Emergent Conformal Symmetry of Quantum Hall States on Singular surfaces

M. Laskin,[1] Y. H. Chiu,[1] T. Can,[2] and P. Wiegmann[1]

[1] *Kadanoff Center for Theoretical Physics, University of Chicago, 5640 South Ellis Ave, Chicago, IL 60637, USA*

[2] *Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794, USA*

(Dated: February 29, 2016)

We show that quantum Hall states on surfaces with conical singularities behave as conformal primaries near the singular points, with a conformal dimension controlled by the gravitational anomaly. We show that the electronic fluid at the cone tip possesses an intrinsic angular momentum equal to the conformal dimension, in units of the Planck constant. Finally, we argue that the gravitational anomaly also controls the fine structure of electronic density at the tip, and the exchange statistics of cones in the Laughlin states, arising from adiabatically braiding conical singularities. Thus, the gravitational anomaly, which appears as a finite size correction on smooth surfaces, dominates geometric transport on singular surfaces.

PACS numbers: 73.43.Cd, 73.43.Lp, 73.43.-f, 02.40.-k, 11.25.Hf

Introduction Early developments in the theory of the quantum Hall effect [1–4] and the recent resurgence [5–14] point to geometric response as a fundamental probe of quantum liquids with topological characterization, complementary to the more familiar electromagnetic response. Such liquids exhibit non-dissipative transport in response to variations of spatial geometry, controlled by quantized transport coefficients. This *geometric transport* is distinct from the transport caused by electromotive forces and geometric transport coefficients characterize the state independently from electromagnetic response.

Surfaces with a singular geometry, such as isolated conical singularities, or disclination defects, highlight the geometric properties of the state. For this reason, they serve as an ideal setting to probe the geometry of QH states. In this paper, we demonstrate this by examining Laughlin states on a singular surface. We compare spatial curvature singularities to magnetic ones (flux tubes), and emphasize the difference. While the QH state imbues both types of singularities with local structure such as charge, spin, and statistics, *only* the curvature singularities reflect the geometric transport.

The gravitational anomaly is central to understanding the geometry of topological states [8–15]. This effect encodes the geometric characterization of such states, and is often referred to as the *central charge*.

On a smooth surface the gravitational anomaly is a sub-leading effect. For example, the central charge, c_H, is a finite size correction to the non-dissipative viscosity η^A introduced in [1]. In [10, 13, 14] it is shown that

$$2\pi\eta^A = \varsigma_H\, e\overline{B} - \hbar\frac{c_H}{48}\overline{R}, \tag{1}$$

where $\overline{B} = \frac{1}{V}\int BdV$ and $\overline{R} = \frac{1}{V}\int RdV$ are mean magnetic field and curvature of a patch of the fluid with the volume V. We take eB to be positive throughout the paper, but do not fix the sign of the charge e. For the j-spin Laughlin states (see [8], and (16) below for the definition of spin) the transport coefficient ς_H and the 'central charge' c_H were found to be

$$c_H = 1 - \frac{3}{\nu}(1-2j\nu)^2, \quad \varsigma_H = \frac{1}{4}(1-2j\nu), \tag{2}$$

where ν is the filling fraction. The last term in (1) represents the gravitational anomaly.

On smooth surfaces, bulk geometric transport is hard to detect, since c_H enters as a small higher order correction. On a flat surface or a torus, where the Euler characteristic is zero, geometric transport vanishes altogether as seen from (1). To observe c_H as a global transport coefficient, we have to study QH states on higher genus surfaces with at least two handles.

We want to identify a setting where the gravitational anomaly c_H is the dominant property, as opposed to being a finite-size correction overshadowed by a larger electromagnetic contribution. We demonstrate that a surface with conical singularities provides this setting, and brings geometric transport to the fore.

One implication of the transport coefficients (2) is that they determine the angular momentum of a parcel of the electronic fluid [5]

$$\mathrm{L} = -2V\eta^A. \tag{3}$$

On a smooth surface, the angular momentum is an *extensive* property. It scales with the volume of the patch.

We show that the fluid parcel near the singularity spins with an *intensive* angular momentum proportional to c_H, independent of the parcel volume. Moreover, near the singularity, the state is a conformal primary. Its conformal dimension equals to the angular momentum in units of $\hbar$.

Singularities elucidate the uneasy relation between QH-states and conformal field theory. In general, QH-states do not possess conformal symmetry. They feature a scale - the magnetic length. As a result, physical observables do not transform conformally. However, the states appear to be conformal in the vicinity of a singularity. Conical singularities are not as exotic as they may seem, and occur naturally in several experimental settings. Disclination defects in a regular lattice can be described by metrics with conical singularities [18], and occur generically in graphene [19]. In a recent photonic experiment, Landau levels on a cone were designed in an optical resonator [20].

