue spectrum for a model sequence. The points with filled dots indicate the degenerate cases, whose categories are indicated by d_c, d_a, d_i, and d_x (see the main text), while the open dots correspond to the non-degenerate eigenmodes. The vertical dashed lines represent the two specific models A and B in the sequence. The eigenmodes of the models A and B are expressed by A_i (i = 1–6) and B_j (j = 1–8), respectively.

this case [cf. equation (64)]. This geometrically means that, if we plot D as a function of λ, the profile of D is tangent from above to the abscissa axis at the value that corresponds to the degenerate eigenvalue λ_d, as we schematically draw in figure 4 [the case of the curve (2)]. If we modify the equilibrium structure infinitesimally, there are the following three possibilities for the local variations in the profile around $\lambda = \lambda_d$: (1) the profile is detached from the abscissa axis; (2) the profile remains to be tangent to the axis; (3) the profile has two crossing points λ_1 and λ_2 with the axis (cf. figure 4). Accordingly, unless the profile stays in the case (2), we may categorize the possible behavior of the profile before and after the degenerate case is found as follows:

- pair creation: case (1) → case (2) → case (3)
- pair annihilation: case (3) → case (2) → case (1)
- isolation: case (1) → case (2) → case (1)
- crossing: case (3) → case (2) → case (3)

If all of these cases are realized, we can observe very peculiar evolution of the eigenmode spectrum along with continuous variation in the equilibrium structure. An example is given in figure 5, in which the cases of the pair creation, the pair annihilation, the isolation, and the crossing are denoted by the points d_c, d_a, d_i, and d_x, respectively. This figure demonstrates how the degenerate cases could complicate the mode classification. If we compare the two models A and B, the eigenmodes with the smallest λ, A_1 and B_1, correspond to each other. The same is true for the modes with the largest λ, A_6 and B_8 though their orders are different from each other because of the influences of the degenerate cases. However, the model B does not have eigenmodes that correspond to A_2 and A_3 of the model A, because these modes annihilate in pair at d_a. It is therefore impossible to allocate continuous integral indexes to the eigenmodes of each of the models A and B in such a way that the eigenmodes with the same index correspond to each other. Similarly, the eigenmodes B_4 and B_7 do not have their counterparts in the model A, because they are created in pair at d_c. The modes B_5 and B_6 of model B, which correspond to the case of the isolation at d_i, are totally isolated so that they are not connected to any modes of the other models in the sequence. On the other hand, it is still possible to regard that the modes A_4 and A_5 of model A turn into the modes B_2 and B_3, respectively, of model B, even though they intersect with each other at d_x. Note that, the reason why the orders of these modes are different in the models A and B is attributed to the influence of the annihilation at d_a, We understand from these observations that it is the cases of the pair creation, the pair annihilation, and the isolation that cause serious complications to the mode classification, while the effect of the crossing is not essential, because the number of modes is conserved. In other words, the scheme of the mode classification must be complicated significantly only if the transition between the cases (1) and (2) in figure 4 is realized by an infinitesimal change in the equilibrium structure.

A.4.3. Conservation of the Number of Eigenmodes

We may regard an infinitesimal change in the equilibrium structure as a perturbation. The question is then how the degenerate eigenmodes are affected by the perturbation. For the rigorous mathematical treatment, we need to invoke rather elaborate argument, which we only outline below. The equations of linear adiabatic oscillations of stars are formally reduced to a single vector equation,

$$\mathcal{L}(\boldsymbol{\xi}) = \lambda \boldsymbol{\xi}\,, \tag{A144}$$

where $\mathcal{L}$ is an integro-differential operator, and $\boldsymbol{\xi}$ is the displacement vector. The operator $\mathcal{L}$ is composed of two parts, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is the operator for the unperturbed structure, and $\mathcal{L}_1$ represents the change in the equilibrium structure. The crucial point of the analysis is that both of the unperturbed operator $\mathcal{L}_0$ and the perturbed operator $\mathcal{L}$ can be extended to

self-adjoint operators, as long as the squared Brunt–Väisälä frequency is bounded below (Dyson & Schutz 1979). As a result, we can apply the general framework of the analytic perturbation theory of the eigenvalue problem of self-adjoint operators, which has originally been developed to provide the mathematical foundation for the perturbative techniques in quantum mechanics (Kato 1976). According to the general theory, the perturbed problem of a discrete eigenvalue λ_0 with finite multiplicity m always has the eigenvalue(s) near λ_0. While the number of the perturbed eigenvalues found is between one and m, the sum of multiplicity of all of the eigenvalues is equal to m. This particularly means that the doubly degenerate eigenmodes are split into two non-degenerate eigenmodes, or remain to be degenerate as a result of the perturbation. Therefore, the transition between the cases (1) and (2) in figure 4 never happens whatever perturbation is applied to the equilibrium structure.

