to rest at the bottom of the shell. We choose our origin to coincide with the initial position of the center of the shell. Figure 11*b* shows that, with respect to this origin, the center of mass of the ball–shell system is located a distance $\frac{1}{2}R$ to the left, halfway between the two particles. Figure 11*d* shows that the displacement of the shell is given by

$$d = \tfrac{1}{2}R.$$

The shell must move to the left through this distance as the ball comes to rest.

The ball is brought to rest by the frictional force that acts between it and the shell. Why does this frictional force not affect the final location of the center of mass?

9-4 LINEAR MOMENTUM OF A PARTICLE

The momentum of a single particle is a vector **p** defined as the product of its mass m and its velocity **v**:

$$\mathbf{p} = m\mathbf{v}. \tag{19}$$

Momentum, being the product of a scalar by a vector, is itself a vector. Because it is proportional to **v**, the momentum **p** of a particle depends on the reference frame of the observer; we must always specify this frame.

Newton, in his famous *Principia,* expressed the second law of motion in terms of momentum (which he called "quantity of motion"). Expressed in modern terminology Newton's second law reads:

> *The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force.*

In symbolic form this becomes

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{20}$$

Here $\Sigma\,\mathbf{F}$ represents the resultant force acting on the particle.

For a single particle of constant mass, this form of the second law is equivalent to the form $\mathbf{F} = m\mathbf{a}$ that we have used up to now. That is, if m is constant, then

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

The relations $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = d\mathbf{p}/dt$ for single particles are completely equivalent in classical mechanics.

A convenient relationship between momentum and kinetic energy is found by combining $K = \frac{1}{2}mv^2$ and $p = mv$, which gives

$$K = \frac{p^2}{2m}. \tag{21}$$

Momentum at High Speeds *(Optional)*

At particle speeds close to the speed of light (a region in which relativity theory must be used in place of Newtonian mechanics), Newton's second law in the form $\mathbf{F} = m\mathbf{a}$ is no longer valid. However, it turns out that Newton's second law in the form $\mathbf{F} = d\mathbf{p}/dt$ is still a valid law if the momentum **p** for a single particle is defined not as $m\mathbf{v}$ but as

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \tag{22}$$

in which c is the speed of light. At ordinary speeds ($v \ll c$), Eq. 22 reduces to Eq. 19.

For relativistic particles, the basic relationship between momentum and kinetic energy can be shown to be

$$K = \sqrt{(pc)^2 + (mc^2)^2} - mc^2. \tag{23}$$

We shall derive this result in Chapter 21. Figure 12 shows a comparison between the classical (Eq. 21) and relativistic (Eq. 23) results for particles of a range of velocities. Obviously the classical result fails at high speed. As expected (see Problem 27), Eq. 23 reduces to Eq. 21 at ordinary speeds.

No matter in what form we write the kinetic energy, it has dimensions of mass times velocity squared, which is the same as momentum times velocity. We can therefore write, using our notation of Section 1-7 to indicate dimensions,

$$[p] = \frac{[K]}{[v]}.$$

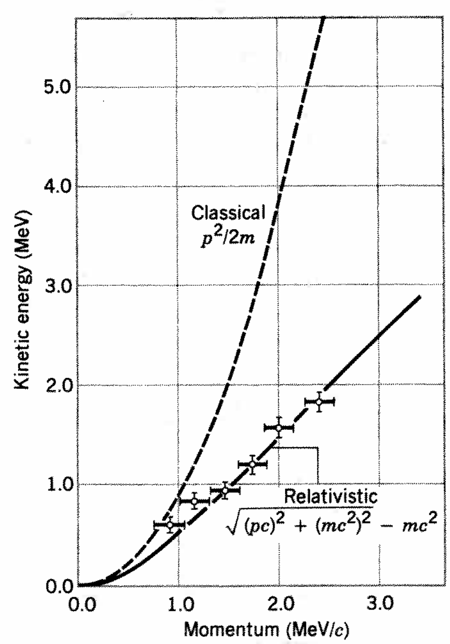

Figure 12 A comparison of the classical (Eq. 21) and relativistic (Eq. 23) relationships between momentum and kinetic energy for electrons emitted in certain radioactive decay processes. The circles represent the experimental measurements; the horizontal and vertical bars passing through the circles represent the range of uncertainty of these measurements. The data obviously favor the relativistic relationship. Notice that at low velocity (small energy and momentum) the two relationships are indistinguishable.