A conical singularity of the order $\alpha < 1$ is an isolated point ξ_0 on the surface with a concentration of curvature

$$R(\xi) = R_0 + 4\pi\alpha\delta(\xi - \xi_0), \tag{4}$$

where R_0 is the background curvature, a smooth function describing the curvature away from the singularity. Locally, if $\alpha > 0$ the singularity is equivalent to an embedded cone with the apex angle $2\arcsin\gamma$, where $2\pi\gamma = 2\pi(1-\alpha)$ is the cone angle (see the Figure). If $\alpha < 0$, the singularity is a branch point of a multi-sheeted Riemann surface. Examples of genus-zero surfaces with constant curvature and conical singularities include: $R_0 > 0$ - an 'american football' with two antipodal conical singularities [16], $R_0 = 0$ - a polyhedron [17, 21], $R_0 < 0$ - a pseudo-sphere (see e.g. [22] and references therein). When γ or $1/\gamma$ is an integer, the surface is also an orbifold, a surface quotiented by a discrete group of automorphisms. Then conical singularities are fixed points of the group action [21].

Conical singularities affect QH states differently than magnetic singularities

$$eB(\xi) = eB_0 - 2\pi\hbar\, a\delta(\xi - \xi_0), \tag{5}$$

To emphasize the difference between geometric and magnetic singularities we consider both simultaneously: a magnetic flux a threaded through the conical singularity α.

Lastly, before discussing our main results, we comment on the inclusion of spin j. As discussed in [9, 11, 14] Laughlin states are characterized not only by the filling fraction but also by the spin. Spin does not enter electromagnetic transport. Nor does it enter local bulk correlation functions, such as the structure factor, either. The spin enters the geometric transport as seen in (2).

To the best of our knowledge, there is no experimental or numerical evidence that determines the spin in QH materials [38], nor are there any arguments that $j = 0$, as it silently assumed in earlier papers. For this reason, we keep spin as a parameter. It affects the physics of the QHE. For example, at the filling $\nu = 1/3$, the central charge vanishes at $j = 1$, and $j = 2$. The central charge equal 1 if $j = \frac{1}{2\nu}$, and equal -2 if $\nu = 1$ and $j = 0$ or 1.

Main results *a. Conformal dimensions.* In [8, 11] it was shown that the magnetic singularity (5) is the conformal primary with the dimension

$$h_a = \frac{1}{2}a(4\varsigma_H - \nu a). \tag{6}$$

In this paper we extend this result and show that the geometric singularity is also conformal primary. In this case its dimension is controlled solely the gravitational anomaly

$$\Delta_\alpha = \frac{c_H}{24}(\gamma^{-1} - \gamma), \quad \gamma = 1 - \alpha. \tag{7}$$

Formula (7) is familiar in conformal field theory: $-\Delta_\alpha$ (mind the opposite sign!) is the dimension of a vertex operator of a conical point in conformal field theory [25, 26]. On singular surfaces, the dimension also appears as a finite-size correction to the free energy [27] and the spectral determinant of the Laplacian [28, 29]. The reason is that near a conical singularity, QH-states and conformal field theory share the same mathematics, but not identical. The conformal dimension of QH states has the opposite sign than that in a conformal field theory with the central charge given by (2).

b. Gyration around the cone We show that the dimension determines transport near the singularity. Near the apex, a small piece of the fluid gyrates around the apex, while the fluid in the bulk does not. The angular momentum of this gyration is an intensive property. It does not depend on the volume of the neighbourhood. We will show that the angular momentum is exactly the dimension (7)

$$\mathrm{L}_\alpha = \hbar\Delta_\alpha. \tag{8}$$

This quantity must be added to Eq.(3). The formulae (7,8) bring us to a conjecture about the value of the total angular momentum on a surface with few conical singularities, thus generalizing (1,3)(cf.[28, 29])

$$\hbar^{-1}\mathrm{L} = -2\varsigma_H N_\Phi + \frac{c_H}{12}\left(\chi + \frac{1}{2}\sum_k\left(\sqrt{\gamma_k} - \frac{1}{\sqrt{\gamma_k}}\right)^2\right),$$

where N_Φ is the total magnetic field in units of the flux quantum $2\pi\hbar/e$. The last term results from the integrated Weyl anomaly and is equivalent to the spectral determinant of the Laplacian. The formula for the spectral determinant obtained for polyhedra in [28, 29]. There it is equivalent to $\sum_k \Delta_{\alpha_k}$ upon using the identity $\chi = \sum_k \alpha_k$.

