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Boundary critical phenomena in the three-state Potts model

Ian Affleck†, Masaki Oshikawa‡ and Hubert Saleur§

† Canadian Institute for Advanced Research and Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, V6T 1Z1, Canada

‡ Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

§ Department of Physics, University of Southern California, Los Angeles, CA90089-0484, USA

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Abstract. Boundary critical phenomena are studied in the three-state Potts model in two dimensions using conformal field theory, duality and renormalization group methods. A presumably complete set of boundary conditions is obtained using both fusion and orbifold methods. Besides the previously known free, fixed and mixed boundary conditions a new one is obtained. This illustrates the necessity of considering fusion with operators that do not occur in the bulk spectrum, to obtain all boundary conditions. It is shown that this new boundary condition is dual to the mixed ones. The phase diagram for the quantum chain version of the Potts model is analysed using duality and renormalization group arguments.

1. Introduction

Recently there has been considerable interest in the behaviour of two-dimensional systems with boundaries, in the context of string theory, classical statistical mechanics and quantum impurity problems. Exact results on the critical behaviour of these systems have been obtained using boundary conformal field theory (CFT) [1, 2]. More complete exact results on universal crossover functions have also been obtained using exact S-matrix methods [3]. One of the simplest examples of such a system is provided by the three-state Potts model. It can be related, via conformal embeddings, [4] to quantum Brownian motion on a hexagonal lattice [5] and to tunnelling in quantum wires [6]. The classical Hamiltonian for this model can be written by introducing an angular variable at each site of a square lattice, θ_i, restricted to take only three values: $0, \pm 2\pi/3$

$$\beta H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j). \tag{1.1}$$

When the model is at its critical coupling, J_c, various universality classes of boundary critical phenomena are possible. These include free boundary conditions (b.c.'s) and (three different) fixed b.c.'s, $\theta_i = 0$ (or $2\pi/3$ or $-2\pi/3$), for i on the boundary. In addition, it was argued [7] that there are also three 'mixed' b.c.'s in which one of the three spin states is forbidden at the boundary so that the Potts spins on the boundary fluctuate between two of the states (for example, between $2\pi/3$ and $-2\pi/3$).

In the following it will also be convenient to consider the standard quantum chain representation. The Hamiltonian is written in terms of unitary matrices, M_i and R_i defined at each site.

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

In fact, these two matrices can be transformed into each other:

$$R = U^{\dagger} M U. \tag{1.3}$$

This is related to the duality symmetry. The Hamiltonian is

$$H = -\sum_i [(M_i + M_i^{\dagger}) + (R_i^{\dagger} R_{i-1} + R_{i-1}^{\dagger} R_i)]. \tag{1.4}$$

Note that the second term corresponds to the classical Potts model with the three different states corresponding to the vectors (1,0,0), (0,1,0) and (0,0,1). The first term flips the spin on each site between the three states. It is like a transverse field in the Ising model. The model has Z_3 symmetry which interchanges the three basis vectors. Decreasing the strength of the transverse field term puts the system in the ordered phase; increasing it gives the disordered phase. As written, these terms exactly balance; the model is at its critical point. One way of seeing this is to observe that, for this value of the coupling constant, the Hamiltonian maps onto itself under the duality transformation:

$$\begin{aligned} R'_{i+1/2} &\equiv \prod_{j=0}^{i} M_j \\ M'_{i+1/2} &\equiv R_{i+1}^{\dagger} R_i. \end{aligned} \tag{1.5}$$

The six fixed and mixed b.c.'s were represented in terms of boundary states [1]. These are defined by a modular transformation of the partition function on a cylinder of circumference β and length l with b.c.'s A and B at the two ends:

$$Z_{AB} = \mathrm{tr}\exp[-\beta H_{AB}^{l}] = \langle A|\exp[-l H_P^{\beta}]|B\rangle. \tag{1.6}$$

Here, H_{AB}^{l} is the Hamiltonian on a strip of length l with b.c.'s A and B at the two ends. H_P^{β} is the Hamiltonian on a circle of circumference β. Z_{AB} may be expanded in characters of the (chiral) Virasoro algebra:

$$Z_{AB} = \sum_k n_{AB}^{k} \chi_k(q). \tag{1.7}$$

Here q is the modular parameter, $q \equiv \exp[-\pi\beta/l]$, k labels (chiral) conformal towers, χ_k are the characters and n_{AB}^{k} are non-negative integers. The boundary states may be expanded in Ishibashi states, constructed out of each conformal tower:

$$|A\rangle = \sum_k |k\rangle\langle k, 0|A\rangle. \tag{1.8}$$

One way of generating a complete set of boundary states (and hence b.c.'s) from an appropriately chosen reference state is by fusion. Beginning with the reference boundary state $|\tilde{0}\rangle$, one constructs a set of boundary states, $|\tilde{j}\rangle$ associated with the conformal towers, j. Its matrix elements are given by:

$$\langle i, 0|\tilde{j}\rangle = \frac{S_j^i}{S_0^i}\langle i, 0|\tilde{0}\rangle \tag{1.9}$$

where S_j^i is the modular S-matrix. This construction gives physically sensible multiplicities, n_{AB}^{i}; that is they are non-negative integers obeying $n_{AA}^{0} = 1$. This construction relies on the Verlinde formula [8] which relates the modular S-matrix to the fusion rule coefficients.

A subtlety arises in the Potts model connected with an extended W-algebra. While there are 10 Virasoro conformal towers for central charge $c = \frac{4}{5}$, labelled by pairs of integers, (n, m) with $1 \leqslant n \leqslant 4$ and $m \leqslant n$, only four larger conformal towers, which are combinations of these ones, occur in the bulk spectrum or with certain pairs of b.c.'s.

Furthermore, two of these conformal towers occur twice in the bulk spectrum corresponding to pairs of operators of opposite charge (± 1) with respect to the Z_3 symmetry of the Potts model. (In general, operators can have charge $q = 0$, 1 or -1, transforming under Z_3 transformations as:

$$\mathcal{O} \rightarrow \mathrm{e}^{\mathrm{i}q\theta}\mathcal{O} \tag{1.10}$$

for $\theta = 0, \pm 2\pi/3$.) These operators are σ, $\sigma^\dagger$ of dimension $\frac{1}{15}$ and ψ, $\psi^\dagger$ of dimension $\frac{2}{3}$. The Potts model also contains an energy operator, ϵ of dimension $\frac{2}{5}$ as well as the identity operator, I. These W-characters are given by:

$$\begin{aligned} \chi_I &= \chi_{11} + \chi_{41} \qquad & \chi_\epsilon &= \chi_{21} + \chi_{31} \\ \chi_\sigma &= \chi_{\sigma^\dagger} = \chi_{33} \qquad & \chi_\psi &= \chi_{\psi^\dagger} = \chi_{43} \end{aligned} \tag{1.11}$$

where χ_{nm} is the Virasoro character for the (n, m) conformal tower. The Potts model has a fusion algebra which closes on these operators. The modular transform of these W-characters can be expressed entirely in terms of W-characters and the corresponding S-matrix and fusion rule coefficients obey the Verlinde formula. Ambiguities in the S-matrix and fusion rules associated with having operators of equal dimension are removed by requiring consistency with the Z_3 symmetry. Cardy constructed a set of boundary states which were linear combinations of the Ishibashi states constructed using the extended W-algebra. The reference state for the fusion process, in this construction, is the boundary state, $|\tilde{I}\rangle$, obeying $Z_{\tilde{I}\tilde{I}} = \chi_I$. It was argued in [1] that it corresponds to one of the fixed b.c.'s, the other two being $|\tilde{\psi}\rangle$ and $|\tilde{\psi}^\dagger\rangle$. Similarly $|\tilde{\epsilon}\rangle$, $|\tilde{\sigma}\rangle$ and $|\tilde{\sigma}^\dagger\rangle$ correspond to the three mixed b.c.'s. All partition functions involving these six b.c.'s can be expressed in terms of W characters. On the other hand, it was observed that partition functions that combine free b.c.'s with fixed or mixed cannot be expressed in terms of W characters and the corresponding free boundary state was not determined.