A.4.4. Expression of the Perturbation to the Degenerate Eigenvalue

As a demonstration, we can construct the formula of the perturbation to the degenerate eigenvalue. The analysis is totally in parallel with the degenerate perturbation theory in quantum mechanics (e.g. Landau & Lifshitz 1977). Let us express the unperturbed eigenvalue and the two corresponding eigenfunction vectors by λ_0 and $\boldsymbol{y}_0^{(i)}$ $(i = 1, 2)$, respectively. The unperturbed problem is then given by

$$\frac{d\boldsymbol{y}_0^{(i)}}{dx} = \mathsf{A}_0(x, \lambda_0)\,\boldsymbol{y}_0^{(i)}\,, \tag{A145}$$

where the subscript 0 is also put to the coefficient matrix A to stress that it is for the unperturbed structure. We may express the perturbed problem as

$$\frac{d\boldsymbol{y}}{dx} = \mathsf{A}^\dagger(x, \lambda)\,\boldsymbol{y}\,, \tag{A146}$$

in which the coefficient matrix $\mathsf{A}^\dagger$ is constructed from the perturbed structure, which is described by the perturbations to the structure variables, as we discuss in appendix 3.4. Accordingly, $\mathsf{A}^\dagger(x, \lambda)$ can be expanded as

$$\mathsf{A}^\dagger(x, \lambda) = \mathsf{A}_0(x, \lambda) + \epsilon\mathsf{A}_1(x, \lambda) + O\left(\epsilon^2\right)\,, \tag{A147}$$

where ϵ is assumed to be small (in amplitude). Note that JA_1 is symmetric, because both of JA_0 and $\mathsf{JA}^\dagger$ are symmetric [cf. equation (27)]. We similarly expand the solutions of the perturbed problem as

$$\boldsymbol{y} = \sum_{i=1}^{2}\alpha_i\boldsymbol{y}_0^{(i)} + \epsilon\boldsymbol{y}_1 + O\left(\epsilon^2\right) \tag{A148}$$

and

$$\lambda = \lambda_0 + \epsilon\lambda_1 + O\left(\epsilon^2\right)\,. \tag{A149}$$

Note that the constant coefficients α_i cannot be determined in advance because we do not know which linear combination of $\boldsymbol{y}_0^{(1)}$ and $\boldsymbol{y}_0^{(2)}$ is reached in the unperturbed limit, $\epsilon \to 0$. Since the coefficient matrix on the right-hand side of equation (A146) is expanded as

$$\mathsf{A}^\dagger(x, \lambda) = \mathsf{A}_0(x, \lambda_0) + \epsilon\left[\mathsf{A}_1(x, \lambda_0) + \lambda_1\mathsf{A}_{0,\lambda}(x, \lambda_0)\right] + O\left(\epsilon^2\right)\,, \tag{A150}$$

equation (A146) yields the equation of the first order quantities as

$$\frac{d\boldsymbol{y}_1}{dx} = \mathsf{A}_0(x, \lambda_0)\,\boldsymbol{y}_1 + \sum_{i=1}^{2}\alpha_i\left[\mathsf{A}_1(x, \lambda_0) + \lambda_1\mathsf{A}_{0,\lambda}(x, \lambda_0)\right]\boldsymbol{y}_0^{(i)}\,. \tag{A151}$$

Because of equation (61), we can arrange $\boldsymbol{y}_0^{(1)}$ and $\boldsymbol{y}_0^{(2)}$ so that they satisfy

$$\int_0^1 \boldsymbol{y}_0^{(i)T}(-\mathsf{JA}_{0,\lambda})\,\boldsymbol{y}_0^{(j)}\,\mathrm{d}x = \delta_{i,j}\,. \tag{A152}$$