A formula similar to (8) holds for the angular momentum of combined magnetic and geometric singularities

$$\mathrm{L}_{\alpha,a} = \hbar\left(\frac{1}{\gamma}h_a + \Delta_\alpha\right). \tag{9}$$

c. Braiding singularities Just like Laughlin quasi-holes (which are closely related to flux tubes (5)) , conical singularities can be braided. The phase acquired by adiabatically exchanging two singularities is called the exchange statistics. Braiding two quasi-holes with charges a_1 and a_2 yields the phase $\Phi_{12} = \pi(\nu a_1 a_2)$. This result is known since early days of QHE [30].

Braiding conical singularities is more involved. We argue that the exchange statistics of braiding of two cones of the order α_1 and α_2 are determined exclusively by the central charge

$$\Phi_{12} = \pi\frac{c_H}{24}\alpha_1\alpha_2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right) = \\ -\pi\left(\alpha_2\Delta_{\alpha_1}+\alpha_1\Delta_{\alpha_2}\right) - \pi\frac{c_H}{12}\alpha_1\alpha_2. \qquad (10)$$

Here, we assume that the path is sufficiently small, so conical singularities are the only contributions to the encompassed solid angle. The first two terms in (10) are the phase acquired by a particle with angular momentum Δ_{α_1} (Δ_{α_2}) going half way around a solid angle $4\pi\alpha_2$(α_1). The last term $\frac{c_H}{12}\alpha_1\alpha_2$ is the exchange statistics. On an orbifold, where either γ or $1/\gamma$ is an integer, the phase for identical cones is $\Phi_{12} = \pi\frac{c_H}{12}\left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)^2$. It appears rational, even in the case of the integer QHE.

The formulae (7-10) are our main results: the braiding statistics of the singularities and the angular momentum of the electronic fluid around a cone are given solely by the gravitational anomaly. Other results such as the transport and the fine structure of the density profile at the singularity are shown below.

d. Moment of inertia The conformal dimension can be also read-off from the fine structure of the density profile in the neighborhood of the singularity. On a singular surface the density changes abruptly on the scale of magnetic length and in the limit of vanishing magnetic length is a singular function. It is properly characterized by the moments

$$m_{2n} = \int (r^2/2l^2)^n(\rho(r)-\rho_\infty)dV. \qquad (11)$$

Here, $\rho_\infty = \nu(e/h)B$ is the asymptotic value of the density away from the singularity and $l = \sqrt{\hbar/(eB)}$. In the integral (11) r is the Euclidean distance to the singularity and $dV = 2\pi\gamma r dr$ is the volume element.

The first moment, the 'charge' m_0, follows from the generalized Středa formula – a number of particles in a patch of the surface is saturated by

$$\bar\rho = \nu(eB/h) + (\varsigma_H/2\pi)R. \qquad (12)$$

We will obtain this relation in the next section. Hence

$$m_0 = \int(\bar\rho-\rho_\infty)\,dV = -\nu a + 2\varsigma_H\alpha. \qquad (13)$$

Eq. (13) says that if $\varsigma_H > 0$, the apex accumulates electrons when $\alpha > 0$. It is an alternative definition of the transport coefficient ς_H.

This result for $j = 0$ is well known (see, e.g., [10, 31, 32]). Recently the charge of the cone has been observed experimentally [20]. However, the gravitational anomaly does not enter here. It emerges in the next moment, the *moment of inertia* of the gyrating parcel m_2. We will see that

$$m_2 = (1-j)m_0 + \gamma^{-1}h_a + \Delta_\alpha, \qquad (14)$$

where h_a and Δ_α are the dimension (6,7). This result can be checked against the integer QH effect, $\nu = 1$, where all the moments are computed in the end of the paper.

This relation between the moment of inertia (14) and the angular moment (8) is not surprising. In QH state the positions of particles determine their velocity. Consequently, the density determines the angular momentum L of the flow. In the case of the Laughlin state this relation reads

$$\mathrm{L} = (eB)\int\frac{r^2}{2}(\rho-\bar\rho)dV + \hbar(j-1)\int\rho dV. \qquad (15)$$

This relation can be extracted from [8, 9]. In the next section we recall its origin. Interpreting this formula we notice that the first term is the diamagnetic effect of fluid gyrating in magnetic field the second term is the paramagnetic contribution. The subtracted term with $\bar\rho$ nulls the orbital moment in the a surface with a constant curvature and a constant magnetic field.

The integral (15) over the bulk of the surface gives the extensive part (3), while the integral over a patch at the singularity is $m_2-(1-j)m_0$. Then (14) yields (9). It remains to compute (6,7).

e. Transport at the singularity. Since the work of Laughlin [35] it was known that an adiabatic change of the magnetic flux $a(t)$ in (5) threading through the puncture of a disk causes a radial electric current flowing outward $\mathrm{I} = -\nu e\dot a$.

