CHAPTER 16

GRAVITATION

Up to now we have discussed the effects of forces, without being too specific about what determines their magnitude and direction. In this chapter we study the details of one particularly important force, gravitation. *In 1665 Newton deduced that the force that governs the fall of apples near the Earth is the same as that which holds the Moon in its orbit. This was the first step toward developing a law of gravitation that could be applied to any pair of bodies in the universe.*

After introducing Newton's law of universal gravitation, we discuss its consequences and its experimental tests. We show that the Earth's gravity can be understood as a particular case of this universal law, and that the motions of the planets can be similarly explained. We conclude with a look at modern gravitational theory, namely Einstein's general theory of relativity, which gives correct results when the gravitational force is strong (where Newton's theory fails) and agrees with Newton's theory when the gravitational force is weak.

As you study this chapter, you should note that many of the basic concepts of dynamics discussed in previous chapters find an application here. We apply basic laws for forces, potential energy, the conservation of energy and angular momentum, harmonic motion, and properties of extended bodies. We also introduce new concepts, including the notion of fields, which will have application in later chapters.

16-1 GRAVITATION FROM THE ANCIENTS TO KEPLER

From at least the time of the ancient Greeks, two problems were the subjects of searching inquiry: (1) the tendency of objects such as stones to fall to Earth when released, and (2) the motions of the planets, including the Sun and the Moon, which were classified with the planets in those times. In early days these problems were thought of as completely separate. One of Newton's great achievements is that he saw them clearly as aspects of a single problem and subject to the same laws.

The earliest serious attempts to explain the kinematics of the solar system were made by the ancient Greeks. Ptolemy (Claudius Ptolemaeus, 2nd century A.D.) developed a geocentric (Earth-centered) scheme for the solar system in which, as the name implies, the Earth remains stationary at the center while the planets, including the Sun and the Moon, revolve around it. This should not be a surprising deduction. The Earth seems to us to be a substantial body. Shakespeare referred to it as "this goodly frame, the Earth. . . . " Even today, in navigational astronomy we use a geocentric reference frame, and in ordinary conversation we use terms such as "sunrise," which implies such a frame.

Because simple circular orbits cannot account for the complicated motions of the planets, Ptolemy had to use the concept of epicycles, in which a planet moves around a circle whose center moves around another circle centered on the Earth (see Fig. 1*a*). He also had to resort to several other geometrical arrangements, each of which preserved the supposed sanctity of the circle as a central feature of planetary motions. We now know that it is not a circle that is fundamental but an ellipse, with the Sun at one focus, as we shall discuss.

In the 16th century Nicolaus Copernicus (1473–1543) proposed a heliocentric (Sun-centered) scheme, in which the Earth (along with the other planets) moves about the Sun (see Fig. 1*b*). Even though the Copernican scheme seems much simpler than that of Ptolemy, it was not immediately accepted. Copernicus still believed in the

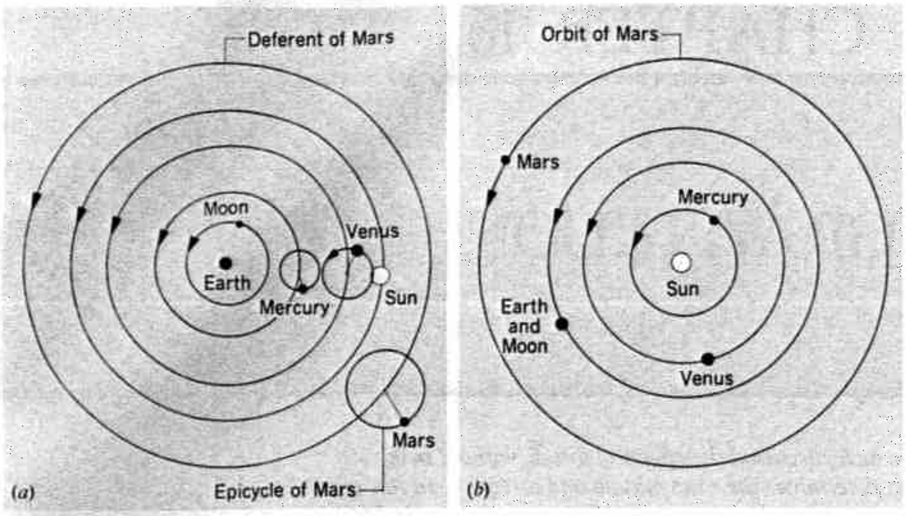

Figure 1 (*a*) The Ptolemaic view of the solar system. The Earth is at the center, and the Sun and planets move around it. The planets move in small circles (epicycles), whose centers travel along large circles (deferents). (*b*) The Copernican view of the solar system. The Sun is at the center, and the planets move around it.

sanctity of circles, and his use of epicycles and other arrangements (which are not shown in Fig. 1*b*) was about as great as that of Ptolemy. However, by putting the Sun at the center of things, Copernicus provided the correct reference frame from which our modern view of the solar system could develop.

To resolve the conflict between the Copernican and Ptolemaic schemes, more accurate observational data were needed. Such data were compiled by Tycho Brahe* (1546 – 1601), who was the last great astronomer to make observations without the use of a telescope. His data on planetary motions were analyzed and interpreted by Johannes Kepler (1571 – 1630), who had been Brahe's assistant. Kepler found important regularities in the motion of the planets, which led him to develop three laws (discussed in Section 16-8) that govern the motion of the planets.

Kepler's laws showed the great simplicity with which planetary motions could be described when the Sun was taken as the central body, if we give up the notion of perfect circles on which both the Ptolemaic and Copernican systems were based. However, Kepler's laws were empirical; they simply described the observed motions of the planets without any basis in terms of forces.† It was therefore a great triumph when Newton was later able to derive Kepler's laws from his laws of motion and his law of gravitation, which specified the force that acts between each planet and the Sun.

