Bounded risk estimation of a linear combination of location parameters in negative exponential distributions via three-stage sampling

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Abstract

The paper deals with the problem of bounded risk point estimation for a linear combination of location parameters of two negative exponential distributions. Isogai and Futschik (2010) considered the situation when the location and scale parameters are all unknown. They proposed purely sequential procedures and gave second order expansions of the average sample sizes and risks.

In this paper we propose three-stage procedures and derive second order expansions of the average sample sizes and risks. Further, we compare the results with those of Isogai and Futschik (2010).

MSC: primary 62L12; secondary 62L10
Keywords: Bounded risk; Location parameters; Negative exponential distributions; Three-stage; Second order expansions

1. Introduction

Let $\{X_{i1}, X_{i2}, X_{i3}, \ldots\}$ $(i = 1, 2)$ be two sequences of independent and identically distributed random variables. We assume both sequences to be exponentially distributed with probability density functions (p.d.f.)

$$f(x\,;\,\mu_i, \sigma_i) = \sigma_i^{-1} \exp\left(-\frac{x-\mu_i}{\sigma_i}\right) I(x > \mu_i)\,, \tag{1.1}$$

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where $I(\cdot)$ denotes the usual indicator function and X_{1j}'s and X_{2j}'s are mutually independent. We abbreviate the distributions of (1.1) as $\mathrm{Exp}(\mu_i, \sigma_i)$. The location or threshold parameters $\mu_i \in (-\infty, \infty)$, and the scale parameters $\sigma_i \in (0, \infty)$ $(i = 1, 2)$ are assumed to be unknown and possibly different. We also define θ to be the parameter vector $\theta := (\mu_1, \mu_2, \sigma_1, \sigma_2)$. Let a_i for $i = 1, 2$ with $a_1 a_2 \neq 0$ be any given constants. The problem is to estimate a linear combination $\delta = a_1\mu_1 + a_2\mu_2$ of two location parameters μ_1 and μ_2 under squared error loss. Let w be a given positive constant and $\delta(m, n)$ be an estimator of δ based on samples $\{X_{11}, \ldots, X_{1m}\}$ and $\{X_{21}, \ldots, X_{2n}\}$. We want to find appropriate sample sizes such that the risk of the estimator $\hat{\delta}(m, n)$ is less than or equal to w. This problem is called a bounded risk point estimation problem.

Regarding negative exponential distributions, the threshold parameter μ_i may stand for the minimum guarantee time. Many authors considered sequential estimation problems for two negative exponential distributions by using purely sequential and/or two-stage procedures. Mukhopadhyay and Hamdy (1984) and Singh and Chaturvedi (1991) dealt with the fixed-width interval estimation problem for the difference between location parameters. Mukhopadhyay and Darmanto (1988) investigated the minimum risk point estimation problem for the difference of means. For the one sample problem, Mukhopadhyay and Zacks (2007) considered bounded risk estimation problems of linear combinations of the location and scale parameters. Isogai and Futschik (2010) dealt with the bounded risk point estimation problem via purely sequential procedures. Hall (1981) proposed a triple sampling method which was designed to combine the operational savings and the efficiency of purely sequential procedures. Three-stage estimation problems for negative exponential distributions were considered by Hamdy et al. (1989) and Mukhopadhyay (1990, 1992), for instance. We refer to Ghosh et al. (1997) and Mukhopadhyay and de Silva (2009) for a general overview on sequential estimation.

In the paper, we propose three-stage procedures to estimate the linear combination of two location parameters. Further, we compare the results with those on the purely sequential procedures of Isogai and Futschik (2010). In Section 2 we propose three-stage procedures. Section 3 provides our main results and the comparison. All the proofs are given in the final section.