Clearly the set of b.c.'s generated by fusion with the primary fields of the bulk Potts spectrum (which are covariant with respect to the W-algebra) is not complete, since it does not include the free b.c.'. In the next section we shall generate a presumably complete set of boundary states including the one corresponding to free boundary conditions and one new boundary state. We do this in two different ways; one method uses fusion and the other uses an orbifold projection. In the final section we shall explore the physical significance of this new boundary condition and the boundary renormalization group flow diagram. The appendix contains a peripherally related result: a general proof that the ground-state entropy always *increases* under fusion.

2. Boundary states

2.1. Fusion approach

In order to determine the 'free' boundary state and check for possible additional boundary states (and conditions) we must work with the larger set of conformal towers not constrained by the W-symmetry. The full modular S-matrix, in the space of all 10 Virasoro conformal towers that can occur in a $c = \frac{4}{5}$ minimal model, is given in table 1. The state $|\tilde{I}\rangle$ may be expanded in terms of W-Ishibashi states as:

$$|\tilde{I}\rangle = N\{|I\rangle + |\psi\rangle + |\psi^\dagger\rangle + \lambda[|\epsilon\rangle + |\sigma\rangle + |\sigma^\dagger\rangle]\} \tag{2.1}$$

where

$$N^4 = \frac{5-\sqrt{5}}{30} \qquad \lambda^2 = \frac{1+\sqrt{5}}{2}. \tag{2.2}$$

Table 1. The modular S-matrix for Virasoro characters (multiplied by $2/N^2$). Characters are labelled by their Kac labels (n, m) (and by their highest weight).

	$11(0)$	$41(3)$	$21(\frac{2}{5})$	$31(\frac{7}{5})$	$43(\frac{2}{3})$	$33(\frac{1}{15})$	$44(\frac{1}{8})$	$42(\frac{13}{8})$	$22(\frac{1}{40})$	$32(\frac{21}{40})$
$11(0)$	1	1	λ^2	λ^2	2	$2\lambda^2$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$
$41(3)$	1	1	λ^2	λ^2	2	$2\lambda^2$	$-\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$
$21(\frac{2}{5})$	λ^2	λ^2	-1	-1	$2\lambda^2$	-2	$-\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$	$\sqrt{3}$	$\sqrt{3}$
$31(\frac{7}{5})$	λ^2	λ^2	-1	-1	$2\lambda^2$	-2	$\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$	$-\sqrt{3}$	$-\sqrt{3}$
$43(\frac{2}{3})$	2	2	$2\lambda^2$	$2\lambda^2$	-2	$-2\lambda^2$	0	0	0	0
$33(\frac{1}{15})$	$2\lambda^2$	$2\lambda^2$	-2	-2	$-2\lambda^2$	2	0	0	0	0
$44(\frac{1}{8})$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$	0	0	$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$
$42(\frac{13}{8})$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$
$22(\frac{1}{40})$	$\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$	$\sqrt{3}$	$-\sqrt{3}$
$32(\frac{21}{40})$	$\sqrt{3}\lambda^2$	$-\sqrt{3}\lambda^2$	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\sqrt{3}\lambda^2$	$\sqrt{3}\lambda^2$	$-\sqrt{3}$	$\sqrt{3}$

The W-Ishibashi states may be expanded in terms of Virasoro Ishibashi states as:

$$|I\rangle = |11\rangle + |41\rangle \qquad |\epsilon\rangle = |21\rangle + |31\rangle. \tag{2.3}$$

Now consider all new boundary states that can be obtained from $|\tilde{I}\rangle$ by fusion with all nine non-trivial Virasoro primaries using equation (1.9). Note that $|\tilde{I}\rangle$ has zero amplitude for the last four Ishibashi states in table 1: (4,4), (4,2), (2,2), (3,2). Also note that the $S_i^{(1,1)} = S_i^{(4,1)}$ for all i except for these last four states. The same statement holds for $S_i^{(2,1)}$ and $S_i^{(3,1)}$. Thus expanding the identity tower with respect to the W-algebra into (1,1) and (4,1) does not lead to any additional boundary states. Neither does expanding the ϵ tower into (2,1) and (4,3). The (4,3) and (3,3) towers just give the states found previously since these are themselves W-towers. However, two additional boundary states can be obtained by fusion with (4,4) and (2,2). On the other hand, fusion with (4,2) gives the same result as (4,4) and (3,2) the same as (2,2) since all but the last four elements in the corresponding rows in table 1 are equal. Thus, the fusion construction, beginning with the W-invariant boundary state $|\tilde{I}\rangle$ but considering the full set of Virasoro primaries leads to two additional boundary states besides the six found previously by considering fusion with W primaries.

Note that we have performed a sort of hybrid construction. Instead we could have begun with the reference boundary state $|\tilde{11}\rangle$ such that $Z_{\tilde{11},\tilde{11}} = \chi_{11}$. In this case we would obtain a larger set of boundary states. However, these states do not occur in the Potts model. One reason is that $|\tilde{11}\rangle$ is not consistent with $|\tilde{I}\rangle$. This follows from the identity

$$|\tilde{I}\rangle = \frac{1}{\sqrt{2}}[|\tilde{11}\rangle + |\tilde{41}\rangle]. \tag{2.4}$$

The factor of $1/\sqrt{2}$ in equation (2.4), necessary to avoid a two-fold degeneracy in the spectrum of $Z_{\tilde{I}\tilde{I}}$, leads to an unphysical partition function $Z_{\tilde{I}\tilde{11}}$, with non-integer multiplicities. Another reason why this larger set of boundary states cannot occur in the Potts model is because they contain Ishibashi states not derived from the bulk spectrum. The eight boundary states discussed here presumably form a complete set of states which are mutually consistent.

We note that the idea of obtaining new boundary states (and conditions) by fusion with operators which do not occur in the bulk spectrum is also fundamental to the solution of the non-Fermi liquid fixed points in the Kondo problem [9]. In that case, the reference state was chosen to give a Fermi liquid b.c. The conformal embedding representing the free fermions restricts the bulk spectrum to contain only certain products of operators from the spin, charge and flavour sectors. Fusion with pure spin operators, not contained in the bulk spectrum, gives the infrared stable fixed points of both Fermi liquid and non-Fermi liquid variety.