In this way Newton was able to account for the motion of the planets in the solar system and of bodies falling near the surface of the Earth with one common concept. He thereby unified into one theory the previously separate sciences of terrestrial mechanics and celestial mechanics. The real scientific significance of Copernicus' work lies in the fact that the heliocentric theory opened the way for this synthesis. Subsequently, on the assumption that the Earth rotates and revolves about the Sun, it became possible to explain such diverse phenomena as the daily and the annual apparent motion of the stars, the flattening of the Earth from a spherical shape, the behavior of the trade-winds, and many other observations that could not have been explained so easily in a geocentric theory.

The historical development of gravitational theory can be viewed as a model example of the way the method of scientific inquiry leads to insight. Copernicus provided the appropriate reference frame for viewing the problem, and Brahe supplied systematic and precise experimental data. Kepler used the data to propose some empirical laws, and Newton proposed a universal force law from which Kepler's laws could be derived. Finally, Einstein was led to a new theory which explained certain small discrepancies in the Newtonian theory.

16-2 NEWTON AND THE LAW OF UNIVERSAL GRAVITATION

In 1665 the 23-year-old Newton left Cambridge University for Lincolnshire when the college was dismissed because of the plague. About 50 years later he wrote: "In the same year (1665) I began to think of gravity extending to the orb of the Moon . . . and having thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the Earth, and found them to answer pretty nearly."

Newton's young friend William Stukeley wrote of having tea with Newton under some apple trees when New-

* See "Copernicus and Tycho," by Owen Gingerich, *Scientific American*, December 1973, p. 86.

† See "How Did Kepler Discover His First Two Laws," by Curtis Wilson, *Scientific American*, March 1972, p. 92.

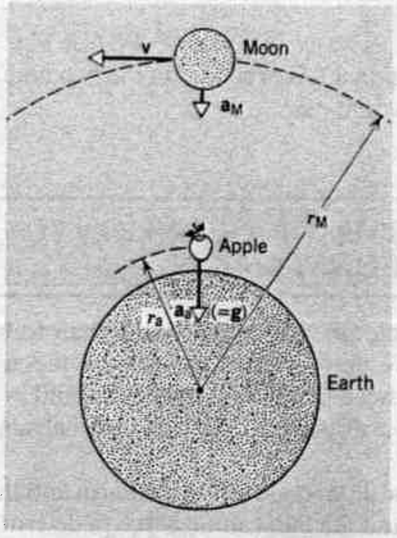

Figure 2 Both the Moon and the apple are accelerated toward the center of the Earth. The difference in their motions arises because the Moon has enough tangential speed v to maintain a circular orbit.

ton said that the setting was the same as when he got the idea of gravitation. "It was occasion'd by the fall of an apple, as he sat in a contemplative mood . . . and thus by degrees he began to apply this property of gravitation to the motion of the Earth and the heavenly bodys . . . " (see Fig. 2).

We can compute the acceleration of the Moon toward the Earth from its period of revolution and the radius of its orbit. We obtain 0.0027 m/s² (see Sample Problem 5, Chapter 4). This value is about a factor of 3600 smaller than g, the free-fall acceleration at the surface of the Earth. Newton, guided by Kepler's third law (see Problem 58), sought to account for this difference by assuming that the acceleration of a falling body is inversely proportional to the square of its distance from the Earth.

The question of what we mean by "distance from the Earth" immediately arises. Newton eventually came to regard every particle of the Earth as contributing to the gravitational attraction it had on other bodies. He made the daring assumption that the mass of the Earth could be treated as if it were all concentrated at its center. (See Section 16-5.)

We can treat the Earth as a particle with respect to the Sun, for example. It is not obvious, however, that we can treat the Earth as a particle with respect to an apple located only a couple of meters above its surface. If we do make this assumption, a falling body near the Earth's surface is a distance of one Earth radius (6400 km) from the effective center of attraction of the Earth. The Moon is about 380,000 km away. The inverse square of the ratio of these distances is $(6400/380{,}000)^2 = 1/3600$, in agreement with the ratio of the accelerations of the Moon and the apple. In Newton's words quoted above, it does indeed "answer pretty nearly."

There are three overlapping realms in which we can discuss gravitation. (1) The gravitational attraction between two bowling balls, for example, although measurable by sensitive techniques, is too weak to fall within our ordinary sense perceptions. (2) The attraction of ourselves and objects around us by the Earth is a controlling feature of our lives from which we can escape only by extreme measures. The designers of our space program have the gravitational force constantly in mind. (3) On the scale of the solar system and of the interaction of stars and galaxies, gravitation is by far the dominant force. It is remarkable that all three situations can be described by the same force law.

This force law, Newton's law of universal gravitation, can be stated as follows:

> *Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of this force is along the line joining the particles.*

Thus the magnitude of the gravitational force F that two particles of masses m_1 and m_2 separated by a distance r exert on each other is

$$F = G\frac{m_1 m_2}{r^2}. \quad (1)$$

Here G, called the gravitational constant, is a universal constant that has the same value for all pairs of particles.

It is important to note that the gravitational forces between two particles are an action–reaction pair. The first particle exerts a force on the second particle that is directed toward the first particle along the line joining them. Likewise, the second particle exerts a force on the first particle that is directed toward the second particle along the line joining them. These forces are equal in magnitude but oppositely directed.

The universal constant G must not be confused with the g that is the acceleration of a body arising from the Earth's gravity. The constant G has the dimensions L^3/MT^2 and is a scalar, while g is the magnitude of a vector, has the dimensions L/T^2, and is neither universal nor constant.

Notice that Newton's law of universal gravitation is not a defining equation for any of the physical quantities (force, mass, or length) contained in it. According to our program for classical mechanics in Chapter 5, force is defined from Newton's second law, $\mathbf{F} = m\mathbf{a}$. The force $\mathbf{F}$ on a particle is assumed to be related in a simple way to the measurable properties of a particle and its environment. The law of universal gravitation is such a simple law. Once G is determined from experiment for any pair of bodies, that value of G can be used in the law of gravitation to determine the gravitational force between any other pair of bodies.