2. Three-stage procedures

Having recorded $X_{11}, \ldots, X_{1m}$ and $X_{21}, \ldots, X_{2n}$, we define

$$X_{1m(1)} = \min\{X_{11}, \ldots, X_{1m}\}, \quad U_{1m} = \frac{1}{m-1}\sum_{j=1}^{m}(X_{1j} - X_{1m(1)}),$$

$$X_{2n(1)} = \min\{X_{21}, \ldots, X_{2n}\}, \quad U_{2n} = \frac{1}{n-1}\sum_{j=1}^{n}(X_{2j} - X_{2n(1)})$$

for $m \geq 2$ and $n \geq 2$. Let a_1 and a_2 be any given constants with $a_1 a_2 \neq 0$. Throughout this paper, we use

$$\hat{\delta}(m,n) = a_1 X_{1m(1)} + a_2 X_{2n(1)} \tag{2.1}$$

as an estimator of $\delta = a_1\mu_1 + a_2\mu_2$. The loss function is $L(\hat{\delta}(m,n),\delta) = A(\hat{\delta}(m,n) - \delta)^2$ with A a known positive constant. The risk associated with $\hat{\delta}(m,n)$ is

$$R_\theta(m,n) = A \cdot E_\theta\{(\hat{\delta}(m,n) - \delta)^2\}\,,$$

where E_θ denotes the expectation under the true parameter $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2)$. For simplicity we drop the parameter θ, and write $E(\cdot)$ and $R(\cdot)$ instead of $E_\theta(\cdot)$ and $R_\theta(\cdot)$. We want $R(m,n)$ to be less than or equal to a given constant w for any fixed θ, that is, $R(m,n) \leq w$. Isogai and Futschik (2010) showed the following lemma.

Lemma 2.1. *Suppose that $a_1 a_2 \neq 0$. Set*

$$\beta = \begin{cases} \sqrt{6A} & \text{if} \quad a_1 a_2 > 0\,, \\ \sqrt{2A} & \text{if} \quad a_1 a_2 < 0, \end{cases} \tag{2.2}$$

and

$$b_i = \beta\,|a_i|,\ \ n_i^* = b_i w^{-1/2}\sigma_i \quad \text{for } i = 1,2. \tag{2.3}$$

Then, we have for $a_1 a_2 > 0$

$$R(m,n) \leq w \quad \text{for all} \ \ m \geq n_1^* \ \text{and}\ n \geq n_2^*\,,$$

and for $a_1 a_2 < 0$

$$\begin{aligned} &R(m,n) > w \quad \text{for all} \ \ m < n_1^* \ \text{and}\ n < n_2^*\,, \quad \text{and} \\ &R(n_1^*,n) \leq (>)\, w \quad \text{for}\ n \geq (\leq)\, n_2^*. \end{aligned}$$

According to Lemma 2.1 we take n_1^* and n_2^* as the optimal fixed sample sizes. Unfortunately, the optimal fixed sample sizes include the unknown scale parameters, and so we cannot compute them in practice. Therefore we propose three-stage procedures to solve this problem. Isogai and Futschik (2010) used purely sequential procedures for this problem

Let any positive number w be fixed. At the first stage we take samples $X_{i1}, \ldots, X_{im}$ $(i = 1,2)$ of the same size m, respectively where the starting sample size $m(\geq 2)$ with $m = O(w^{-1/r})$ for some $r > 2$ is given in advance, and compute U_{im}. We also choose and fix any two numbers $0 < \rho_i < 1$ $(i = 1,2)$. Our three-stage procedures are defined as

$$T_i = T_i(w) = \max\{m,\ \langle \rho_i b_i w^{-1/2} U_{im}\rangle + 1\}, \tag{2.4}$$

where $\langle x \rangle$ is the largest integer less than x. If $T_i > m$ then we sample the difference in the second stage. Based on $X_{i1}, \ldots, X_{iT_i}$ we define

$$N_i = N_i(w) = \max\{T_i,\ \langle b_i w^{-1/2} U_{iT_i} \rangle + 1\}. \tag{2.5}$$

If $N_i > T_i$ then we take more samples $X_{i,T_i+1}, \ldots, X_{iN_i}$ and the total sample size is N_i. At the stopped stage the estimator of δ is given by

$$\delta(\underline{N}) = a_1 X_{1N_1(1)} + a_2 X_{2N_2(1)} \tag{2.6}$$

from (2.1) where $\underline{N} = (N_1, N_2)$.