The two additional boundary states for the Potts model, found above, are:

$$\begin{aligned}|\tilde{44}\rangle &= N\sqrt{3}[(|11\rangle - |41\rangle) - \lambda(|21\rangle - |31\rangle)] \\ |\tilde{22}\rangle &= N\sqrt{3}[\lambda^2(|11\rangle - |41\rangle) + \lambda^{-1}(|21\rangle - |31\rangle)].\end{aligned} \tag{2.5}$$

The partition functions for any pair of b.c.'s can be determined from the boundary states using:

$$\langle i|\exp -lH_P^\beta|j\rangle = \delta_{ij}\chi_i(\tilde{q}) \tag{2.6}$$

where $\tilde{q} = \mathrm{e}^{-4\pi l/\beta}$. Finally we perform a modular transformation to the q-representation:

$$\chi_i(\tilde{q}) = \sum_j S_i^j \chi_j(q). \tag{2.7}$$

Alternatively, we may determine these partition functions from the fusion rule coefficients. For a b.c., $\tilde{i}$ obtained by fusion with primary operator, $\mathcal{O}_i$ from $|\tilde{I}\rangle$ and some other b.c., $\tilde{j}$,

$$n_{\tilde{i}\tilde{j}}^k = \sum_l N_{il}^k n_{\tilde{I}\tilde{j}}^l. \tag{2.8}$$

Here N_{il}^k is the number of times that the primary operator $\mathcal{O}_k$ occurs in the operator product expansion of $\mathcal{O}_i$ with $\mathcal{O}_l$. The needed fusion rule coefficients are given in table 2. These are derived from the fusion rules of the tetracritical Ising model. For instance, to obtain the first box in the table we use:

$$\mathcal{O}_{44} \cdot I = \mathcal{O}_{44} \cdot [\mathcal{O}_{11} + \mathcal{O}_{41}] \to \mathcal{O}_{44} + \mathcal{O}_{42}. \tag{2.9}$$

In cases where two dimension $\frac{2}{3}$ ($\frac{1}{15}$) operators occur in the operator product expansion (OPE) we have interpreted them as $\psi + \psi^\dagger$ ($\sigma + \sigma^\dagger$). This calculation shows that all partition functions involving $|\tilde{44}\rangle$ and any of the fixed or mixed boundary states are the same as those determined previously for the free b.c. Hence we conclude that $|\tilde{44}\rangle$ is the free boundary state. On the other hand, $|\tilde{22}\rangle$ is a new boundary state corresponding to a new b.c. whose physical interpretation is so far unclear. In the next section we investigate the nature of this new boundary fixed point. First, however, we obtain this set of boundary states by an interesting different method.

2.2. Orbifold approach

An alternative way of producing the complete set of boundary states for the Potts model is based on obtaining the Potts model from an orbifold projection on the other $c = \frac{4}{5}$ conformal field theory, the tetracritical Ising model, which has a diagonal bulk partition function [10]. A Z_2 Ising charge, q_i, can be assigned to each primary operator, $\mathcal{O}_i$ of the tetracritical Ising model which is 0 for the first six entries in table 1 and one for the remaining four.

Table 2. Fusion rules for extended operator algebra. The fusion rules not shown are the standard ones for the Potts model [1].

	I or ψ or $\psi^\dagger$	ϵ or σ or $\sigma^\dagger$	$\mathcal{O}_{44} + \mathcal{O}_{42}$	$\mathcal{O}_{22} + \mathcal{O}_{32}$
$\mathcal{O}_{44}$ or $\mathcal{O}_{42}$	$\mathcal{O}_{44} + \mathcal{O}_{42}$	$\mathcal{O}_{22} + \mathcal{O}_{32}$	$I + \psi + \psi^\dagger$	$\epsilon + \sigma + \sigma^\dagger$
$\mathcal{O}_{22}$ or $\mathcal{O}_{32}$	$\mathcal{O}_{22} + \mathcal{O}_{32}$	$\mathcal{O}_{44} + \mathcal{O}_{42} + \mathcal{O}_{22} + \mathcal{O}_{32}$	$\epsilon + \sigma + \sigma^\dagger$	$I + \psi + \psi^\dagger + \epsilon + \sigma + \sigma^\dagger$

(This is a special case of a general construction for minimal models. Choosing a different fundamental domain for Kac labels, (n, m) with

$$1 \leqslant n \leqslant p' - 1 \qquad 1 \leqslant m \leqslant p - 1 \qquad n + m = 0 \bmod 2 \tag{2.10}$$

the charge is:

$$q = n + 1. \tag{2.11}$$

The $c = \frac{4}{5}$ case corresponds to $p = 6$, $p' = 5$. This identification is consistent with the Landau–Ginsburg description of the tetracritical Ising model [11]. The charge 1 operators $\mathcal{O}_{22}$, $\mathcal{O}_{44}$ and $\mathcal{O}_{32}$ correspond to ϕ, $:\phi^3:$ and $:\phi^5:$ respectively. The charge 0 operators $\mathcal{O}_{33}$, $\mathcal{O}_{21}$ and $\mathcal{O}_{43}$ correspond to $:\phi^2:$, $:\phi^4:$ and $:\phi^6:$ respectively. The other three operators other than the identity presumably could be identified with operators in the Landau–Ginsburg description containing derivatives with the number of powers of ϕ even for $\mathcal{O}_{41}$ and $\mathcal{O}_{31}$ and odd for $\mathcal{O}_{42}$.) The tetracritical Ising model has the diagonal bulk partition function:

$$Z_{\mathrm{TC}} \equiv Z_{++} = \sum_{i=1}^{10} |\chi_i|^2 \tag{2.12}$$

where we number the conformal towers from 1 to 10 in the order in table 1. We may define a twisted partition function:

$$Z_{+-} \equiv \sum_{i=1}^{10} (-1)^{q_i} |\chi_i|^2. \tag{2.13}$$

We also define two other twisted partition functions by the modular transforms of Z_{+-}:

$$Z_{-+} \equiv \mathcal{S} Z_{+-} \qquad Z_{--} = \mathcal{T} Z_{-+} \tag{2.14}$$

where $\mathcal{S}$ is the modular transformation $\tau \to -1/\tau$ and $\mathcal{T}$ is the modular transformation $\tau \to \tau + 1$. It can be shown that:

$$Z_{\mathrm{Potts}} = Z_{\mathrm{orb}} = (\tfrac{1}{2})[Z_{++} + Z_{+-} + Z_{-+} + Z_{--}]. \tag{2.15}$$