Then, if we left-multiply equation (A151) by $-\boldsymbol{y}_0^{(j)T}\mathsf{J}$, and integrate the result from $x = 0$ to $x = 1$, we obtain the matrix equation,

$$\begin{pmatrix} \langle 1|\mathsf{A}_1|1\rangle & \langle 1|\mathsf{A}_1|2\rangle \\ \langle 2|\mathsf{A}_1|1\rangle & \langle 2|\mathsf{A}_2|2\rangle \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = -\lambda_1\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\,, \tag{A153}$$

where we have defined

$$\langle i|\mathsf{A}_1|j\rangle = \int_0^1 \boldsymbol{y}_0^{(i)T}\left[-\mathsf{JA}_1(x, \lambda_0)\right]\boldsymbol{y}_0^{(j)}\,\mathrm{d}x\,. \tag{A154}$$

We have also utilized

$$\int_0^1 \boldsymbol{y}_0^{(j)T}(-\mathsf{J})\frac{d\boldsymbol{y}_1}{dx}\,\mathrm{d}x = \left[\boldsymbol{y}_0^{(j)T}(-\mathsf{J})\,\boldsymbol{y}_1\right]_0^1 + \int_0^1 \frac{d\boldsymbol{y}_0^{(j)T}}{dx}\mathsf{J}\boldsymbol{y}_1\,\mathrm{d}x = \int_0^1 \boldsymbol{y}_0^{(j)T}\mathsf{A}_0^T\mathsf{J}\boldsymbol{y}_1\,\mathrm{d}x = \int_0^1 \boldsymbol{y}_0^{(j)T}(-\mathsf{JA}_0)\,\boldsymbol{y}_1\,\mathrm{d}x\,, \tag{A155}$$

where the surface terms vanish because of equations (28), (A140), and (A143), and $d\boldsymbol{y}_0^{(j)T}/dx$ is evaluated by equation (A145). Because $\langle j|\mathsf{A}_1|i\rangle = \langle i|\mathsf{A}_1|j\rangle$ due to the symmetry of JA_1, λ_1 satisfies the secular equation,

$$\lambda_1^2 + (\langle 1|\mathsf{A}_1|1\rangle + \langle 2|\mathsf{A}_1|2\rangle)\lambda_1 + \langle 1|\mathsf{A}_1|1\rangle\langle 2|\mathsf{A}_1|2\rangle - \langle 1|\mathsf{A}_1|2\rangle^2 = 0\,. \tag{A156}$$

Note that the discriminant $\mathfrak{D}$ of this quadratic equation is non-negative because

$$\mathfrak{D} = (\langle 1|\mathsf{A}_1|1\rangle - \langle 2|\mathsf{A}_1|2\rangle)^2 + 4(\langle 1|\mathsf{A}_1|2\rangle)^2 \geq 0\,. \tag{A157}$$

Therefore we always have the real solution(s)

$$\lambda_1 = \frac{1}{2}\left(-\sum_{i=1}^{2}\langle i|\mathsf{A}_1|i\rangle \pm \sqrt{\mathfrak{D}}\right)\,. \tag{A158}$$

Depending on $\mathfrak{D} > 0$ or $\mathfrak{D} = 0$, we have two distinct solutions or a double solution, respectively. We should stress that, if we regard the double solution as the degenerate two solutions, we always find two real values of λ_1 for any perturbation A_1 to the equilibrium structure. We can show that the perturbed eigenmodes stay degenerate ($D = 0$ and $D_\lambda = 0$) up to the first order when $\mathfrak{D} = 0$.