3. Main results

In this section we provide results concerning the second order asymptotic expansions of average sample size and risk for the estimator $\hat{\delta}(\underline{N})$ in (2.6).

The following theorem gives the second order expansions of the expected sample size.

Theorem 3.1. *Suppose that $a_1 a_2 \neq 0$. For $i = 1, 2$ we have*

$$E(N_i) = n_i^* + \eta_i + o(1) \quad \text{as } w \to 0,$$

where

$$\eta_i = \frac{1}{2} - \rho_i^{-1}.$$

Comparison 3.1. Isogai and Futschik (2010) proposed the purely sequential procedures M_i $(i = 1, 2)$ corresponding to N_i in (2.5) defined by $M_i = \inf\{m_i \geq k_i,\ U_{im} \leq b_i^{-1} w^{1/2} m(1 + l_i m^{-1})\}$ where $k_i \geq 3$ is any given integer and l_i is any fixed constant satisfying $l_i > -k_i$. Theorem 3.1 there provided that $E(M_i) = n_i^* + (\nu/2) - 2 - l_i + o(1)$ as $w \to 0$, where $\nu \approx 1.494$. Let $\rho_i = 0.5$ for instance. Since l_i can be chosen as $l_i > -k_i$ for any given integer $k_i \geq 3$, we get for $i = 1, 2$

$$E(M_i) - E(N_i) \approx 0.247 - l_i \quad \text{as } w \to 0.$$

Thus we have

$$E(M_i) - E(N_i) \begin{cases} > 0 & \text{if} \quad -3 < l_i < 0.247, \\ \leq 0 & \text{if} \quad l_i \geq 0.247. \end{cases}$$

The following theorem provides the second order asymptotic expansion of the risks.

Theorem 3.2. *Suppose that $a_1 a_2 \neq 0$. Then we have*

$$R(\underline{N}) = w + c^{-1} \left\{ \sum_{i=1}^{2} h_i (|a_i|\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}) \quad \text{as } w \to 0,$$

where for $i = 1, 2$

$$h_i = \begin{cases} 14\rho_i^{-1} - 3 & \text{when } a_1 a_2 > 0, \\ 6\rho_i^{-1} - 1 & \text{when } a_1 a_2 < 0 \end{cases}$$

and

$$c = \begin{cases} 6\sqrt{6A} & \text{when } a_1 a_2 > 0, \\ 2\sqrt{2A} & \text{when } a_1 a_2 < 0. \end{cases}$$

Remark 3.1. (i) Since $h_i > 0$ for $0 < \rho_i < 1$, we have that $R(\underline{N}) > w$ for sufficiently small w. Thus the condition for boundedness is not satisfied. Let $\underline{M} = (M_1, M_2)$ and any fixed constant l_i in M_i satisfies $l_i > -k_i$ for $k_i \geq 7$. Choosing $-7 < l_i < -3.252$, Remarks 3.1 and 3.2 of Isogai and Futschik (2010) showed that $R(\underline{M}) \leq w$ for sufficiently small w.
(ii) Let us define the modified three-stage procedures with "fine-tuning" factors ε_i $(i = 1, 2)$ instead of N_i in (2.5) as

$$N_i' = N_i'(w) = \max\{T_i, \langle b_i w^{-1/2} U_{iT_i} + \varepsilon_i \rangle + 1\},$$

where T_i is the same as in (2.4). Then, choosing suitable factors ε_i, we might be able to show that $R(\underline{N}') \leq w$ for sufficiently small w.