We may think of the first two terms as representing the contribution of the untwisted sector of the Hilbert space, with the Z_2 invariant states projected out. The second two terms represent the contribution of the twisted sector of the Hilbert space, corresponding to twisted b.c.'s on the circle. (For the simpler case of the $c = 1$ bosonic orbifold the twisted b.c.'s are simply $\phi(0) = -\phi(l)$.) These contributions are explicitly:

$$\begin{aligned} (\tfrac{1}{2})[Z_{++} + Z_{+-}] &= |\chi_{11}|^2 + |\chi_{41}|^2 + |\chi_{21}|^2 + |\chi_{31}|^2 + |\chi_{43}|^2 + |\chi_{33}|^2 \\ (\tfrac{1}{2})[Z_{-+} + Z_{--}] &= \bar{\chi}_{11}\chi_{41} + \bar{\chi}_{41}\chi_{11} + \bar{\chi}_{21}\chi_{31} + \bar{\chi}_{31}\chi_{21} + |\chi_{43}|^2 + |\chi_{33}|^2. \end{aligned} \tag{2.16}$$

There are two types of Ishibashi states which may be used to construct boundary states in the orbifold model. We may take states from the untwisted sector, projecting out the Z_2 invariant parts or we may take states from the twisted sector. The first set of Ishibashi states are labelled by the first six (Z_2 even) conformal towers in table 1. We refer to these untwisted $|43\rangle$ and $|33\rangle$ states as $|\psi_u\rangle$ and $|\sigma_u\rangle$ respectively. There are two additional Ishibashi states from the twisted sector, $|\psi_t\rangle$ and $|\sigma_t\rangle$. We then define:

$$\begin{aligned} |\psi\rangle &\equiv (1/\sqrt{2})[|\psi_u\rangle + i|\psi_t\rangle] \\ |\psi^\dagger\rangle &\equiv (1/\sqrt{2})[|\psi_u\rangle - i|\psi_t\rangle] \end{aligned} \tag{2.17}$$

and similarly for $|\sigma\rangle$. One way of constructing consistent boundary states, using only the untwisted sector, is by projecting out the Z_2 even parts of the tetracritical Ising boundary states. From inspecting table 1 we see that the various tetracritical Ising boundary states are mapped into each other by the Z_2 transformation. We have ordered them in table 1 so that successive pairs are interchanged, apart from $|\tilde{43}\rangle$ and $|\tilde{33}\rangle$ which are invariant. We expect the conjugate pairs to correspond to various generalized spin-up and spin-down b.c.'s. We may formally write the transformed states as:

$$(-1)^{\hat{Q}}|A_{\mathrm{TC}}\rangle. \tag{2.18}$$

With each boundary state, $|A_{\mathrm{TC}}\rangle$, of the tetracritical Ising model, we may associate a boundary state, $|A_{\mathrm{Potts}}\rangle$ of the Potts model using:

$$\langle i,0|A_{\mathrm{Potts}}\rangle = \frac{1+(-1)^{q_i}}{\sqrt{2}}\langle i,0|A_{\mathrm{TC}}\rangle. \tag{2.19}$$

Formally we may write:

$$|A_{\mathrm{Potts}}\rangle = \frac{1+(-1)^{\hat{Q}}}{\sqrt{2}}|A_{\mathrm{TC}}\rangle. \tag{2.20}$$

It is necessary to divide by $\sqrt{2}$ in order that the identity operator appear only once in the diagonal partition functions. In this way we obtain the following Potts boundary states from each tetracritical Ising boundary state:

$$\begin{aligned}
|\tilde{11}_{\mathrm{TC}}\rangle &\to |\tilde{I}\rangle\\
|\tilde{41}_{\mathrm{TC}}\rangle &\to |\tilde{I}\rangle\\
|\tilde{21}_{\mathrm{TC}}\rangle &\to |\tilde{\epsilon}\rangle\\
|\tilde{31}_{\mathrm{TC}}\rangle &\to |\tilde{\epsilon}\rangle\\
|\tilde{43}_{\mathrm{TC}}\rangle &\to |\tilde{\psi}\rangle + |\tilde{\psi}^{\dagger}\rangle\\
|\tilde{33}_{\mathrm{TC}}\rangle &\to |\tilde{\sigma}\rangle + |\tilde{\sigma}^{\dagger}\rangle\\
|\tilde{44}_{\mathrm{TC}}\rangle &\to |\tilde{44}\rangle\\
|\tilde{42}_{\mathrm{TC}}\rangle &\to |\tilde{44}\rangle\\
|\tilde{22}_{\mathrm{TC}}\rangle &\to |\tilde{22}\rangle\\
|\tilde{32}_{\mathrm{TC}}\rangle &\to |\tilde{22}\rangle.
\end{aligned} \tag{2.21}$$

(Note that the states $|\tilde{11}_{\mathrm{TC}}\rangle$ and $|\tilde{41}_{\mathrm{TC}}\rangle$ are the same states simply labelled $|\tilde{11}\rangle$ and $|\tilde{41}\rangle$ in equation (2.4).) We observe that this construction gives us a sum of Potts boundary states in the 43 and 33 cases because the corresponding tetracritical Ising boundary states are Z_2 invariant. We may remedy this situation by forming linear combinations of the projected tetracritical boundary states with the twisted Ishibashi states:

$$\begin{aligned}
|\tilde{\psi}\rangle &= (\tfrac{1}{2})\frac{1+(-1)^{\hat{Q}}}{\sqrt{2}}|\tilde{43}_{\mathrm{TC}}\rangle - N\sqrt{3/2}[|\psi_t\rangle + \lambda|\sigma_t\rangle]\\
|\tilde{\psi}^{\dagger}\rangle &= (\tfrac{1}{2})\frac{1+(-1)^{\hat{Q}}}{\sqrt{2}}|\tilde{43}_{\mathrm{TC}}\rangle + N\sqrt{3/2}[|\psi_t\rangle + \lambda|\sigma_t\rangle]\\
|\tilde{\sigma}\rangle &= (\tfrac{1}{2})\frac{1+(-1)^{\hat{Q}}}{\sqrt{2}}|\tilde{33}_{\mathrm{TC}}\rangle - N\sqrt{3/2}[\lambda^2|\psi_t\rangle - \lambda^{-1}|\sigma_t\rangle]\\
|\tilde{\sigma}^{\dagger}\rangle &= (\tfrac{1}{2})\frac{1+(-1)^{\hat{Q}}}{\sqrt{2}}|\tilde{33}_{\mathrm{TC}}\rangle + N\sqrt{3/2}[\lambda^2|\psi_t\rangle - \lambda^{-1}|\sigma_t\rangle].
\end{aligned} \tag{2.22}$$

This construction is rather reminiscent of the one used to obtain orbifold boundary states to describe a defect line in the Ising model [12] where it was also necessary to add a contribution from the twisted sector when the periodic boson boundary states were invariant under the Z_2 transformation. This is presumably an important element of a general prescription for constructing boundary states for orbifold models.