Adiabatically evolving the order of the conical singularity $\alpha(t)$ also induces a current. It follows from (13) that the current flowing away from the apex $\mathrm{I} = e\dot m_0$ is $\mathrm{I} = 2e\varsigma_H\dot\alpha$. More interestingly, evolving the cone angle accelerates the gyration of the fluid, which produces torque near the singularity. The torque is the the rate of change of the angular momentum $\mathrm{M} = \dot{\mathrm{L}}$. From (8) it then follows that the torque is proportional to the rate of change of the conformal dimension. We collect the formulae for electric and geometric transport

$$\begin{aligned} &\text{e-transport: current} = -e\nu\dot a, \quad &&\text{torque} = \hbar\dot h_a, \\ &\text{g-transport: current} = 2e\varsigma_H\dot\alpha, \quad &&\text{torque} = \hbar\dot\Delta_\alpha. \end{aligned}$$

In the remaining part of the paper we obtain the dimensions (6,7) and the statistics (10) by employing the conformal Ward identity, a framework developed in [8, 36]. It would be instructive to obtain the results by two complimentary methods of [13] and [11] based on the field theory.

QH-states on a Riemann surface Before turning to singular surfaces, we recall some major facts about Laughlin states on a Riemann surface [7, 8].

The most compact form of the state appears in locally chosen complex coordinates $(z,\bar z)$, where the metric is conformal $ds^2 = e^\phi|dz|^2$. In these coordinates the Laplace-Beltrami operator is $\Delta = 4e^{-\phi}\partial_z\partial_{\bar z}$ and the volume form is $dV = e^\phi d^2z$.

In the conformal metric we can always choose coordinates such that, the unnormalized state reads

$$\Psi_L = \prod_{1\le i<j}^{N} (z_i - z_j)^{\beta} \exp \sum_{i=1}^{N} \frac{1}{2} \left[Q(z_i, \bar z_i) - j\phi(z_i, \bar z_i)\right] \quad (16)$$

where, the integer $\beta = \nu^{-1}$ is the inverse filling fraction and Q is the magnetic potential defined by $-\hbar\Delta Q = 2eB$.

While the wave function (16) explicitly depends on the choice of coordinates, the normalization factor

$$\mathcal{Z}[Q, \phi] = \int |\Psi_L|^2 \prod_i \exp\left[\phi(z_i, \bar z_i)\right] d^2 z_i \quad (17)$$

does not. It is an invariant functional depending on the geometry of the surface, and in particular on the positions and orders of singularities.

Eq. (17) is referred to as the generating functional as it encodes the correlations and the transport properties of the state. A variation of the generating functional over the magnetic potential Q at a fixed conformal factor ϕ is the particle density

$$\rho = e^{-\phi} \delta \log \mathcal{Z} / \delta Q$$

In [14] it was shown that the variation over the metric at a fixed magnetic field gives the angular momentum

$$\mathrm{L} = -\hbar \int \frac{\delta \log \mathcal{Z}}{\delta \phi} d^2 z - \overline{\mathrm{L}} \quad (18)$$

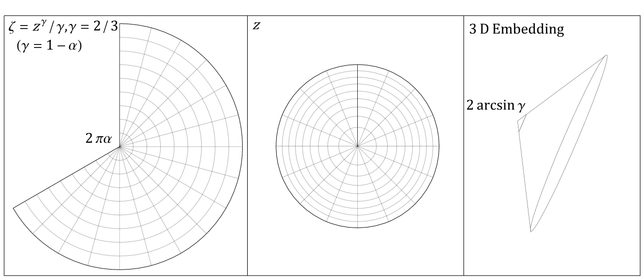
Subtracting $\overline{\mathrm{L}} = (eB) \int \frac{r^2}{2} \bar\rho dV$ assures the vanishing of L on surfaces with constant curvature and magnetic field.

Now we can obtain the relation (15). It follows from the observation that the magnetic potential and the conformal factor appear in (16,17) almost in equal footing, except that the variation over the conformal factor is taken at a fixed magnetic field. Under this condition the magnetic potential also varies $-\hbar\Delta\delta Q = 2\delta\phi(eB)$.

QH-state on a cone A surface has a conical singularity of order $-\alpha\,(\alpha < 1)$ if in the neighborhood of the conical point z_0 the conformal factor behaves as

$$\phi \sim -\alpha \log |z - z_0|^2. \quad (19)$$

Locally a cone is thought as a wedge of a plane with the deficit angle $2\pi\alpha$, whose sides are isometrically glued together (see the figure).