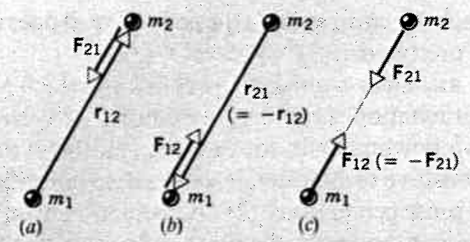

Figure 3 (*a*) The force $\mathbf{F}_{21}$ exerted on m_2 (by m_1) is directed opposite to the displacement, $\mathbf{r}_{12}$, of m_2 from m_1. (*b*) The force $\mathbf{F}_{12}$ exerted on m_1 (by m_2) is directed opposite to the displacement, $\mathbf{r}_{21}$, of m_1 from m_2. (*c*) $\mathbf{F}_{12} = -\mathbf{F}_{21}$, the forces being an action–reaction pair.

Notice also that Eq. 1 expresses the force between *particles.* If we want to determine the force between extended bodies, as, for example, the Earth and the Moon, we must regard each body as composed of particles. Then the interaction between all particles must be computed. Integral calculus makes such a calculation possible. Newton's motive in developing the calculus arose in part from a desire to solve such problems. Although it is in general incorrect to assume that all the mass of a body can be concentrated at its center of mass for gravitational purposes, this assumption *is* correct for spherically symmetric bodies. We often use this result, which we prove in Section 16-5.

Experiment strongly suggests that the gravitational force between two particles is independent of the presence of other bodies and of the properties of the medium in which the particles are immersed. The gravitational force between two bowling balls remains unchanged whether the balls are in free space, are under water, or are separated by a brick wall. The "gravity screens" of science fiction have no basis in fact.

The law of universal gravitation is a *vector law,* which can be expressed as follows. Let the displacement vector $\mathbf{r}_{12}$ point from the particle of mass m_1 to the particle of mass m_2, as Fig. 3*a* shows. The gravitational force $\mathbf{F}_{21}$, exerted on m_2 by m_1, is given in direction and magnitude by the vector relation

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{r_{12}^3}\mathbf{r}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\frac{\mathbf{r}_{12}}{r_{12}}, \qquad (2a)$$

in which r_{12} is the magnitude of $\mathbf{r}_{12}$. The minus sign in Eq. 2*a* shows that $\mathbf{F}_{21}$ points in a direction opposite to $\mathbf{r}_{12}$; that is, the gravitational force is attractive, m_2 experiencing a force directed toward m_1. The displacement vector divided by its own magnitude, $\mathbf{r}_{12}/r_{12}$, is simply a unit vector $\mathbf{u}_r$ in the direction of the displacement, so the last part of Eq. 2*a* shows the inverse-square nature of the force.

The force exerted on m_1 by m_2 (see Fig. 3*b*) is similarly

$$\mathbf{F}_{12} = -G\frac{m_2 m_1}{r_{21}^3}\mathbf{r}_{21} = -G\frac{m_2 m_1}{r_{21}^2}\frac{\mathbf{r}_{21}}{r_{21}}. \qquad (2b)$$

Note in Eqs. 2*a* and 2*b* that $\mathbf{r}_{21} = -\mathbf{r}_{12}$ (see Figs. 3*a* and 3*b*) so that, as we expect, $\mathbf{F}_{12} = -\mathbf{F}_{21}$ (see Fig. 3*c*); that is, the gravitational forces acting on the two bodies form an action–reaction pair.

16-3 THE GRAVITATIONAL CONSTANT *G*

Determining the value of G would seem to be a simple task. All we need to do is to measure the gravitational force F between two known masses m_1 and m_2 separated by a known distance r. We can then calculate G from Eq. 1.

A large-scale system such as the Earth and the Moon or the Earth and the Sun cannot serve to determine G. The distances are large enough that the objects can be regarded as approximately point masses, but the values of the masses are not determined independently. In fact, the masses of these bodies, as we shall soon discuss, are determined using the value of G.

Instead, we must turn to a small-scale measurement, in which we use two laboratory objects of known mass and measure the force between them. The force is very weak, and the masses must be placed close together to make the force as large as possible. When we do this, we can usually no longer regard the masses as point particles, and Eq. 1 may not be applicable. There is, however, one special case in which we can use Eq. 1 for large objects. As we prove in Section 16-5, for spherical mass distributions we can regard the object as a point mass concentrated at its center. This is *not* an approximation; it is an exact relationship.

The first laboratory determination of G from the force between spherical masses at close distance was done by Henry Cavendish in 1798. He used a method based on the torsion balance, illustrated in Fig. 4. Two small balls, each of mass m, are attached to the ends of a light rod. This rigid "dumbbell" is suspended, with its axis horizontal, by a fine vertical fiber. Two large balls each of mass M are placed near the ends of the dumbbell on opposite sides. When the large masses are in the positions A, they attract the small masses according to the law of gravitation, and a torque is exerted on the dumbbell, rotating it counterclockwise as viewed from above. The rod reaches an equilibrium position under the opposing actions of the gravitational torque exerted by the masses M and the restoring torque exerted by the fiber. When the large masses are in the positions B, the dumbbell rotates clockwise to a new equilibrium position. The angle 2θ, through which the fiber is twisted when the balls are moved from one position (AA) to the other (BB), is measured by observing the deflection of a beam of light reflected from the small mirror attached to the fiber. From the value of θ and the torsional constant of the fiber (determined by measuring its period of oscillation—see Section 15-5), the torque

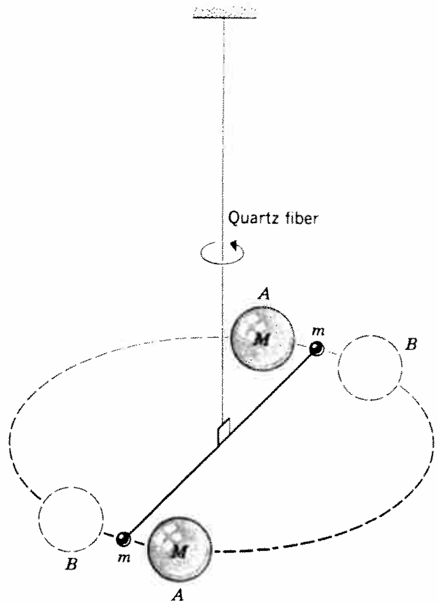

Figure 4 A schematic view of the apparatus used in 1798 by Henry Cavendish to measure the gravitational constant G. The large spheres of mass M, shown in location AA, can also be moved to location BB.

can be determined and the gravitational force can be obtained. Knowing the values of the masses m and M and the separation of their centers, we can calculate G. (See Sample Problem 1.)