3. The new boundary condition

The various partition functions involving the new b.c. are given below. Henceforth, to simplify our notation, we will refer to the fixed b.c.'s as A, B and C (corresponding to the three possible states of the Potts variable) the mixed b.c.'s as AB, AC, BC, the free b.c. as 'free' and the new b.c. corresponding to the $|\tilde{22}\rangle$ boundary state as 'new'. (In [1] the notation '$A+B$' was used rather than 'AB'.)

$$\begin{aligned}
&Z_{\text{new},A} = Z_{\text{new},B} = Z_{\text{new},C} = \chi_{22} + \chi_{32} = Z_{\text{free},AB}\\
&Z_{\text{new},AB} = Z_{\text{new},BC} = Z_{\text{new},AC} = \chi_{44} + \chi_{42} + \chi_{22} + \chi_{32}\\
&Z_{\text{new,free}} = \chi_\epsilon + \chi_\sigma + \chi_{\sigma^\dagger} = Z_{AB,A} + Z_{AB,B} + Z_{AB,C}\\
&Z_{\text{new,new}} = \chi_I + \chi_\epsilon + \chi_\sigma + \chi_{\sigma^\dagger} + \chi_\psi + \chi_{\psi^\dagger} = Z_{AB,AB} + Z_{AB,BC} + Z_{AB,AC}.
\end{aligned} \tag{3.1}$$

Several clues to the nature of the new fixed point are provided by these partition functions. The equality of the three partition functions on the first line of equation (3.1) and on the second line strongly suggests that the new b.c. is Z_3 invariant. This is also probably implied by the fact that their is only one new b.c., and not three. In general, the diagonal partition functions, $Z_{\alpha\alpha}$ give the boundary operator content with b.c. α, with the usual relation between the finite-size energies and the scaling dimensions of operators. This in turn gives information about the renormalization group stability of the boundary fixed point. We give all diagonal partition functions below:

$$\begin{aligned}
&Z_{AA} = \chi_I\\
&Z_{AB,AB} = \chi_I + \chi_\epsilon\\
&Z_{\text{free,free}} = \chi_I + \chi_\psi + \chi_{\psi^\dagger}\\
&Z_{\text{new,new}} = \chi_I + \chi_\epsilon + \chi_\sigma + \chi_{\sigma^\dagger} + \chi_\psi + \chi_{\psi^\dagger}.
\end{aligned} \tag{3.2}$$

We see that the fixed boundary fixed point is completely stable. Apart from the identity operator it only contains operators of dimensions $\geqslant 2$. The mixed fixed point has one relevant operator of dimension $\frac{2}{5}$ while the free fixed point has two relevant operators, both of dimension $\frac{2}{3}$. It is easy to see, on physical grounds, what these operators are. Consider adding a boundary 'magnetic field' to the free b.c.:

$$\beta H \to \beta H - {\sum_j}' [h\mathrm{e}^{\mathrm{i}\theta_j} + \text{c.c.}]. \tag{3.3}$$

Here the sum runs over the spins on the boundary only. h is a complex field and c.c. denotes complex conjugate. The two relevant operators at the free fixed point correspond to the real and imaginary parts of h. If we assume that $|h|$ renormalizes to ∞ then it would enforce a fixed b.c. for generic values of $\arg(h)$. For instance, a real positive h picks out $\theta_j = 0$. There are three special directions, $\arg(h) = \pi$, $\pm\pi/3$ for which two of the Potts states remain degenerate. For instance, for h real and negative, $\theta = \pm 2\pi/3$. These values of $\mathrm{Im}(h)$ are invariant under renormalization owing to a Z_2 symmetry. We expect the system to renormalize to the mixed fixed point for these values of $\arg(h)$. $\mathrm{Im}(h)$ corresponds to the single relevant coupling constant at the mixed fixed point with $\mathrm{Im}(h) = 0$. Giving h a small imaginary part at this fixed point will select one of the two Potts states $2\pi/3$ or

$-2\pi/3$, corresponding to a renormalization group flow from mixed to fixed. Since the free fixed point has Z_3 symmetry we can classify the relevant operators by their Z_3 charge. The two operators at the free fixed point, $e^{\pm i\theta_j}$, have charge ± 1, corresponding to ψ and $\psi^\dagger$.

We see from equation (3.2) that there are five relevant operators at the new fixed point. Two with charge 1, two with charge -1 and one with charge 0. The charged operators presumably arise from applying a magnetic field. However, even if we preserve the Z_3 symmetry, there still remains one relevant operator, ϵ of dimension $\frac{2}{5}$. Thus, we might expect the new fixed point to be unstable, even in the presence of Z_3 symmetry, with a renormalization group flow to the free fixed point occurring.

It turns out that there is a simple physical picture of the new b.c. within the quantum Potts chain realization. The corresponding classical model can also be constructed but involves negative Boltzmann weights. Therefore we first discuss the quantum model and turn to the classical model at the end.

We now consider the quantum chain model on a finite interval, $0 \leqslant i \leqslant l$. In order to explore the Z_3 symmetric part of the phase diagram it is convenient to consider the model with a complex transverse field, h_T at the origin and a free b.c. at l:

$$H = -(h_T M_0 + h_T^* M_0^\dagger) - \sum_{i=1}^{l}[(M_i + M_i^\dagger) + (R_i^\dagger R_{i-1} + R_{i-1}^\dagger R_i)]. \tag{3.4}$$

We can effectively map out the phase diagram by considering the duality transformation of equation (1.5). The dual lattice consists of the points $i + \frac{1}{2}$ for $i = 0, 1, \ldots l$. Note that, from equation (1.5):

$$R'_{\frac{1}{2}} \equiv M_0. \tag{3.5}$$

The exactly transformed Hamiltonian is:

$$H = -(h_T R'_{\frac{1}{2}} + h_T^* R'^\dagger_{\frac{1}{2}}) - \sum_{i=0}^{l}(R'^\dagger_{i+\frac{1}{2}} R'_{i-\frac{1}{2}} + \text{h.c.}) - \sum_{i=0}^{l-1} M'_{i+\frac{1}{2}}. \tag{3.6}$$

We have a longitudinal field at site $\frac{1}{2}$, as well as a transverse field. Also note that, at the last site, $l + \frac{1}{2}$ there is no field of either kind.

First consider the case where h_T is real and positive, for example, $h_T = 1$ corresponding to standard free b.c.'s. The dual model has the longitudinal field term at $\frac{1}{2}$:

$$-h_T \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{3.7}$$

which favours the third (C) Potts state. We expect this Hamiltonian to renormalize to the fixed (C) b.c. The spin at site $l + \frac{1}{2}$ cannot flip. We may fix it in the A, B or C state. This corresponds to a sum of three fixed b.c.'s A, B or C. From the dual viewpoint the partition function at low energies is

$$Z_{C,A} + Z_{C,B} + Z_{C,C} = \chi_I + \chi_\psi + \chi_{\psi^\dagger} = Z_{\text{free,free}}. \tag{3.8}$$

This is obviously the correct answer when $h_T = 1$ and is a useful check on duality. It implies that the dual of free is fixed. Now consider the case where h_T is real and negative. The dual model has a longitudinal field which favours states A and B equally. It should flow to the mixed b.c. AB. Thus the partition function is:

$$Z_{AB,A} + Z_{AB,B} + Z_{AB,C} = \chi_\epsilon + \chi_\sigma + \chi_{\sigma^\dagger}. \tag{3.9}$$