It is often convenient to express momentum in units of energy divided by velocity, and convenient choices in working with particles are eV/c, MeV/c, and so on. This allows us to express the quantity pc in energy units such as MeV, which makes it much more convenient in working with expressions like Eq. 23. For an electron with a momentum given as 1.5 MeV/c, for example, the term pc in Eq. 23 is 1.5 MeV and the kinetic energy of the electron can easily be calculated from that equation to be 1.1 MeV.

In the region of *very* high particle speeds, the particle momentum p can be so great that the term pc in Eq. 23 becomes much larger than the term mc^2, and that equation then reduces to $K = pc$ to a good approximation. Expressing momentum in units of energy divided by c is especially useful in this region. For example, an electron whose momentum is given as 500 MeV/c has a kinetic energy very close to 500 MeV. (Note that this approximation is a very poor one for the 1.5-MeV electron considered above.) ■

9-5 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Suppose that instead of a single particle we have a system of N particles, with masses $m_1, m_2, \ldots, m_N$. We assume that no mass enters or leaves the system, so that the total mass M ($= \Sigma\, m_n$) of the system remains constant with time. The particles may interact with each other, and external forces may act on them as well. Each particle has a certain velocity and momentum in the particular reference frame being used. The system as a whole has a total momentum **P**, which is defined to be simply the vector sum of the momenta of the individual particles in that same frame, or

$$\begin{aligned}\mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N \\ &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_N\mathbf{v}_N. \end{aligned} \qquad (24)$$

If we compare this relation with Eq. 13, we see at once that

$$\mathbf{P} = M\mathbf{v}_{\rm cm}, \qquad (25)$$

which is an equivalent definition for the momentum of a system of particles:

The total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its center of mass.

If we differentiate Eq. 25 with respect to time we obtain, for an assumed constant mass M,

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{v}_{\rm cm}}{dt} = M\mathbf{a}_{\rm cm}. \qquad (26)$$

Comparison of Eq. 26 with Eq. 16, $\Sigma\,\mathbf{F}_{\rm ext} = M\mathbf{a}_{\rm cm}$, allows us to write Newton's second law for a system of particles in the form:

$$\sum \mathbf{F}_{\rm ext} = \frac{d\mathbf{P}}{dt}. \qquad (27)$$

Equation 27 states that, in a system of particles, the net external force equals the rate of change of the linear momentum of the system. This equation is the generalization of the single-particle equation $\Sigma\,\mathbf{F} = d\mathbf{p}/dt$ (Eq. 20) to a system of many particles, when no mass enters or leaves the system. Equation 27 reduces to Eq. 20 for the special case of a single particle, since only external forces can act on a one-particle system. In Section 9-8 we consider modifications of Eq. 27 for systems of variable mass.

9-6 CONSERVATION OF LINEAR MOMENTUM

Suppose that the sum of the external forces acting on a system is zero. Then, from Eq. 27,

$$\frac{d\mathbf{P}}{dt} = 0 \quad \text{or} \quad \mathbf{P} = \text{a constant}.$$

When the net external force acting on a system is zero, the total vector momentum of the system remains constant.

This simple but quite general result is called the law of *conservation of linear momentum.* Like the law of conservation of energy, the law of conservation of linear momentum applies to a vast range of physical situations and has no known exceptions.

Conservation laws (such as those of energy and linear momentum, which we have already encountered, and those of angular momentum and electric charge, which we shall encounter later in the text) are of theoretical and practical importance in physics because they are simple and universal. The laws of conservation of energy and of linear momentum, for example, go beyond the limitations of classical mechanics and remain valid in both the relativistic and quantum realms.

Conservation laws all have the following form. While the system is changing there is one aspect of the system that remains unchanged. Different observers, each in a different reference frame, would all agree, if they watched the same changing system, that the conservation laws applied to the system. For the conservation of linear momentum, for example, observers in different inertial reference frames would assign different values of **P** to the linear momentum of the system, but each would agree (assuming $\Sigma\,\mathbf{F}_{\rm ext} = 0$) that the value of **P** remained unchanged as the particles that make up the system move about. The force **F** is an invariant with respect to Galilean transformations (all inertial observers agree on its measurement). If $\Sigma\,\mathbf{F}_{\rm ext} = 0$ in *any* inertial frame, then *all* inertial observers will also find $\Sigma\,\mathbf{F}_{\rm ext} = 0$ and will conclude that momentum is conserved.

The total momentum of a system can be changed only by external forces acting on the system. The internal forces, being equal and opposite, produce equal and opposite changes in momentum, which cancel each other. For a system of particles on which no net external force acts,

$$\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \text{a constant}. \tag{28}$$

The momenta of the individual particles may change, but their sum remains constant if there is no external force.