Cavendish's original experiment gave a value for G of $6.75 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. In the nearly 200 years since the time of Cavendish, the same basic technique using the torsion balance has been used to repeat this measurement many times, leading to the presently accepted value of G,

$$G = 6.67259 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2},$$

with an uncertainty of $\pm 0.00085 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ or about $\pm 0.013\%$. Compared with the results of measuring other physical constants, this precision is not impressive; for example, the speed of light was measured to a precision of about $10^{-8}\%$ before its value was set as a standard. It is difficult to improve substantially on the precision of the measured value of G because of its small magnitude and the correspondingly small value of the force between the two objects in our laboratory experiments. If we use two lead spheres of diameter 10 cm (and mass 6 kg), the maximum gravitational force between them when they are as close as possible is about 2×10^{-7} N, corresponding roughly to the weight of a piece of paper of area 1 $\mathrm{mm^2}$.

This difficulty of measuring G is unfortunate, because gravitation has such an essential role in theories of the origin and structure of the universe. For example, we would like to know if G really is a constant. Does it change with time? Does it depend on the chemical or physical state of the masses? Does it depend on their temperature? Despite many experimental searches, no such variations in G have so far been unambiguously confirmed, but measurements continue to be refined and improved, and the experimental tests continue.*

The large gravitational force exerted by the Earth on all bodies near its surface is due to the large mass of the Earth. In fact, the mass of the Earth can be determined from the law of universal gravitation and the value of G calculated from the Cavendish experiment. For this reason Cavendish is said to have been the first person to "weigh" the Earth. (In fact, the title of the paper written by Cavendish describing his experiments referred not to measuring G but instead to determining the density of the Earth from its weight and volume.) Consider the Earth, of mass M_E, and an object on its surface of mass m. The force of attraction is given both by

$$F = mg \quad \text{and} \quad F = \frac{GmM_E}{R_E^2}.$$

Here R_E is the radius of the Earth, which is the separation of the two bodies, and g is the free-fall acceleration at the Earth's surface. Combining these equations we obtain

$$M_E = \frac{gR_E^2}{G} = \frac{(9.80\ \mathrm{m/s^2})(6.37 \times 10^6\ \mathrm{m})^2}{6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}}$$
$$= 5.97 \times 10^{24}\ \mathrm{kg}.$$

Dividing the mass of the Earth by its volume, we obtain the average density of the Earth to be 5.5 $\mathrm{g/cm^3}$, or about 5.5 times the density of water. The average density of the rocks on the Earth's surface is much less than this value. We conclude that the interior of the Earth contains material of density greater than 5.5 $\mathrm{g/cm^3}$. The Cavendish experiment has given us information about the Earth's core! (See Problem 26.)

Sample Problem 1 In the Cavendish apparatus illustrated in Fig. 4, suppose $M = 12.7$ kg and $m = 9.85$ g. The length L of the rod connecting the two small spheres is 52.4 cm. The rod and the fiber form a torsion pendulum whose rotational inertia I about the central axis is $1.25 \times 10^{-3}\ \mathrm{kg \cdot m^2}$ and whose period of oscillation T is 769 s. The angle 2θ between the two equilibrium positions of the rod is 0.516° when the distance R between the centers of the large and small spheres is 10.8 cm. What is the value of the gravitational constant resulting from these data?

Solution Let us first find κ, the torsional constant of the fiber.

* For a list of references to measurements of G, see "The Newtonian Gravitational Constant," by George T. Gillies, *Metrologia*, Vol. 24, p. 1, 1987. A discussion of these experiments and others testing the inverse-square law can be found in "Experiments on Gravitation," by Alan Cook, *Reports on Progress in Physics*, Vol. 51, p. 707, 1988.

The period of torsional oscillation is given by Eq. 21 of Chapter 15,

$$T = 2\pi\sqrt{\frac{I}{\kappa}}.$$

Solving for κ yields

$$\kappa = \frac{4\pi^2 I}{T^2} = \frac{(4\pi^2)(1.25 \times 10^{-3}\ \text{kg}\cdot\text{m}^2)}{(769\ \text{s})^2} = 8.34 \times 10^{-8}\ \text{N}\cdot\text{m}.$$

The rod is in equilibrium under the influence of two opposing torques resulting from the actions of the fiber and of the large spheres. The magnitude of the torque exerted by the fiber is related to the angular displacement θ according to Eq. 17 of Chapter 15,

$$\tau = \kappa\theta = (8.34 \times 10^{-8}\ \text{N}\cdot\text{m})\left(\frac{0.516^\circ}{2} \times \frac{2\pi\ \text{rad}}{360^\circ}\right)$$
$$= 3.75 \times 10^{-10}\ \text{N}\cdot\text{m}.$$

This torque is balanced by the total torque due to the gravitational force exerted by each large sphere on the nearby small sphere. The force F on each small sphere is equal to GMm/R^2, and the moment arm is one-half the length L of the rod. The total gravitational torque is then

$$\tau = (2F)(L/2) = FL = \frac{GMmL}{R^2}.$$

Solving for G yields

$$G = \frac{\tau R^2}{MmL} = \frac{(3.75 \times 10^{-10}\ \text{N}\cdot\text{m})(0.108\ \text{m})^2}{(12.7\ \text{kg})(0.00985\ \text{kg})(0.524\ \text{m})}$$
$$= 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2.$$

Sample Problem 2 Calculate the gravitational forces (*a*) between two 7.3-kg bowling balls separated by 0.65 m between their centers and (*b*) between the Earth and the Moon.