We see from equation (3.1) that this is $Z_{\text{new,free}}$. This indicates that we obtain the new b.c. by reversing the sign of the transverse field at the boundary. We see that the dual of mixed is new. This is consistent with $Z_{\text{new,new}}$ in equation (3.1). This new BC is stable provided that h_T is real and negative. There is a discrete symmetry associated with h_T being real, time reversal. Now let us break this symmetry and give h_T a small imaginary part, $h_T \to h_T + \mathrm{i}h'_T$. Note that we have not broken the Z_3 symmetry (in the original formulation). In the dual picture the longitudinal field at site $\frac{1}{2}$ is:

$$\begin{pmatrix} h_T + \sqrt{3}h'_T & 0 & 0 \\ 0 & h_T - \sqrt{3}h'_T & 0 \\ 0 & 0 & -2h_T \end{pmatrix}. \tag{3.10}$$

For $h_T > 0$ and small h'_T the C state is still favoured, but for $h_T < 0$ the h'_T term breaks the degeneracy between A and B. We then obtain a flow from mixed (AB) to fixed (A or B) in the dual picture. In the original formulation we obtain a flow from new to free. In either picture, the flow is driven by an $x = \frac{2}{5}$ boundary operator. This explains the Z_3 symmetric relevant operator at the new fixed point that we were discussing. Importantly there is a different symmetry, time reversal, which forbids it. In the complex h-plane the phase diagram can be easily constructed. There are three completely stable free fixed points (in the original formulation) at equal distances from the origin on the positive real axis and at angles $\pm 2\pi/3$. There are three new fixed points at equal distances from the origin on the negative real axis and at angles $\pm\pi/3$. These are attractive for flows along rays from the origin but repulsive for flows perpendicular to these rays. One can easily connect up these critical points and draw sensible looking flows for the whole complex plane, as shown in figure 1. Although three 'free' fixed points occur in this phase diagram, they all correspond to the same boundary state. In fact, $\arg(h_T)$ can be rotated by $2\pi/3$ by a unitary transformation at site 0 by the matrix R_0. Thus the three finite-size ground states (and all excited states) for h_T at the three 'free' fixed point values, are rigorously identical except for a local change at site 0. The spectra, with any given b.c. at l is the same in all three cases. Clearly all three cases have the same long distance, low-energy properties and should thus be thought of as corresponding to the same fixed point. Similarly all three 'new' fixed points are equivalent. It might, in fact, be more appropriate to draw the new fixed point

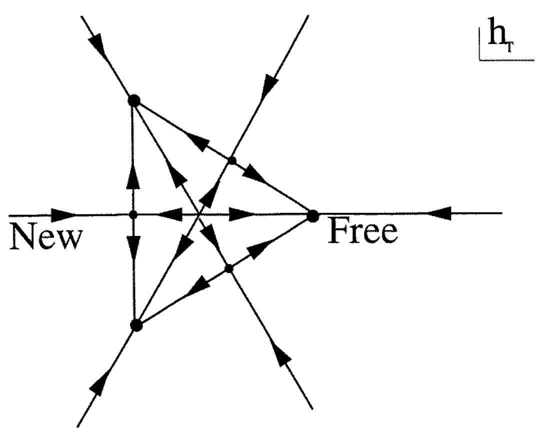

Figure 1. Schematic phase diagram of the quantum chain version of the Potts model with a complex boundary transverse field. Arrows indicate direction of renormalization group flows as the energy scale is decreased.

at $|h_T| = \infty$ rather than at a finite distance from the origin, as in figure 1. This follows since, in the dual picture, we obtain the mixed b.c. by eliminating one of the classical Potts states and hence taking the longitudinal field to ∞. An infinite real negative transverse field eliminates the symmetric state (1,1,1) at the first site and projects onto the two orthogonal states with basis $(1, e^{i2\pi/3}, e^{-i2\pi/3})$, $(1, e^{-i2\pi/3}, e^{i2\pi/3})$.

The origin, $h_T = 0$, corresponds to a sum of A, B and C boundary conditions. We may specify a value for the Potts variable at 0 and it is unchanged by the action of the Hamiltonian. The Hilbert space breaks up into three sectors depending on which value is chosen. One way of checking the consistency of this is the duality transformation. For $h_T = 0$ the dual model still has a transverse field at site $\frac{1}{2}$ but no longitudinal field. Thus it corresponds to a free b.c. However, in the dual model the b.c. at $l + \frac{1}{2}$ is a sum of A, B, and C b.c.'s. Thus we obtain the same partition function from either picture

$$Z_{\mathrm{free},A} + Z_{\mathrm{free},B} + Z_{\mathrm{free},C}.$$

The set of boundary operators at $h_T = 0$, is given by the finite-size spectrum with a sum of A, B and C boundary conditions at each end of the system. This gives the partition function:

$$Z = 3(Z_{A,A} + Z_{A,B} + Z_{A,C}) = 3(\chi_I + \chi_\psi + \chi_{\psi^\dagger}). \tag{3.11}$$

Note that there are three zero-dimensional boundary operators for $h_T = 0$. One is the identity. The other two correspond to a longitudinal field $h_L R_0 + h_L^* R_0^\dagger$. This should pick out one of the three b.c.'s A, B or C (for generic phases of h_L). $\langle R_0 \rangle$ takes on a finite value for infinitesimal h_L corresponding to a 1st order transition. This becomes especially obvious by again using duality but now running the argument backwards. That is, let us now study the dual model with 0 transverse field and a small non-zero longitudinal field, h_L. This corresponds to the original model with a transverse field $h_L M_0 + \mathrm{h.c.}$ but zero classical Potts interaction $R_0^\dagger R_1 + \mathrm{h.c.}$ Clearly, h_L produces a first-order transition in this model since the first site is exactly decoupled. We can diagonalize M_0 and 1 or the other of the three eigenstates will be the ground state depending on the phase of h_L (for generic values of this phase). In the dual model this corresponds to first-order transitions between eigenstates of $R_{\frac{1}{2}}$ when a longitudinal field is turned on (with a non-zero classical Potts interaction of order 1). It is also clear that there are special values for the phase of h_L for which two ground states remain degenerate so another first-order transition occurs, across the negative real h_L axis (and the two other axes rotated by $\pm 2\pi/3$).

From equation (3.11), there are six relevant boundary operators of dimension $\frac{2}{3}$ at the $h_T = 0$ fixed point. We may identify these with these with the six tunneling processes $A \to B$, $B \to A$, etc. Imposing Z_3 symmetry, no dimension 0 operators and only two-dimension $\frac{2}{3}$ operators are allowed. The latter couple to the complex transverse field, h_T. Thus we see that the flow away from $h_T = 0$ to the new or free fixed points is driven by $x = \frac{2}{3}$ operators.