Momentum is a vector quantity. Equation 28 is therefore equivalent to three scalar equations, one for each coordinate direction. Hence the conservation of linear momentum supplies us with three conditions on the motion of a system to which it applies. The conservation of energy, on the other hand, supplies us with only one condition on the motion of a system to which it applies, because energy is a scalar.

If our system of particles consists of only a single particle, then Eq. 28 reduces to a statement that when no net force acts on it the momentum of the particle is a constant, which (for a single particle) is equivalent to stating that its velocity is a constant. This is simply a restatement of Newton's first law.

Sample Problem 6 A stream of bullets whose mass m is each 3.8 g is fired horizontally with a speed v of 1100 m/s into a large wooden block of mass M (= 12 kg) that is initially at rest on a horizontal table; see Fig. 13. If the block is free to slide without friction across the table, what speed will it acquire after it has absorbed 8 bullets?

Solution Equation 28 ($\mathbf{P}$ = constant) is valid only for closed systems, in which no particles leave or enter. Thus our system must include both the block and the 8 bullets, taken together. In Fig. 13, we have identified this system by drawing a closed curve around it.

For the moment we consider only the horizontal direction. No external horizontal force acts on the system of block + bullets. The forces that act when the bullets strike the block are internal forces and do not contribute to $\mathbf{F}_{\text{ext}}$, which has no horizontal component.

Because no (horizontal) external forces act, we can apply the law of conservation of momentum (Eq. 28). The initial (horizontal) momentum, measured while the bullets are still in flight and the block is at rest, is

$$P_i = N(mv),$$

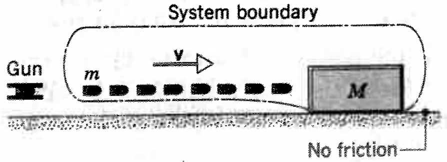

Figure 13 Sample Problem 6. A gun fires a stream of bullets toward a block of wood. We analyze the system that we define to be the block plus the bullets in flight.

in which mv is the momentum of an individual bullet and $N = 8$. The final momentum, measured when all the bullets are in the block and the block is sliding over the table with speed V, is

$$P_f = (M + Nm)V.$$

Conservation of momentum requires that

$$P_i = P_f$$

or

$$N(mv) = (M + Nm)V.$$

Solving for V yields

$$V = \frac{Nm}{M + Nm}\,v = \frac{(8)(3.8 \times 10^{-3}\ \text{kg})}{12\ \text{kg} + (8)(3.8 \times 10^{-3}\ \text{kg})}(1100\ \text{m/s})$$
$$= 2.8\ \text{m/s}.$$

With the choice of system that we made, we did not have to consider the forces exerted when the bullets hit the block. Those forces are all internal.

In the vertical direction, the external forces are the weight of the bullets, the weight of the block, and the normal force on the block. While the bullets are in flight, they acquire a small vertical momentum component as a result of the action of gravity. When the bullets strike the block, the block must exert on each bullet a force with both horizontal and vertical components. Along with the vertical force on the bullet, which is necessary to change its vertical momentum to zero, there must (according to Newton's third law) be a corresponding increase in the normal force exerted on the block by the horizontal surface. This increase is not only from the weight of the imbedded bullet; it has an additional contribution arising from the rate of change of the vertical momentum of the bullet. When all the bullets have come to rest relative to the block, the normal force will equal the combined weights of block and imbedded bullets.

For simplicity in solving this problem, we have assumed that the bullets are fired so rapidly that all 8 are in flight before the first bullet strikes the block. Can you solve this problem without making this assumption?

Suppose the system boundary is enlarged so that it includes the gun, which is fixed to the Earth. Does the horizontal momentum of this system change before and after the firing? Is there a horizontal external force?

Sample Problem 7 As Fig. 14 shows, a cannon whose mass M is 1300 kg fires a 72-kg ball in a horizontal direction with a muzzle speed v of 55 m/s. The cannon is mounted so that it can recoil freely. (*a*) What is the velocity V of the recoiling cannon with respect to the Earth? (*b*) What is the initial velocity v_E of the ball with respect to the Earth?

Solution (*a*) We choose the cannon plus the ball as our system. By doing so, the forces associated with the firing of the cannon are internal to the system, and we do not have to deal with them. The external forces acting on the system have no horizontal components. Thus the horizontal component of the total linear momentum of the system must remain unchanged as the cannon is fired.