Examples. Let $\rho_i = 0.5$ for $i = 1, 2$. (i) For the case $a_1 a_2 < 0$ we consider the difference of two location parameters $\delta = \mu_1 - \mu_2$ and the estimator is $\hat{\delta}(\underline{N}) = X_{1N_1(1)} - X_{2N_2(1)}$. Then Theorem 3.2 gives

$$R(\underline{N}) = w + \frac{11}{2\sqrt{2A}} (\sigma_1^{-1} + \sigma_2^{-1}) w^{3/2} + o(w^{3/2}) \quad \text{as } w \to 0.$$

(ii) For the case $a_1 a_2 > 0$ we consider the mean of two parameters $\delta = \frac{1}{2}(\mu_1 + \mu_2)$ and the estimator is $\hat{\delta}(\underline{N}) = \frac{1}{2}(X_{1N_1(1)} + X_{2N_2(1)})$. Then we have

$$R(\underline{N}) = w + \frac{25}{3\sqrt{6A}} (\sigma_1^{-1} + \sigma_2^{-1}) w^{3/2} + o(w^{3/2}) \quad \text{as } w \to 0.$$

Comparison 3.2. Theorems 3.2 and 3.3 of Isogai and Futschik (2010) provided

$$R(\underline{M}) = w + c'^{-1} \left\{ \sum_{i=1}^{2} h_i' (|a_i|\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}) \quad \text{as } w \to 0,$$

where for $i = 1, 2$

$$h_i' = \begin{cases} 10 - 3\nu/2 + 3l_i & \text{when } a_1a_2 > 0, \\ 4 - \nu/2 + l_i & \text{when } a_1a_2 < 0, \end{cases}$$

and $l_i > -k_i$ for any given integer $k_i \geq 7$ and

$$c' = \begin{cases} 3\sqrt{6A} & \text{when } a_1a_2 > 0, \\ \sqrt{2A} & \text{when } a_1a_2 < 0. \end{cases}$$

Theorem 3.2 implies

$$R(\underline{M}) - R(\underline{N}) \approx \left\{ \sum_{i=1}^{2} ({c'}^{-1}h_i' - c^{-1}h_i)(|a_i|\sigma_i)^{-1} \right\} w^{3/2}$$

for sufficiently small w. Let us compare the three-stage procedure with the purely sequential one from the risk point of view. Choose $\rho_i = 0.5$ for $i = 1, 2$. When $a_1a_2 > 0$ we have

$$\begin{aligned} 6\sqrt{6A}w^{-3/2}(R(\underline{M}) - R(\underline{N})) &\approx \sum_{i=1}^{2} (6l_i - 3\nu - 5)(|a_i|\sigma_i)^{-1} \\ &\approx \sum_{i=1}^{2} (6l_i - 9.482)(|a_i|\sigma_i)^{-1}. \end{aligned}$$

Taking $k_i = 7$ and $l_i = 2 > -k_i$ for $i = 1, 2$, we have

$$6\sqrt{6A}w^{-3/2}(R(\underline{M}) - R(\underline{N})) \approx 2.518 \sum_{i=1}^{2} (|a_i|\sigma_i)^{-1} > 0,$$

which shows that $R(\underline{M}) > R(\underline{N})$. On the other hand, Comparison 3.1 gives

$$E(M_i) - E(N_i) \approx 0.247 - l_i = -1.753,$$

which shows that $E(M_i) < E(N_i)$ for $i = 1, 2$. Let $k_i = 7$ and $l_i = 0$ for $i = 1, 2$. Then

$$\begin{aligned} &6\sqrt{6A}w^{-3/2}(R(\underline{M}) - R(\underline{N})) \approx -9.482 \sum_{i=1}^{2} (|a_i|\sigma_i)^{-1} \quad \text{and} \\ &E(M_i) - E(N_i) \approx 0.247 - l_i = 0.247. \end{aligned}$$