Further insight into the nature of the new fixed point can be gained by considering again the model with no classical Potts interaction between sites 0 and 1, $h_T \neq 0$ and no longitudinal field. Thus we have a Potts chain with a free b.c. at one and an additional decoupled Potts spin at 0. For real positive h_T, the decoupled Potts spin has a unique symmetric ground state, (1,1,1). In this case, we expect that turning on the classical Potts interaction with the first site leads to the free fixed point. The end spin is simply adsorbed, with a flow from free to free. On the other hand, for real and negative h_T, the ground state of the decoupled spin at 0 is two-fold degenerate. These two states can be chosen to be $(1, e^{2\pi i/3}, e^{-2\pi i/3})$ and $(1, e^{-2\pi i/3}, e^{2\pi i/3})$. Turning on the classical Potts interaction should now produce a flow to the new fixed point from the above discussion. Thus we obtain a flow from a free b.c. with a decoupled system with a two-fold degeneracy, to the new b.c. This is somewhat like the renormalization group flow in the $S = \frac{1}{2}$ Kondo problem, with the two states of the decoupled spin in the Potts model corresponding to spin up or down in the Kondo model. The flow to the new fixed point is analogous to Kondo-screening of the impurity. A related problem, an impurity with triangular symmetry coupled to conduction electrons, was discussed in [13]. The two-fold degeneracy of the ground states of the impurity is guaranteed by the Z_3 symmetry (for the appropriate sign of the tunneling term) and can lead to two-channel Kondo behaviour (when electron spin is taken into account) without the fine-tuning necessary for ordinary two-level impurities. We note that both of these problems correspond to a Z_3 symmetric impurity coupled to a dissipative environment. In [13] this environment is the conduction electrons; in our model it is the rest of the Potts chain.

The dual version of this last renormalization group flow is easily constructed. At site $\frac{1}{2}$ there is originally a longitudinal field but no transverse field. Thus the system is in a sum of two states, $A + B$. Upon turning on the transverse field we expect a flow to the mixed state AB.

Now let us consider the classical Potts model of equation (1.1). We may again construct the new boundary fixed point using duality. The first step is to Fourier transform the factor associated with each link, ij in the partition sum. Thus we introduce a new angular variable, ϕ_{ij}, taking on values 0, $\pm 2\pi/3$ associated with the link ij by:

$$\mathrm{e}^{J\cos(\theta_i-\theta_j)} = \sum_{\phi_{ij}} \mathrm{e}^{\mathrm{i}3\phi_{ij}(\theta_i-\theta_j)/2\pi} A \mathrm{e}^{K\cos(\phi_{ij})} \tag{3.12}$$

where A is a normalization constant. We now sum over the original Potts variables, θ_i. Ignoring the boundaries, for the moment, the sum over the Potts variable at each site gives a constraint on the four link variables associated with the links terminating at the site. (See figure 2.)

$$\sum_{\pm}(\phi_{i,i\pm\hat{x}} + \phi_{i,i\pm\hat{y}}) = 0 \qquad (\mathrm{mod}\, 2\pi) \tag{3.13}$$

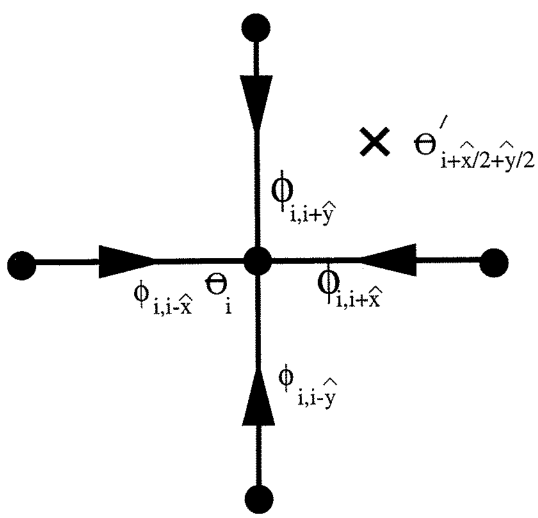

Figure 2. Site, link and dual lattice variables.

where $\phi_{ji} \equiv -\phi_{ij}$. We may solve these constraints by introducing new angular variables, θ'_i (also restricted to the values $0, \pm 2\pi/3$) on the dual lattice, i.e. the centres of the squares of the original lattice (see figure 2). Explicitly:

$$\begin{aligned}
\phi_{i,i+\hat{y}} &= \theta'_{i+\hat{x}/2+\hat{y}/2} - \theta'_{i-\hat{x}/2+\hat{y}/2} \\
\phi_{i,i+\hat{x}} &= \theta'_{i+\hat{x}/2-\hat{y}/2} - \theta'_{i+\hat{x}/2+\hat{y}/2}.
\end{aligned} \tag{3.14}$$

The partition function is transformed into:

$$Z \propto \prod_i \sum_{\theta'_i} \mathrm{e}^{\sum_{\langle i,j\rangle} K \cos(\theta'_i - \theta'_j)}. \tag{3.15}$$

Thus we retreive the original Potts model with a dual coupling constant, K. The critical coupling is given by the self-duality condition, $J = K$, which gives:

$$J_c = \tfrac{2}{3} \ln(1 + \sqrt{3}). \tag{3.16}$$

Now consider the system with a free boundary along the x-axis with a boundary Potts interaction J_B (and no fields at the boundary). Consider summing over the Potts variable θ_i at the boundary, as indicated in figure 3. This gives the constraint:

$$\phi_{i,i+\hat{y}} + \phi_{i,i+\hat{x}} + \phi_{i,i-\hat{x}} = 0. \tag{3.17}$$

Writing $\phi_{i,i+\hat{y}}$ in terms of the dual variables, this becomes:

$$\theta'_{i-\hat{x}/2+\hat{y}/2} - \theta'_{i+\hat{x}/2+\hat{y}/2} + \phi_{i,i+\hat{x}} + \phi_{i,i-\hat{x}} = 0. \tag{3.18}$$

We may solve this equation for all sites, i, along the boundary by:

$$\phi_{i,i+\hat{x}} = \theta'_{i+\hat{x}/2+\hat{y}/2}. \tag{3.19}$$

Thus the edge of the dual lattice is at $y = \frac{1}{2}$. In addition to the bulk Potts interaction of strength K, given by equation (3.12), there is an additional classical boundary term in the dual Hamiltonian:

$$-\beta H_{\mathrm{field}} = h \sum_j \cos \theta'_{j,0}, \tag{3.20}$$

with the dual boundary field, h, determined by the boundary interaction:

$$\mathrm{e}^{J_B \cos(\theta_i - \theta_{i+\hat{x}})} = \sum_{\phi_{i,i+\hat{x}}} \mathrm{e}^{\mathrm{i}3\phi_{i,i+\hat{x}}(\theta_i - \theta_{i+\hat{x}})/2\pi} C \mathrm{e}^{h \cos(\phi_{i,i+\hat{x}})} \tag{3.21}$$

for some constant, C. This gives the condition:

$$\mathrm{e}^{3J_B/2} = \frac{\mathrm{e}^h + 2\mathrm{e}^{-h/2}}{\mathrm{e}^h - \mathrm{e}^{-h/2}}. \tag{3.22}$$

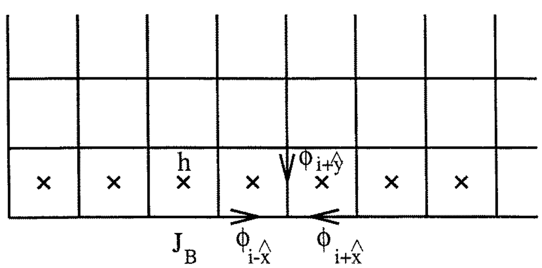

Figure 3. Boundary variables.