Solution (*a*) Using Eq. 1, we have

$$F = \frac{Gm_1m_2}{r^2} = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(7.3\ \text{kg})(7.3\ \text{kg})}{(0.65\ \text{m})^2}$$
$$= 8.4 \times 10^{-9}\ \text{N}.$$

(*b*) Using data for the Earth and the Moon from Appendix C, we find

$$F = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})(7.36 \times 10^{22}\ \text{kg})}{(3.82 \times 10^{8}\ \text{m})^2}$$
$$= 2.01 \times 10^{20}\ \text{N}.$$

16-4 GRAVITY NEAR THE EARTH'S SURFACE

Let us assume, for the time being, that the Earth is spherical and that its density depends only on the radial distance from its center. The magnitude of the gravitational force acting on a particle of mass m, located at an external point a distance r from the Earth's center, can then be written, from Eq. 1, as

$$F = G\frac{M_E m}{r^2},$$

in which M_E is the mass of the Earth. This gravitational force can also be written, from Newton's second law, as

$$F = mg_0.$$

Here g_0 is the free-fall acceleration due only to the gravitational pull of the Earth. Combining the two equations above gives

$$g_0 = \frac{GM_E}{r^2}. \quad (3)$$

TABLE 1 VARIATION OF g_0 WITH ALTITUDE

Altitude (km)	g_0 (m/s²)
0	9.83
5	9.81
10	9.80
50	9.68
100	9.53
400[a]	8.70
35,700[b]	0.225
380,000[c]	0.0027

[a] A typical space shuttle altitude.
[b] The altitude of communication satellites.
[c] The distance to the Moon.

Table 1 shows some values of g_0 at various altitudes above the surface of the Earth, calculated from this equation. Note that, contrary to the impression that gravity drops to zero in an orbiting satellite, we find $g_0 = 8.7\ \text{m/s}^2$ at typical space shuttle altitudes.

The real Earth differs from our model Earth in three ways.

1. *The Earth's crust is not uniform.* There are local density variations everywhere. The precise measurement of local variations in the free-fall acceleration gives information that is useful, for example, for oil prospecting. Figure 5 shows a gravity survey over an underground salt dome. The contours connect points with the same free-fall acceleration, plotted as deviations from a convenient reference value. The unit, named to honor Galileo, is the milligal, where 1 gal $= 10^3$ mgal $= 1\ \text{cm/s}^2$.

2. *The Earth is not a sphere.* The Earth is approximately an ellipsoid, flattened at the poles and bulging at the equator. The Earth's equatorial radius is greater than its polar radius by 21 km. Thus a point at the poles is closer to the dense core of the Earth than is a point on the equator. We would expect that the free-fall acceleration would increase as one proceeds, at sea level, from the equator toward the

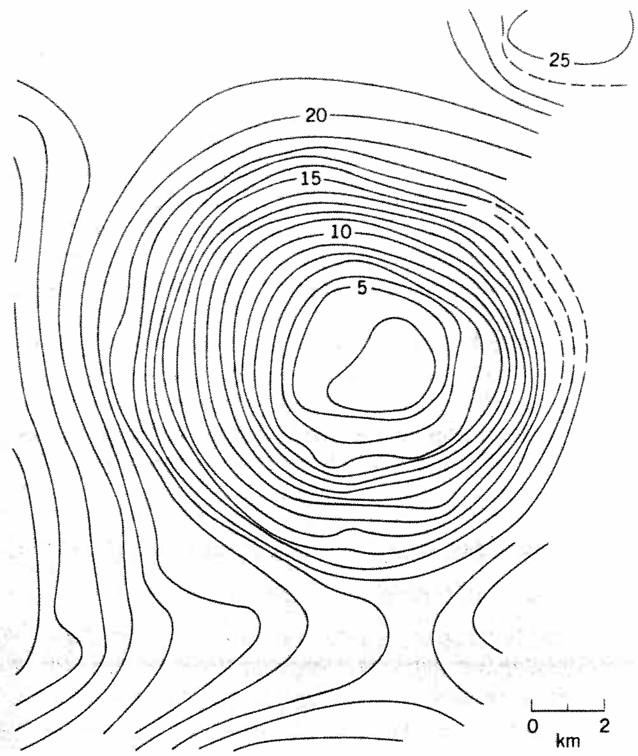

Figure 5 A surface gravity survey over an underground salt dome in Denmark. The lines connect points with the same value of g. The difference between the value of g on a contour and the value at the center is in units of milligal, equivalent to 10^{-5} m/s² or about 10^{-6} g. It is clear that something buried here is exerting a force centered in this region. Oil is often found in such formations.

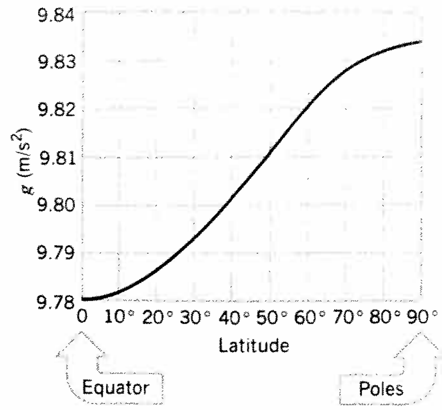

Figure 6 The variation of g with latitude at sea level. About 65% of the effect is due to the rotation of the Earth, with the remaining 35% coming from the Earth's slightly flattened shape.

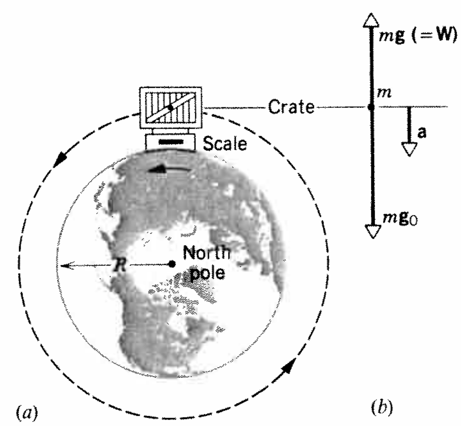

Figure 7 (*a*) A crate on the rotating Earth, resting on a platform scale at the equator. The view is along the Earth's rotational axis, looking down on the north pole. (*b*) A free-body diagram of the crate. The crate is in uniform circular motion and is thus accelerated toward the center of the Earth.

poles. Figure 6 shows that this is indeed what happens. The measured values of g in this figure include both the equatorial bulge effect and effects resulting from the rotation of the Earth.