Hence we cannot decide which procedure is better when $a_1a_2 > 0$. We consider the case $a_1a_2 < 0$. Choosing $k_i = 7$ and $l_i = 3$ for $i = 1, 2$, we have

$$\begin{aligned} \sqrt{2A}w^{-3/2}(R(\underline{M}) - R(\underline{N})) &\approx \sum_{i=1}^{2} (l_i - 2.247)(|a_i|\sigma_i)^{-1} \\ &\approx 0.753 \sum_{i=1}^{2} (|a_i|\sigma_i)^{-1} \end{aligned}$$

and

$$E(M_i) - E(N_i) \approx 0.247 - l_i = -2.753.$$

Taking $k_i = 7$ and $l_i = 0$ for $i = 1, 2$, we get

$$6\sqrt{6A}w^{-3/2}(R(\underline{M}) - R(\underline{N})) \approx -2.247 \sum_{i=1}^{2} (|a_i|\sigma_i)^{-1} \quad \text{and}$$

$$E(M_i) - E(N_i) \approx 0.247 - l_i = 0.247.$$

Thererfore in general we cannot declare which procedure is superior.

4. Proofs

In this section we will give the proofs of two theorems in Section 3. Without any loss of generality we can write $\delta = a_1\mu_1 + a_2\mu_2$ as $\delta = a_1\sigma_1 - a_2\sigma_2$ for $a_1a_2 < 0$ and $\delta = a_1\sigma_1 + a_2\sigma_2$ for $a_1a_2 > 0$ where $a_1 > 0$ and $a_2 > 0$. Let $\mu_i' = a_i\mu_i$ and $\sigma_i' = a_i\sigma_i$ for $i = 1, 2$. Then we can use the form

$$\delta = \begin{cases} \mu_1' - \mu_2' & \text{when} \quad a_1a_2 < 0, \\ \mu_1' + \mu_2' & \text{when} \quad a_1a_2 > 0 \end{cases} \tag{4.1}$$

and throughout this section we consider the fundamental forms (4.1) and M denotes a generic positive constant, not depending on w.

Let $X_{ij}' = (a_iX_{ij} - \mu_i')/\sigma_i'$ for $i = 1, 2$. Then $X_{i1}', X_{i2}', \ldots$ are independent and identically distributed (i.i.d.) random variables according to the exponential distribution $\mathrm{Exp}(0, 1)$. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables according to $\mathrm{Exp}(0, 1)$ and let $\{X_{1j}', X_{2j}' : j \geq 1\}$ and $\{Y_j : j \geq 1\}$ be independent. For $i = 1, 2$ set

$$X_{in(1)}' = \min\{X_{i1}', \ldots, X_{in}'\}, \; U_{in}' = \frac{1}{n-1} \sum_{j=1}^{n} (X_{ij}' - X_{in(1)}') \quad \text{and}$$

$$\overline{Y}_n = \frac{1}{n-1} \sum_{j=2}^{n} Y_j \quad \text{for } n \geq 2.$$

The estimator $\hat{\delta}(\underline{N})$ in (2.6) of δ is give by

$$\hat{\delta}(\underline{N}) = \begin{cases} \sigma_1'X_{1N_1(1)}' - \sigma_2'X_{2N_2(1)}' + \delta & \text{for } a_1a_2 < 0\,, \\ \sigma_1'X_{1N_1(1)}' + \sigma_2'X_{2N_2(1)}' + \delta & \text{for } a_1a_2 > 0. \end{cases} \tag{4.2}$$

Let

$$\lambda_i = n_i^* = \beta w^{-1/2}\sigma_i' \quad \text{for } i = 1, 2. \tag{4.3}$$

From (2.3), (2.4) and (2.5) we get

$$T_i = \max\{m, \langle \rho_i\lambda_iU_{im}' \rangle + 1\} \quad \text{and} \quad N_i = \max\{T_i, \langle \lambda_iU_{iT_i}' \rangle + 1\}.$$

Lemma 6.1 of Lomberd and Swanepoel (1978) implies that $\{(n-1)U'_{in},\ n \geq 2\}$ and $\{(n-1)\overline{Y}_n,\ n \geq 2\}$ are identically distributed. Let us define for $i = 1, 2$

$$\begin{aligned} R_i &= R_i(w) = \max\left\{m,\ \langle \rho_i \lambda_i \overline{Y}_m \rangle + 1\right\} \quad \text{and} \\ S_i &= S_i(w) = \max\left\{R_i,\ \langle \lambda_i \overline{Y}_{R_i} \rangle + 1\right\}. \end{aligned} \tag{4.4}$$

Then we get the following lemma.