Let us set the apex of the cone at the origin and denote the complex coordinate on the plane as ξ and the cone angle $2\pi\gamma = 2\pi(1-\alpha)$. The wedge is a domain $0 \le \arg\xi < 2\pi\gamma$ with a Euclidean metric $ds^2 = |d\xi|^2$. A pullback of a singular conformal map

$$z \to \xi(z) = z^{\gamma}/\gamma \quad (20)$$

maps the wedge to a punctured disk. The map introduces the complex coordinates $(z, \bar z)$ where the metric is conformal

$$ds^2 = |z|^{-2\alpha} |dz|^2. \quad (21)$$

The quantum mechanics on the cone assumes the 'wedge-periodic' condition. The lowest Landau level on a cone is spanned by the holomorphic polynomials of z (see (34)) in the metric (21).

Eq. (16) is valid on any genus-zero surface. Specifically, in the neighborhood of the conical singularity the the conformal factor in (16) behaves as (19).

A singularity can be interpreted as an insertion of the 'vertex operator' at the marked point of the surface. Then the generating functional $\mathcal{Z}_\alpha$ is the expectation value of this operator. We will show that this operator is a conformal primary. This means that under a dilatation, the functional transforms as $-\delta \log \mathcal{Z}_\alpha = \Delta_\alpha \delta\phi$, where Δ_α is the conformal dimension. Eq. (18) identifies the conformal dimension with the angular momentum (8). We compute it in the remaining part of the paper.

Conformal Ward identity Moments of the density and the angular momentum are computed via the Ward identity. The Ward identity reflects the invariance of the integral (17) under the infinitesimal holomorphic change of variables $z_i \to z_i + \epsilon/(z - z_i)$. It claims that the function of coordinates z_i and a complex parameter z

$$\sum_i \frac{\partial_{z_i} Q + (1-j)\partial_{z_i}\phi}{z - z_i} + \frac{\beta}{2} \left(\sum_i \frac{1}{z - z_i}\right)^2 + \sum_i \frac{1 - \frac{\beta}{2}}{(z - z_i)^2}$$

vanishes under averaging over the state .

Ward identity in terms of the 'Bose' field

$$\varphi = -\beta \sum_i \log |z - z_i|^2 - Q. \quad (22)$$

Then the identity yields a form that resembles the Ward identity in the context of conformal field theory

$$\frac{1}{\hbar} \int \frac{\mathrm{i}P_{z'} - \frac{\varsigma_H}{\pi} \partial_{z'}(eB)}{z - z'} dV_{z'} = T. \quad (23)$$

Here P_z and T are holomorphic components of the momentum of the fluid and the conformal 'stress tensor,'

$$\mathrm{i}P_z = \frac{\nu}{2\pi}(eB)\partial_z(\langle\varphi\rangle - \bar\varphi) + (1-j)\hbar\partial_z\rho, \quad (24)$$

$$T = \frac{\nu}{2}\langle(\partial_z\varphi)^2\rangle - 2\varsigma_H\langle\partial_z^2\varphi\rangle, \quad (25)$$

and we denoted $-\nu\Delta\bar{\varphi} = 4\pi(\bar{\rho} - \nu eB/2\pi\hbar)$.

The first moment m_0 follows from the differential form of the Ward identity (23), obtained by acting on both sides by $e^{-\phi}\partial_{\bar{z}}$. This allows one to see that away from singularities, $\oint P_z dz = \int(\nabla\times P)dV = 0$. Then the identity $\int(\nabla\times P)dV = -eB\int(\rho-\bar{\rho})\,dV$ yields (13). This is closely related to the translation invariance $z_i \to z_i + \epsilon$ of the integrand (17).

The angular momentum L follows from dilatation invariance $z_i \to \lambda^{-1/2} z_i$ of the Ward identity. Multiplying (23) by $\frac{zdz}{2\pi i}$ and integrating over the boundary of a singular patch yields $\hbar^{-1}\int \mathrm{Im}(zP)dV = \mathrm{res}(z\,T)$. The LHS of this equation is proportional to the angular momentum since $\int \mathrm{Im}(zP)dV = -\gamma\int r\times P dV = -\gamma L$, so that

$$L = \frac{\hbar}{\gamma}\,\mathrm{res}(zT). \qquad (26)$$

Gravitational Anomaly The source of the gravitational anomaly is the two-point function $\langle(\partial_z\varphi)^2\rangle$ in the stress tensor T. Evaluated at coincident points, the connected part of the two-point function $T^A = \frac{\nu}{2}\langle(\partial_z\varphi)^2\rangle_c = \frac{\nu}{2}\left[\langle(\partial_z\varphi)^2\rangle - \langle\partial_z\varphi\rangle^2\right]$ must be regularized. In [8] it was shown that

$$T^A = \frac{1}{12}\mathcal{S}[\phi], \quad \mathcal{S}[\phi] = -\frac{1}{2}(\partial_z\phi)^2 + \partial_z^2\phi. \qquad (27)$$

Thus $T = T^C + T^A$ consists of the 'classical' part

$$T^C = \frac{\nu}{2}\langle(\partial_z\varphi)^2\rangle - 2\varsigma_H\partial_z^2\langle\varphi\rangle \qquad (28)$$

and the anomalous part (27). This explicit representation of T completes the Ward identity.