3. *The Earth is rotating.* Figure 7*a* shows the rotating Earth from a position in space above the north pole. A crate of mass m rests on a platform scale at the equator. This crate is in uniform circular motion because of the Earth's rotation and is accelerated toward the center of the Earth. The resultant force acting on it must then point in that direction.

Figure 7*b* is a free-body diagram for the crate. The Earth exerts a downward gravitational pull of magnitude mg_0. The scale platform pushes up on the crate with a force mg, the weight of the crate. These two forces do not quite balance, and we have, from Newton's second law,

$$F = mg_0 - mg = ma$$

or

$$g_0 - g = a,$$

in which a is the centripetal acceleration of the crate. For a we can write $\omega^2 R_E$, where ω is the Earth's angular rotation rate and R_E is its radius. Making this substitution leads to

$$g_0 - g = \omega^2 R_E = \left(\frac{2\pi}{T}\right)^2 R_E, \tag{4}$$

in which $T = 24$ h, the Earth's period of rotation. Substituting numerical values in Eq. 4 yields

$$g_0 - g = 0.034 \text{ m/s}^2.$$

We see that g, the measured free-fall acceleration on the equator of the rotating Earth, is less than g_0, the expected result if the Earth were not rotating, by only 0.034/9.8 or 0.35%. The effect decreases as one goes to higher latitudes and vanishes at the poles.

Sample Problem 3 (*a*) A neutron star is a collapsed star of extremely high density. The blinking pulsar in the Crab nebula is the best known of many examples. Consider a neutron star with a mass M equal to the mass of the Sun, 1.99×10^{30} kg, and a radius R of 12 km. What is the free-fall acceleration at its surface? Ignore rotational effects. (*b*) The asteroid Ceres has a mass of 1.2×10^{21} kg and a radius of 470 km. What is the free-fall acceleration at its surface?

Solution (*a*) From Eq. 3 we have

$$g_0 = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2})(1.99 \times 10^{30}\ \mathrm{kg})}{(12{,}000\ \mathrm{m})^2}$$
$$= 9.2 \times 10^{11}\ \mathrm{m/s^2}.$$

Even though pulsars rotate extremely rapidly, rotational effects have only a small influence on the value of g, because of the small size of pulsars.

(*b*) In the case of the asteroid Ceres, we have

$$g_0 = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2})(1.2 \times 10^{21}\ \mathrm{kg})}{(4.7 \times 10^5\ \mathrm{m})^2}$$
$$= 0.36\ \mathrm{m/s^2}.$$

There is quite a contrast between the gravitational forces on the surfaces of these two bodies!

16-5 GRAVITATIONAL EFFECT OF A SPHERICAL DISTRIBUTION OF MATTER *(Optional)*

We now prove a result we have already used: *a spherically symmetric body attracts particles outside it as if its mass were concentrated at its center.* We begin by considering a uniformly dense spherical shell of mass M whose thickness t is small compared to its radius R (Fig. 8). We seek the gravitational force it exerts on an external particle P of mass m.

We assume that each particle of the shell exerts on P a force that is proportional to the mass of the particle, inversely proportional to the square of the distance between that particle of the shell and P, and directed along the line joining them. We must then obtain the resultant force on P, attributable to all parts of the spherical shell.

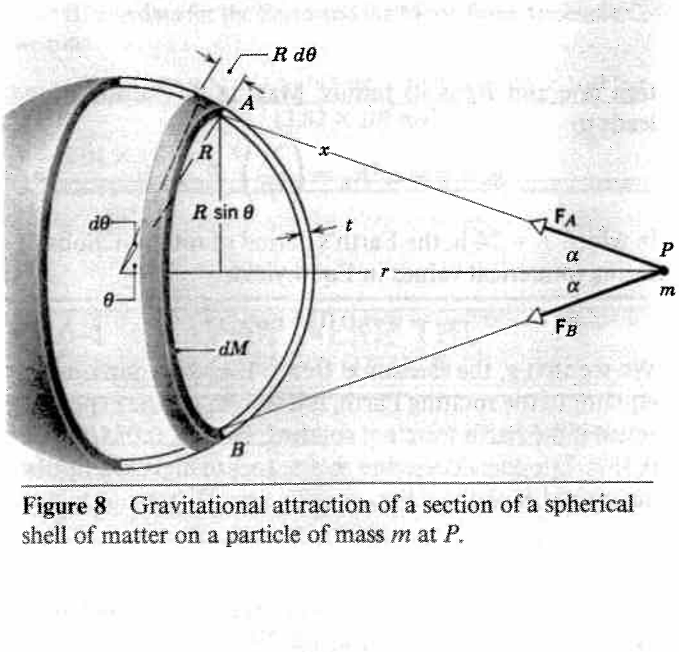

Figure 8 Gravitational attraction of a section of a spherical shell of matter on a particle of mass m at P.

A small part of the shell at A attracts m with a force $\mathbf{F}_A$. A small part of equal mass at B, equally far from m but diametrically opposite A, attracts m with a force $\mathbf{F}_B$. The resultant of these two forces on m is $\mathbf{F}_A + \mathbf{F}_B$. Each of these forces has a component $F \cos \alpha$ along the symmetry axis and a component $F \sin \alpha$ perpendicular to the axis. The perpendicular components of $\mathbf{F}_A$ and $\mathbf{F}_B$ cancel, as they do for all such pairs of opposite points. To find the resultant force on P for all points on the shell, we need consider only the components parallel to the axis.