Appendix 5. Calibration of the Mode Index by the Case of Radial Oscillations

The purpose of this section is to show that equation (71) provides $\Theta = \pi$, hence the mode index N of 1, for the radial fundamental mode. When $l = 0$, we can show that the solutions that satisfy the central boundary conditions follow

$$U y_1^{\rm D} + y_4^{\rm D} = 0 \,, \tag{A159}$$

which is obtained by integrating the linearized Poisson equation of the gravitational potential (e.g. Takata 2006b). Since equation (A159) means

$$p_2 + q_2 = 0 \tag{A160}$$

[cf. equations (A13) and (A15)], we obtain from equation (42)

$$v_1^{\rm c} = \frac{1}{2}\left[(p_1 + q_1) \wedge p_2\right]^{\rm c} \tag{A161}$$

and

$$v_2^{\rm c} = \frac{1}{2}\left[(p_1 - q_1) \wedge p_2\right]^{\rm c} \,. \tag{A162}$$

Therefore equation (41) tells us

$$u_1^{\rm c} = \frac{1}{2}\left[(p_1 - q_1) \wedge p_2\right]^{\rm c} = v_2^{\rm c} \,, \tag{A163}$$

$$u_2^{\rm c} = -\frac{1}{2}\left[(p_1 + q_1) \wedge p_2\right]^{\rm c} = -v_1^{\rm c} \,, \tag{A164}$$

and

$$u_3^{\rm c} = 0 \tag{A165}$$

[cf. equation (28)]. Based on equations (A163)–(A165), we can derive from equation (43) the system of differential equations for the vector

$$\boldsymbol{w}^{\rm c} = \begin{pmatrix} v_1^{\rm c} - v_2^{\rm c} \\ v_1^{\rm c} + v_2^{\rm c} \end{pmatrix} = \sqrt{2} A^{\rm c} \begin{pmatrix} \cos\left(\theta^{\rm c} + \frac{\pi}{4}\right) \\ \sin\left(\theta^{\rm c} + \frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} (q_1 \wedge p_2)^{\rm c} \\ (p_1 \wedge p_2)^{\rm c} \end{pmatrix} \tag{A166}$$

as

$$x \frac{d\boldsymbol{w}^{\rm c}}{dx} = \begin{pmatrix} U - 3 & -\frac{V_g}{U} \\ U(\lambda c_1 + 4) & 4 - U \end{pmatrix} \boldsymbol{w}^{\rm c} \,. \tag{A167}$$

The expansion of $\boldsymbol{w}^{\rm c}$ near the center is obtained from equation (A67) as

$$\boldsymbol{w}^{\rm c} = x \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(x^3\right) \,, \tag{A168}$$

whereas the condition of $D \to 0$ as $x_{\rm f} \to 1-$ is written as

$$w_2^{\rm c}(x) \to 0 \qquad \text{as } x \to 1- \tag{A169}$$

[cf. equations (49) and (A90)]. The system of equations for $\boldsymbol{w}^{\rm c}$ given by (A167) and the associated boundary conditions provided by equations (A168) and (A169) are actually the same as those for $\sqrt{x}(q_1\ p_1)^T$. The problem for $\boldsymbol{w}^{\rm c}$ is therefore mathematically equivalent to that of radial oscillations of stars in the standard formulation, to which the Strum–Liouville analysis can be applied. Based on the analogy, we understand that, for the radial fundamental mode, which has no node in the eigenfunctions except at the boundaries, the polar angle of the point $\left(w_1^{\rm c}, w_2^{\rm c}\right)$ changes from $\pi/2$ at the center to π at the surface. Since we may regard that the polar angle of $\left(w_1^{\rm c}, w_2^{\rm c}\right)$ is larger by $\pi/4$ than that of $\left(v_1^{\rm c}, v_2^{\rm c}\right)$ [cf. equation (A166)], we find $\theta^{\rm c} \to 3\pi/4$ as $x_{\rm f} \to 1-$. Then, because of equation (A93), we obtain

$$\Delta\theta^{\rm c} \to \frac{3\pi}{4} \qquad \text{as } x_{\rm f} \to 1- \,. \tag{A170}$$

Similarly, because of equations (A163) and (A164), we may consider the polar angle of the point $\left(u_1^{\rm c}, u_2^{\rm c}\right)$ is smaller by $\pi/2$ than that of $\left(v_1^{\rm c}, v_2^{\rm c}\right)$. We therefore obtain from equation (A90)

$$\Delta\phi^{\rm c} \to \frac{3\pi}{4} \qquad \text{as } x_{\rm f} \to 1- \,. \tag{A171}$$

Substituting equations (A170) and (A171) into equation (71), we certainly find $\Theta = \pi$ for the radial fundamental mode.

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