Geometric singularity The Ward identity consists of terms of a different order in magnetic length and has to be solved iteratively. The leading approximation, where $\rho \approx \bar{\rho}$ suffices. Then as it follows from (22) $\langle\varphi\rangle \approx 2(\varsigma_H/\nu)\phi$. Up to this order, the classical part of the stress tensor is $T^C = -4(\varsigma_H^2/\nu)\left[-\frac{1}{2}(\partial_z\phi)^2 + \partial_z^2\phi\right]$. Together with the anomalous part (27) the stress tensor reads

$$T = \frac{c_H}{12}\mathcal{S}[\phi]. \qquad (29)$$

We compute the singular part of the stress tensor by evaluating the Schwarzian derivative on the singular metric (19). Equivalently, we treat a conical singularity as a conformal map (20) and compute the Schwarzian derivative of the map

$$\mathcal{S}[\sigma] \equiv \{w,\, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 = \frac{\alpha(2-\alpha)}{2z^2}.$$

Resulting in

$$T = \frac{c_H}{24}\frac{\alpha(2-\alpha)}{z^2}. \qquad (30)$$

Using (26), we arrive at our main result (8).

Magnetic singularity In this case, the gravitational anomaly does not appear in the singular scaling of the stress tensor near the location of the flux tube. Rather, the stress tensor receives an additional contribution from the magnetic potential of the flux tube $Q_a = 2a\log|z|$, such that

$$T = -\frac{\nu}{2}(\partial_z Q_a)^2 - 2\varsigma_H\partial_z^2 Q_a = \frac{h_a}{z^2}, \qquad (31)$$

where h_a is the conformal dimension (6).

Finally, when the flux tube sits on top of a conical singularity, the stress tensor is the sum of (31) and (25). Near the singularity , $T \sim (\gamma\Delta_\alpha + h_a)/z^2$. This implies the relation (9).

Exchange statistics When adiabatically exchanging two singularities, located at say z^1 and z^2 of the z=plane, the state acquires a phase. Since it is a holomorphic function of singularity position, its holonomy is encoded by the normalization factor. The phase is then $\Phi_{12} = -\frac{i}{2}\oint d\log\mathcal{Z}$, where the integral goes along the adiabatic path in parameter space (z^1, z^2). The adiabatic connection $d\log\mathcal{Z}$ has a pole when two singularities coincide, so the phase is the residue of the pole $\Phi_{12} = \pi\,\mathrm{res}\,[d\log\mathcal{Z}]$.

For conical singularities, the residue arises entirely from the gravitational anomaly. Calculation of the normalization is equivalent to computing the determinant of the Laplacian $\mathrm{Det}(-\Delta)$ on singular surfaces [9]. A reason for this is that, on singular surfaces, $\log\mathcal{Z}$ is equivalent to $\frac{c_H}{2}\log\mathrm{Det}(-\Delta)$, viewed as a gravitational effective action, since it encodes the same stress tensor T. In order to capture the leading singularity, a piece-wise flat surfaces suffices. The result can be borrowed from e.g., [29]

$$\log\mathcal{Z}|_{z^1\to z^2} = \frac{c_H}{12}\alpha_1\alpha_2\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\log|z^1 - z^2|. \qquad (32)$$

Then the adiabatic connection is

$$d\log\mathcal{Z} = \frac{c_H}{24}\alpha_1\alpha_2\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\frac{dz^1 - dz^2}{z^1 - z^2} \qquad (33)$$

which prompts the formula (10) for the exchange statistics.

Integer QH-state on a cone The formulae for the charge of the singularity (13) and the moment of inertia (14) readily checked against the direct calculations for the integer case $\nu = 1$. See, [37], and more recently [31] for study of Landau levels on a cone. In the case where a flux $(2\pi\hbar/e)a$ threads the cone the Landau level is spanned by one-particle states

$$\psi_k = \frac{e^{-|\xi|^2/4l^2}}{l\sqrt{2\pi\gamma\Gamma(\frac{k}{\gamma} - j + 1)}}\left(\frac{\xi}{\sqrt{2}l}\right)^{\frac{k}{\gamma}-j} \qquad (34)$$

with $k = a+j, a+j+1, \ldots, a+j+N-1$, and the density is the sum of densities of each one-particle state $\rho = \sum_k |\psi_k|^2$.