Let us take as our element of mass of the shell a circular strip dM. Its radius is $R \sin \theta$, its length is $2\pi(R \sin \theta)$, its width is $R\, d\theta$, and its thickness is t. Hence it has a volume

$$dV = 2\pi t R^2 \sin \theta\, d\theta.$$

Let the density of the shell be ρ, so that the mass within the strip is

$$dM = \rho\, dV = 2\pi t \rho R^2 \sin \theta\, d\theta.$$

Every particle in the ring, such as one of mass dm_A at A, attracts P with a force that has an axial component

$$dF_A = G\frac{m\, dm_A}{x^2} \cos \alpha.$$

Adding the contributions for all the particles in the ring gives

$$dF_A + dF_B + \cdots = \frac{Gm}{x^2}(\cos \alpha)(dm_A + dm_B + \cdots)$$

or

$$dF = \frac{Gm\, dM}{x^2} \cos \alpha,$$

where dM is the total mass of the ring and dF is the total force on m exerted by the ring.

Substituting for dM, we obtain

$$dF = 2\pi G t \rho m R^2 \frac{\sin \theta\, d\theta}{x^2} \cos \alpha. \qquad (5)$$

The variables x, α, and θ are related. From the figure we see that

$$\cos \alpha = \frac{r - R \cos \theta}{x}. \qquad (6)$$

Using the law of cosines, $x^2 = r^2 + R^2 - 2rR \cos \theta$, we obtain

$$R \cos \theta = \frac{r^2 + R^2 - x^2}{2r}. \qquad (7)$$

Differentiating Eq. 7 gives

$$\sin \theta\, d\theta = \frac{x}{rR}\, dx. \qquad (8)$$

We now put Eq. 7 into Eq. 6 and then put Eqs. 6 and 8 into Eq. 5. As a result we eliminate θ and α and obtain

$$dF = \frac{\pi G t \rho m R}{r^2}\left(\frac{r^2 - R^2}{x^2} + 1\right) dx. \qquad (9)$$

This is the force exerted by the circular strip dM on the particle m.

We must now consider every element of mass in the shell by summing over all the circular strips in the entire shell. This involves an integration over the shell with respect to the variable x, which ranges from a minimum value of $r - R$ to a maximum value $r + R$. The needed integral is

$$\int_{r-R}^{r+R} \left(\frac{r^2 - R^2}{x^2} + 1 \right) dx = \left[\frac{-(r^2 - R^2)}{x} + x \right] \Bigg|_{r-R}^{r+R} = 4R,$$

which gives for the force, using Eq. 9,

$$F = \int_{r-R}^{r+R} dF = \frac{\pi G t p m R}{r^2} (4R) = G \frac{Mm}{r^2}, \qquad (10)$$

where

$$M = 4\pi R^2 tp$$

is the total mass of the shell. Equation 10 is exactly the same result we would obtain for the force between *particles* of mass M and m separated by a distance r. We have therefore proved the following important general result:

A uniformly dense spherical shell attracts an external point mass as if all the mass of the shell were concentrated at its center.

A solid sphere can be regarded as composed of a large number of concentric shells. If each spherical shell has a uniform density, even though different shells may have different densities, the same result applies to the solid sphere. Hence a body such as the Earth, the Moon, or the Sun, to the extent that they are such spheres, may be regarded gravitationally as point particles to bodies outside them.

Keep in mind that our proof applies only to spheres and only when the density is uniform over the sphere or a function of radius alone.

Force on an Interior Particle

We now prove another important result: the force exerted by a spherical shell on a particle *inside* it is *zero*. Figure 9 shows the particle at point P inside the shell. Notice that r is now smaller than R. The integration over x, now with the limits $R - r$ to $r + R$, gives

$$\int_{R-r}^{r+R} \left(\frac{r^2 - R^2}{x^2} + 1 \right) dx = \left[\frac{-(r^2 - R^2)}{x} + x \right] \Bigg|_{R-r}^{r+R} = 0,$$

and so $F = 0$. Thus we obtain another general result:

A uniform spherical shell of matter exerts no gravitational force on a particle located inside it.

This last result, although not obvious, is plausible because the mass elements of the shell to the left and to the right of m in Fig. 9 now exert forces of opposite directions on m. There is more mass on the left that pulls m to the left, but the smaller mass on the right is closer to m; the two effects exactly cancel only if the force varies precisely as an inverse square of the separation distance of two particles. (See Problem 29.) Important consequences of this result will be discussed in the chapters on electricity. There we shall see that the electrical force between charged particles also depends inversely on the square of the distance between them.

The above result for a particle inside a spherical shell implies that the gravitational force exerted by the Earth on a particle decreases as the particle goes deeper into the Earth, assuming a constant density for the Earth. As the particle goes deeper, more of the Earth's mass is in shells that are external to the location of the particle, and the net force on the particle from those shells is zero. The gravitational force becomes zero at the center of the Earth. Hence g would be a maximum at the Earth's surface and decrease both outward and inward from that point if the Earth had constant density. Can you imagine a spherically symmetric distribution of the Earth's mass which would not give this result? (See Problem 26.)

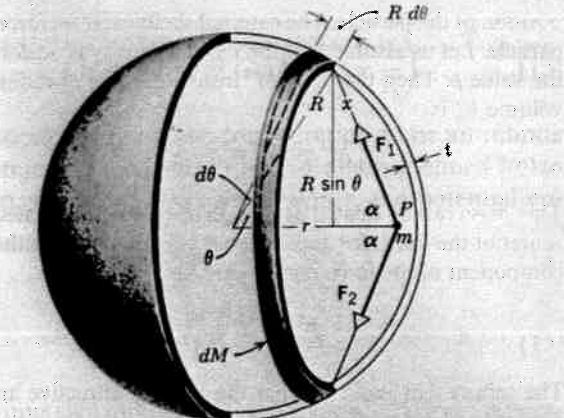

Figure 9 Gravitational attraction of a section of a spherical shell of matter on a particle of mass m at a point P inside the shell.

Sample Problem 4 Suppose a tunnel could be dug through the Earth from one side to the other along a diameter, as shown in Fig. 10. (*a*) Show that the motion of a particle dropped into the tunnel is simple harmonic motion. Neglect all frictional forces and assume that the Earth has a uniform density. (*b*) If mail were delivered through this chute, how much time would elapse between deposit at one end and delivery at the other end?