Lemma 4.1 *For $i = 1, 2$ (T_i, N_i) and (R_i, S_i) are identically distributed.*

Proof. Let any integers $m \leq k \leq n$ be fixed. Then

$$\begin{aligned} &P\{T_i \leq k,\ N_i \leq n\} \\ &= \sum_{t=m}^{k} P\{T_i = t = \max\{m,\ \langle \rho_i \lambda_i U'_{im} \rangle + 1\},\ \max\{t,\ \langle \lambda_i U'_{it} \rangle + 1\} \leq n\} \\ &= \sum_{t=m}^{k} P\{t = \max\{m,\ \langle \rho_i \lambda_i \overline{Y}_m \rangle + 1\} = R_i,\ \max\{t,\ \langle \lambda_i \overline{Y}_t \rangle + 1\} \leq n\} \\ &= \sum_{t=m}^{k} P\{R_i = t,\ S_i = \max\{R_i,\ \langle \lambda_i \overline{Y}_{R_i} \rangle + 1\} \leq n\} = P\{R_i \leq k,\ S_i \leq n\}, \end{aligned}$$

which shows that (T_i, N_i) and (R_i, S_i) are identically distributed. This completes the proof.

The three-stage procedure $(R_i,\ S_i)$ in (4.4) is the same as $(T,\ N)$ of Mukhopadhyay (1990). Thus we can use Theorems 2 and 3 of Mukhopadhyay (1990). Since $E(Y_2^p) < \infty$ for all $p > 0$, the Doob maximal inequality for reversed martingales (see Gut (2005), page 543 and 545) implies

$$E\left[\left(\sup_{n \geq 2} \overline{Y}_n\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p E(Y_2^p) \leq M \tag{4.5}$$

for $p > 1$. In view of Hölder's inequality (4.5) holds for $0 < p \leq 1$. From (4.5) we can see that $E(\overline{Y}_{R_i}^p) \leq E\left[\left(\sup_{n \geq 2} \overline{Y}_n\right)^p\right] < \infty$ for all $p > 0$, because of $P\{R_i < \infty\} = 1$. Thus from Theorems 2 and 3 of Mukhopadhyay (1990) we get

Lemma 4.2 *Let $i = 1, 2$. For any positive integer k we have as $w \to 0$*

$$\begin{aligned} E(S_i^k) &= (n_i^*)^k + \frac{1}{2} k (n_i^*)^{k-1} \{(k-3) + \rho_i\}/\rho_i + o((n_i^*)^{k-1}) \quad \textit{and} \\ E(S_i^{-k}) &= (n_i^*)^{-k} - \frac{1}{2} k (n_i^*)^{-k-1} \{\rho_i - (k+3)\}/\rho_i + o((n_i^*)^{-k-1}). \end{aligned}$$

Proof of Theorem 3.1. Let $\eta_i = \frac{1}{2} - \rho_i^{-1}$ for $i = 1, 2$. Theorem 3.1 is an immediate consequence of Lemmas 4.1 and 4.2.