We find the moments from the conformal Ward identity. Under re-scaling the magnetic length $l^2 \to \lambda^{-1} l^2$ the state (34) scales $\psi_k \to \lambda^{\frac{1}{2}(1-j+\frac{k}{\gamma})} e^{(1-\lambda)|\xi|^2/4l^2} \psi_k$. Then the normalization condition for the new state yields the identity

$$\int e^{(1-\lambda)\frac{|\xi|^2}{2l^2}} |\psi_k|^2 dV = \lambda^{(j-1)} \lambda^{-\frac{k}{\gamma}}.$$

Then, summing over all modes and taking $N \to \infty$, we obtain a generating function of moments (11)

$$\int e^{(1-\lambda)\frac{|\xi|^2}{2l^2}} (\rho - \rho_\infty) dV = \frac{\lambda^{-\frac{1}{\gamma}(a+j)} \lambda^{j-1}}{1 - \lambda^{-\frac{1}{\gamma}}} - \frac{\gamma}{1-\lambda},$$

where $\rho_\infty = 1/2\pi l^2$. Expanding it around $\lambda = 1$ yields the charge m_0 (13) and the moment of inertia m_2 (14).

We thank A. Abanov, R. Biswas, A. Gromov, S. Klevtsov, J. Simon and D. T. Son for discussions. TC thanks S. Ganeshan, J. Ross, A. Waldron, S. Zelditch for many illuminating discussions. The work was supported in part by the NSF under the Grants NSF DMR-1206648, NSF DMR-MRSEC 1420709, and NSF PHY11-25915.

---

[1] J. E. Avron, R. Seiler, and P. G. Zograf, *Viscosity of quantum Hall fluids,* Phys. Rev. Lett., **75**:697–700, 1995, arXiv:9502011

[2] P. Lévay, *Berry phases for Landau Hamiltonians on deformed tori,* J. Math. Phys., **36(6)**:2792–2802, 1995; P. Lévay, *Berry's phase, chaos, and the deformations of Riemann surfaces*, Phys. Rev. E **56**(5):6173–6176, 1997.

[3] X. G. Wen and A. Zee, *Shift and spin vector: New topological quantum numbers for the Hall fluids,* Phys. Rev. Lett., **69**:953–956, 1992.

[4] J. Fröhlich and U. M. Studer, *U(1)×SU(2)-gauge invariance of non-relativistic quantum mechanics, and generalized Hall effects,* Commun. Math. Phys., **148(3)**: 553–600, 1992.

[5] N. Read, *Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids,* Phys. Rev. B, **79(4)**: 045308, 2009, arXiv:0805.2507v3
N. Read, E. H. Rezayi, *Hall viscosity, orbital spin, and geometry: Paired superfluids and quantum Hall systems,* Phys. Rev. B, **84**: 085316, 2011, arXiv:1008.0210

[6] C. Hoyos, D. T. Son, *Hall Viscosity and Electromagnetic Response* Phys. Rev. Lett., **108**: 066805, 2012, arXiv:1109.2651

[7] M. R. Douglas and S. Klevtsov, *Bergman kernel from path integral,* Commun. Math. Phys., **293(1)**:205–230, 2010, arXiv:0808.2451v1
S. Klevtsov, *Random normal matrices, Bergman kernel and projective embeddings,* JHEP, **2014(1)**:1–19, 2014, arXiv:1309.7333v2

[8] T. Can, M. Laskin, and P. Wiegmann, *Fractional quantum Hall effect in a curved space: Gravitational anomaly and electromagnetic response,* Phys. Rev. Lett., **113**:046803, 2014, arXiv:1402.1531v2

[9] T. Can, M. Laskin, and P. B. Wiegmann, *Geometry of quantum Hall states: Gravitational anomaly and transport coefficients*, Ann. of Physics, **362**:752–794, 2015, arXiv:1411.3105v3

[10] A. G. Abanov and A. Gromov, *Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field.* Phys. Rev. B, **90(1)**: 014435, 2014, arXiv:1401.3703v1
A. Gromov and A. G. Abanov, *Density-Curvature Response and Gravitational Anomaly*, Phys. Rev. Lett., **113(26)**:266802, 2014, arXiv:1403.5809v1
A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, *Framing anomaly in the effective theory of the fractional quantum Hall effect,* Phys. Rev. Lett., **114**: 016805, 2015, arXiv:1410.6812v3

[11] M. Laskin, T. Can, and P. Wiegmann, *Collective field theory for quantum hall states,* Phys. Rev. B, **92**: 235141, 2015, arXiv:1412.8716v2

[12] B. Bradlyn and N. Read, *Topological central charge from Berry curvature: gravitational anomalies in trial wave functions for topological phases,* Phys. Rev. B, **91**: 165306, 2015, arXiv:1502.04126v2

[13] F. Ferrari and S. Klevtsov, *FQHE on curved backgrounds, free fields and large N*, JHEP, **1412** 086, 2014, arXiv:1410.6802v3

[14] S. Klevtsov and P. Wiegmann, *Precise adiabatic transport and geometry of quantum Hall states*, Phys. Rev. Lett., **115(8)**: 086801, 2015, arXiv:1504.07198v2

[15] C. Vafa, *Fractional quantum hall effect and M-theory,* arXiv:1511.03372v3.