Solution (*a*) The gravitational attraction of the Earth for the particle at a distance r from the center of the Earth arises entirely from that portion of matter of the Earth in shells internal to the

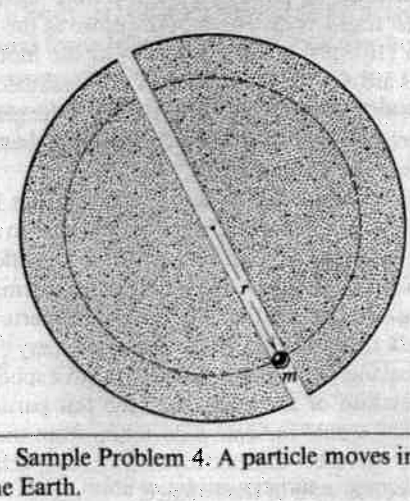

Figure 10 Sample Problem 4. A particle moves in a tunnel through the Earth.

position of the particle. The external shells exert no force on the particle. Let us assume that the Earth's density is uniform with the value ρ. Then the mass M' inside a sphere of radius r and volume V' is

$$M' = \rho V' = \rho \frac{4\pi r^3}{3}.$$

This mass can be treated as though it were concentrated at the center of the Earth for gravitational purposes. Hence the radial component of the force on the particle of mass m is

$$F = -\frac{GM'm}{r^2}.$$

The minus sign indicates that the force is attractive and thus directed toward the center of the Earth.

Substituting for M', we obtain

$$F = -G\frac{\rho 4\pi r^3 m}{3r^2} = -\left(G\rho\frac{4\pi m}{3}\right)r = -kr.$$

Here $G\rho 4\pi m/3$ is a constant, which we have called k. The force is therefore proportional to the displacement r but oppositely directed. This is exactly the criterion for simple harmonic motion.

(*b*) The period of this simple harmonic motion is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3m}{G\rho 4\pi m}} = \sqrt{\frac{3\pi}{G\rho}}.$$

With $\rho = 5.51 \times 10^3$ kg/m^3, we have

$$T = \sqrt{\frac{3\pi}{G\rho}} = \sqrt{\frac{3\pi}{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.51 \times 10^3\ \text{kg/m}^3)}}$$
$$= 5060\ \text{s} = 84.4\ \text{min}.$$

The time for delivery is one-half period, or about 42 min. This time is *independent of the mass of the mail.* It can be shown that the same period results if the tunnel is dug along any chord instead of along a diameter.

The Earth's density is not really uniform. What would be the effect on this problem if we took ρ to be some function of r, rather than a constant?

Testing the Inverse-Square Law

As we discuss in Section 16-8, Kepler's laws give direct evidence for a $1/r^2$ gravitational force. We can therefore regard the $1/r^2$ law to be well tested at distances of the order of the size of the solar system (10^{13} m). Small exceptions in the motion of the inner planets are explained by Einstein's general theory of relativity, which supersedes Newton's law when the gravitational force is intense but which reduces to Newton's law when the force is weaker; see Section 16-10.

We would therefore like to test the $1/r^2$ law at laboratory distances. Because the force is so weak, it is difficult to make such a test by repeating the Cavendish experiment with different separations between the masses. A more precise method makes use of the vanishing of the gravitational force on a test particle inside a spherical shell. If we could isolate a test particle, say on one arm of a torsion balance, and then surround it with a spherical shell, any slight rotation of the balance as the test particle moves within the shell would indicate a deviation from the $1/r^2$ law. The rotation could be detected by a suitable mechanism attached to the other arm of the balance.

Unfortunately, surrounding a test mass with a spherical shell and moving it inside present great technical difficulties; as an alternative a long cylinder is used instead. From a calculation similar to the one we used for the spherical shell, it can be shown that the gravitational force exerted by a long cylindrical shell on a test mass inside the cylinder vanishes if the cylinder is infinitely long; for a cylinder of finite length a small but easily calculable correction must be applied.

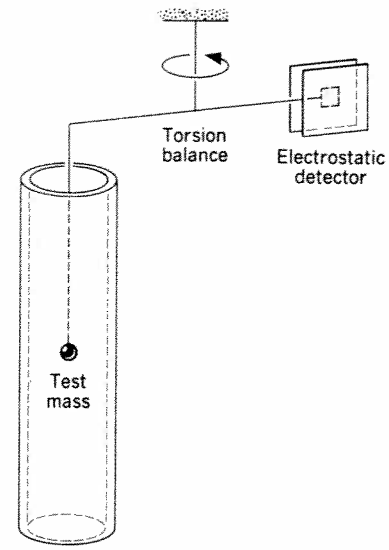

Figure 11 A test mass inside a long cylinder. For a $1/r^2$ force, the gravitational attraction between the test mass and the cylinder should vanish (neglecting effects of the ends). A torsion balance allows changes in the force on the test mass to be measured at different locations inside the cylinder.

Figure 11 shows the geometry for a typical experiment. As the test mass is moved in a horizontal plane, variations in the gravitational force between the cylinder and the test mass would be detectable with the torsion balance. If the gravitational force between particles had a variation other than $1/r^2$, the force on the test mass would not vanish and would vary as the test mass moves in the horizontal plane.

Such experiments show that the force is indeed of the form $1/r^2$ at laboratory distances (centimeters to meters). One way of expressing the results of these experiments is to assume the force to be of the form $1/r^{2+\delta}$ where $\delta = 0$ in the Newtonian theory, and then to show that the experiment places a small upper limit on δ. The present upper limit on δ is about 10^{-4}; at the best precision obtainable from laboratory studies, there appears to be no deviation from the $1/r^2$ form of the law of gravitation. By comparison, experiments testing the $1/r^2$ force between electric charges (see Section 29-6) give an upper limit of about 10^{-16} on δ in that case. ■

16-6 GRAVITATIONAL POTENTIAL ENERGY

In Chapter 8 we discussed the gravitational potential energy of a particle (mass m) and the Earth (mass M). We considered only the special case in which the particle re-