This equation has the annoying feature that, for real J_B and h, there are only solutions for h and $J_B\rangle 0$, with h running from ∞ to 0 as J_B runs from 0 to ∞. From our analysis of the quantum model we expect that the new critical point occurs when the dual model has a real negative h. This requires a complex J_B, $\mathrm{Im}(J_B) = 2\pi/3$. Noting that the ratio of Boltzmann weights for $\theta_i - \theta_{i+\hat{x}} = 0$ or $\pm 2\pi/3$ is $\mathrm{e}^{3J_B/2}$, we see that this implies real but negative Boltzmann weights. In particular, we may regard the new fixed point as corresponding to an infinite negative h; this corresponds to

$$\mathrm{e}^{3J_B/2} = -2. \tag{3.23}$$

In the quantum model, discussed above, this limit eliminates the symmetric state (1,1,1) on the first site, projecting onto the two orthogonal states. The same projection is realized in the standard transfer matrix formalism for the classical Potts model. The Potts model with negative Boltzmann weights in the bulk occurs quite naturally in the cluster formulation based on the high-temperature expansion [14].

We note that the values of the 'ground state degeneracies' of the various fixed points, $\langle\alpha|0,0\rangle$ are given by:

$$g_A = N \qquad g_{AB} = N\lambda^2 \qquad g_{\mathrm{free}} = N\sqrt{3} \qquad g_{\mathrm{new}} = N\sqrt{3}\lambda^2. \tag{3.24}$$

Noting that

$$1 < \lambda^2 = \frac{1+\sqrt{5}}{2} < \sqrt{3} \tag{3.25}$$

we see that:

$$g_A < g_{AB} < g_{\mathrm{free}} < g_{\mathrm{new}} < 3g_A < 2g_{\mathrm{free}}. \tag{3.26}$$

Thus all renormalization group flows that we have discussed are consistent with the 'g-theorem' [15] (or 'g-conjecture' as it is more accurately referred to). g always decreases under an renormalization group flow. We also note that the various flows which are related by duality have the same ratios of g-factors:

$$\begin{aligned}
\frac{g_{\mathrm{new}}}{g_{\mathrm{free}}} &= \frac{g_{AB}}{g_A} = \lambda^2 \\
\frac{3g_A}{g_{\mathrm{free}}} &= \frac{g_{\mathrm{free}}}{g_A} = \sqrt{3} \\
\frac{3g_A}{g_{\mathrm{new}}} &= \frac{g_{\mathrm{free}}}{g_{AB}} = \frac{\sqrt{3}}{\lambda^2} \\
\frac{2g_{\mathrm{free}}}{g_{\mathrm{new}}} &= \frac{2g_A}{g_{AB}} = \frac{2}{\lambda^2}.
\end{aligned} \tag{3.27}$$

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Appendix. g-theorem for fusion

At present, the most general and systematic method used to construct a new boundary state is fusion. For rational CFTs (with a finite number of conformal towers), fusion is quite a powerful method. Empirically it has been recognized that fusion is a kind of irreversible process: when a boundary state B is obtained by fusion from another boundary state A, fusion on B does not generally give A. (Although sometimes it does.) This irreversibility reminds us of the 'g-theorem' which states that the ground-state degeneracy g of the system always decreases along the boundary renormalization group flow [15]. Actually, here we prove that the irreversibility of fusion is also related to the ground-state degeneracy g. Amusingly, the 'direction' is opposite to that of the renormalization group flow. We state the following.

Theorem. We consider a unitary rational CFT. Let B be a boundary state obtained by fusion from the boundary state A. The ground-state degeneracy of B is always greater than or equal to that of A.

Proof. To prove the theorem, first let us give the definition of the ground-state degeneracy. Given a boundary state $|X\rangle$, the ground-state degeneracy of the state g_X is given by the following

$$g_X = \langle 0|X\rangle \tag{A.1}$$

where $|0\rangle$ is the ground state of the system and we choose the overall phase of $|X\rangle$ so that g_X is positive. For unitary CFTs, the ground state corresponds to the identity operator with conformal weight 0. We primarily denote this identity as 0. The definition of equation (A.1) follows from the fact that the partition function is proportional to this matrix element in the limit of an infinite length system.

On the other hand, a general relation for fusion [1, 15] reads:

$$\langle a|B\rangle = \langle a|A\rangle \frac{S_c^a}{S_0^a} \tag{A.2}$$

where a represents an aribitrary primary field, c is the primary used for fusion from A to B, and S_y^x is the modular S-matrix element for primaries x and y. The special case $a = 0$ (identity) gives the relation between the degeneracies:

$$\frac{g_B}{g_A} = \frac{S_c^0}{S_0^0}. \tag{A.3}$$

Now we employ the Verlinde formula [8, 1]:

$$\sum_b S_b^a N_{cd}^b = \frac{S_c^a S_d^a}{S_0^a}. \tag{A.4}$$

The special case $a = 0$ and $d = c'$ (c' is the conjugate of c), combined with equation (A.3) gives

$$\begin{aligned} \left|\frac{g_B}{g_A}\right|^2 &= \frac{1}{S_0^0} \sum_b S_b^0 N_{c'c}^b \\ &= 1 + \sum_{b\neq 0} \frac{S_b^0}{S_0^0} N_{c'c}^b \end{aligned} \tag{A.5}$$

where we used the fact that the operator product expansion between c and its conjugate c' always contains the identity operator.

Since the fusion rule coefficients N^b_{cc} are non-negative integers, the theorem follows if $S^0_b/S^0_0 > 0$. Actually, it is known that $S^0_b > 0$ for any primary b, proved as follows [10]. Consider the character $\chi_b(\tilde{q})$. By modular transformation,

$$\chi_b(\tilde{q}) = \sum_e S^e_b \chi_e(q). \tag{A.6}$$

When evaluating the limit $q \to 0$, the right-hand side is dominated by the lowest power of q. Thus $\chi_b(\tilde{q}) \sim S^0_b q^{-c/24}$. (Here c in the exponent is the central charge of the CFT.) Since the left-hand side and $q^{-c/24}$ are both positive, $S^0_b > 0$. Thus the theorem is proved. □

Our theorem is of course consistent with all known cases, including boundary states of the Ising and Potts models. When there is a renormalization group flow between two boundary states, our theorem implies that the direction of the fusion rule construction is opposite. Namely, we can obtain an unstable boundary state from a more stable boundary state, but the reverse is not possible. However, there can be an exception: if there are some extra degrees of freedom, the renormalization group flow can be in the same direction as the fusion. An example of this is the Kondo effect; the screened state, which has larger ground-state degeneracy than the original state, is constructed by fusion. However, if we take the degeneracy owing to the impurity spin into account, the total degeneracy is smaller in the screened state. Thus the renormalization group flow occurs from the unscreened to screened state.

That fusion generates rather opposite 'flow' to the renormalization group one which makes it somewhat difficult to understand the physical meaning of fusion, which is a more or less abstract mathematical manipulation. Perhaps the best intuition is gained again from the example of the Kondo effect. Namely, fusion roughly corresponds to an absorption of some degree of freedom by the boundary. Considering the generality of this result, it is tempting to imagine some deeper connection with the 'g-theorem' on the renormalization group flow.

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