[16] M. Troyanov, *Metrics of constant curvature on a sphere with two conical singularities,* Lecture Notes in Math., **1410**: 286–306, 1989.

[17] M. Troyanov, *On the Moduli Space of Singular Euclidean Surfaces,* IRMA Lectures in Math. and Theor. Phys. **11**, 507-540, 2007, arXiv:1504.07198

[18] M. Katanev, I. Volovich *Theory of defects in solids and three-dimensional gravity,* Annals of Physics, **216**:1 1992,

[19] M.A.H. Vozmediano and M.I. Katsnelson and F. Guinea *Gauge fields in graphene* Phys. Rep.**496**: 109 - 148, 2010

[20] N. Schine, A. Ryou, A. Gromov, A. Sommer and J. Simon, *Synthetic Landau levels for photons*, arXiv:1511.07381v1

[21] W. P. Thurson, *Shapes of polyhedra and triangulations of the sphere*, Geom. Topol. Monogr., **1**: 511-549, 1998, arXiv:math/9801088v2

[22] L. Takhtajan and P. Zograf, *Hyperbolic 2-spheres with conical singularities, accessory parameters and Kaehler metrics on $\mathcal{M}_{0,n}$,* Trans. Amer. Math. Soc., **355**: 1857-1867, 2002, arXiv:math/0112170

[23] D. T. Son, *Is the Composite Fermion a Dirac Particle?*, Phys. Rev. X, **5(3)**:031027, 2015, arXiv:1502.03446v3

[24] R. B. Laughlin. *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations. Phys. Rev. Lett.*, 50:1395–1398, 1983.

[25] V.G. Knizhnik, *Analytic fields on Riemannian surfaces II*, Commun. Math. Phys, **112**: 567–590, 1987.

[26] M A Bershadsky and A O Radul, *Conformal field theories with additional Z(N) symmetry*, Sov. J. Nucl. Phys., **47**: 363–369, 1988.

[27] J. L. Cardy and I. Peschel, *Finite-size dependence of the free energy in two-dimensional critical systems,* Nucl. Phys. B, **300**: 377–392, 1988.

[28] Kokotov, A, *Polyhedral surfaces and determinant of Laplacian,* Proc. Math. Soc., **141**: 725–735, 2013.

[29] E. Aurell and P. Solomonsin, *Further results on Functional Determinants of Laplacians in Simplicial Complexes,* arXiv:9405140

[30] D. Arovas, J. R. Schrieffer and F. Wilczek, *Fractional Statistics and the Quantum Hall Effect,* Phys. Rev. Lett., **53**: 722–723, 1984.

[31] R. R. Biswas and D. T. Son, *Fractional charge and inter-Landau level states at points of singular curvature*, arXiv:1412.3809v1

[32] P. Wiegmann, *Anomalous hydrodynamics of fractional quantum Hall states,* JETP, **117(3)**: 538–550, 2013. arXiv:1305.6893v2

[33] P. Wiegmann, *Quantum Hydrodynamics of Fractional Hall Effect: Quantum Kirchhoff Equations,* arXiv:1211.5132v1

[34] P. Wiegmann, *Hydrodynamics of Euler incompressible fluid and the fractional quantum Hall effect,* Phys. Rev. B, **88(24)**: 241305, 2013. arXiv:1309.5992v1

[35] R. B. Laughlin, *Quantized Hall conductivity in two dimensions,* Phys. Rev. B, **23(10)**: 5632–5633(R), 1981.

[36] A. Zabrodin and P. Wiegmann, *Large-N expansion for the 2D Dyson gas,* J. Phys. A, **39(28)**: 8933–8964, 2006, arXiv:0601009v3

[37] C. Furtado, B.G.C. da Cunha, F. Moraes, E.R. Bezerra de Mello, V.B. Bezzerra *Landau levels in the presence of disclinations,* Phys. Rev. A, **195**:90–94, 1994,
G.A. Marques, C. Furtado, V.B. Bezerra, and F. Moraes *Landau Levels in the Presence of Topological Defects,* J. Phys. A, **34**:5945–5954, 2001, arXiv:9405017v1
A. Poux, L. R. S. Araujo, C. Filgueiras, and F. Moraes *Landau levels, self-adjoint extensions and Hall conductivity on a cone,* Eur. Phys. J., **129**:100 2014, arXiv:1405.6599

[38] The recent paper [23] argues that $\nu = \frac{n}{2n+1}$ filling fractions exhibit Dirac fermions, i.e., $j = 1/2$, but the arguments of [15] assign $j = 1$ to these fractions.