Proof of Theorem 3.2. First we consider the case $a_1 a_2 < 0$. Set $R'(\underline{N}) = R(\underline{N})/A$. (4.2) gives

$$R'(\underline{N}) = \sum_{i=1}^{2} \sigma_i'^2 E(X_{iN_i(1)}'^2) - 2\prod_{i=1}^{2} \sigma_i' E(X_{iN_i(1)}'), \tag{4.6}$$

for $X_{1N_1(1)}'$ and $X_{2N_2(1)}'$ are independent. We will evaluate the term $E(X_{iN_i(1)}'^2)$. Since $X_{in(1)}'$ and $\{U_{ij}', 2 \le j \le n\}$ are independent for any $n \ge m$, $X_{in(1)}'$ and $I(N_i = n)$ are independent. Thus, taking $P\{N_i < \infty\} = 1$ and Lemma 4.1 into account, we have

$$E(X_{iN_i(1)}'^2) = \sum_{n=m}^{\infty} E(X_{in(1)}'^2) P\{N_i = n\} = \sum_{n=m}^{\infty} 2n^{-2} P\{S_i = n\} = 2E(S_i^{-2}) \quad \text{and}$$
$$E(X_{iN_i(1)}') = E(S_i^{-1}).$$

By using $\sigma_i' = a_i \sigma_i$ and (4.3), Lemma 4.2 implies that as $w \to 0$

$$\sigma_i'^2 E(X_{iN_i(1)}'^2) = 2\sigma_i'^2 E(S_i^{-2}) = \beta^{-2} w - \beta^{-3}(1 - 5\rho_i^{-1})(a_i\sigma_i)^{-1} w^{3/2} + o(w^{3/2}) \quad \text{and}$$
$$\sigma_i' E(X_{iN_i(1)}') = \sigma_i' E(S_i^{-1}) = \beta^{-1} w^{1/2} - \frac{1}{2}\beta^{-2}(1 - 4\rho_i^{-1})(a_i\sigma_i)^{-1} w + o(w). \tag{4.7}$$

From (4.6) and (4.7) we obtain

$$\begin{aligned} R'(\underline{N})/2 &= \sum_{i=1}^{2} \sigma_i'^2 E(S_i^{-2}) - \prod_{i=1}^{2} \sigma_i' E(S_i^{-1}) \\ &= 2\beta^{-2} w - \beta^{-3} \left\{ \sum_{i=1}^{2} (1 - 5\rho_i^{-1})(a_i\sigma_i)^{-1} \right\} w^{3/2} \\ &\qquad -\beta^{-2} w + \frac{1}{2}\beta^{-3} \left\{ \sum_{i=1}^{2} (1 - 4\rho_i^{-1})(a_i\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}) \\ &= \beta^{-2} w + \frac{1}{2}\beta^{-3} \left\{ \sum_{i=1}^{2} (6\rho_i^{-1} - 1)(a_i\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}), \end{aligned} \tag{4.8}$$

which, together with $\beta = \sqrt{2A}$ in (2.2) and $a_i > 0\,(i = 1, 2)$, yields

$$R(\underline{N}) = AR'(\underline{N}) = w + \frac{1}{2\sqrt{2A}} \left\{ \sum_{i=1}^{2} (6\rho_i^{-1} - 1)(|a_i|\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}).$$

Therefore Theorem 3.2 is proved when $a_1 a_2 < 0$. In the same way as (4.8) we have

for $a_1 a_2 > 0$

$$R'(\underline{N})/2 = \sum_{i=1}^{2} \sigma_i'^2 E(S_i^{-2}) + \prod_{i=1}^{2} \sigma_i' E(S_i^{-1})$$
$$= 3\beta^{-2} w + \frac{1}{2}\beta^{-3} \left\{ \sum_{i=1}^{2} (14\rho_i^{-1} - 3)(a_i\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}),$$

which, together with $\beta = \sqrt{6A}$ and $a_i > 0\,(i = 1, 2)$, yields

$$R(\underline{N}) = AR'(\underline{N}) = w + \frac{1}{6\sqrt{6A}} \left\{ \sum_{i=1}^{2} (14\rho_i^{-1} - 3)(|a_i|\sigma_i)^{-1} \right\} w^{3/2} + o(w^{3/2}).$$

This shows Theorem 3.2 when $a_1 a_2 > 0$. Therefore the proof of Theorem 3.2 is complete.

Acknowledgments

The first author was supported by JSPS KAKENHI Grant-Number 26400193.

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