We choose a reference frame fixed with respect to the Earth, and we assume that all velocities are positive if they point to the right in Fig. 14.

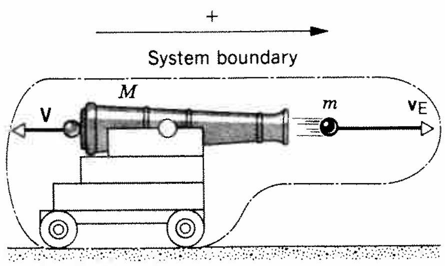

Figure 14 Sample Problem 7. A cannon of mass M fires a ball of mass m. The velocities of the ball and the recoiling cannon are shown in a reference frame fixed with respect to the Earth. Velocities are taken as positive to the right.

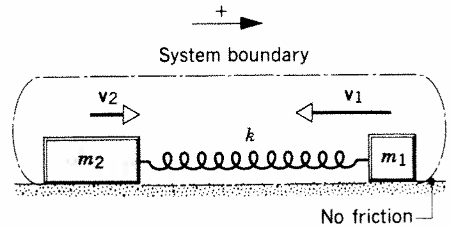

Figure 15 Sample Problem 8. Two blocks, resting on a frictionless surface and connected by a spring, have been pulled apart and then released from rest. The initial total momentum is zero, and so it must remain zero at all subsequent times. Velocities are taken as positive to the right.

Before the cannon is fired, the system has an initial momentum P_i of zero. After the cannon has fired, the ball has a horizontal velocity v with respect to the recoiling cannon, v being the ball's muzzle speed. In the reference frame of the Earth, however, the horizontal velocity of the ball is $v + V$. Thus, the total linear momentum of the system after firing is

$$P_f = MV + m(v + V),$$

in which the first term on the right is the momentum of the recoiling cannon and the second term that of the speeding ball.

Conservation of linear momentum in the horizontal direction requires that $P_i = P_f$, or

$$0 = MV + m(v + V).$$

Solving for V yields

$$V = -\frac{mv}{M + m} = -\frac{(72\text{ kg})(55\text{ m/s})}{1300\text{ kg} + 72\text{ kg}} = -2.9\text{ m/s}.$$

The minus sign tells us that the cannon recoils to the left in Fig. 14, as we expect it should.

(*b*) The velocity of the ball with respect to the (recoiling) cannon is its muzzle speed v. With respect to the Earth, the velocity of the ball is

$$\begin{aligned} v_E &= v + V \\ &= 55\text{ m/s} + (-2.9\text{ m/s}) = 52\text{ m/s}. \end{aligned}$$

Because of the recoil, the ball is moving a little slower with respect to the Earth than it otherwise would. Note the importance in this problem of choosing the system (cannon + ball) wisely and being absolutely clear about the reference frame (Earth or recoiling cannon) to which the various measurements are referred.

Sample Problem 8 Figure 15 shows two blocks, connected by a spring and free to slide on a frictionless horizontal surface. The blocks, whose masses are m_1 and m_2, are pulled apart and then released from rest. What fraction of the total kinetic energy of the system will each block have at any later time?

Solution We take the two blocks and the spring (assumed massless) as our system and the horizontal surface on which they slide as our reference frame. We assume that velocities are positive if they point to the right in Fig. 15.

The initial momentum P_i of the system before the blocks are released is zero. The final momentum, at any time after the blocks are released, is

$$P_f = m_1 v_1 + m_2 v_2,$$

in which v_1 and v_2 are the velocities of the blocks. Conservation of momentum requires that $P_i = P_f$, or

$$0 = m_1 v_1 + m_2 v_2 .$$

Thus we have

$$\frac{v_1}{v_2} = -\frac{m_2}{m_1}, \tag{29}$$

the minus sign telling us that the two velocities always have opposite directions. This holds at every instant after release, no matter what the individual speeds of the blocks.

The kinetic energies of the blocks are $K_1 = \frac{1}{2}m_1 v_1^2$ and $K_2 = \frac{1}{2}m_2 v_2^2$. The fraction we seek, for the block of mass m_1, is

$$f_1 = \frac{K_1}{K_1 + K_2} = \frac{\frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2}.$$

Substituting $v_2 = -v_1 (m_1/m_2)$ leads, after a little algebra, to

$$f_1 = \frac{m_2}{m_1 + m_2}.$$

Similarly, for the block of mass m_2,

$$f_2 = \frac{m_1}{m_1 + m_2}.$$

Thus, although the kinetic energy of the oscillating system varies with time, the division of this energy between the two blocks is a constant, independent of time, the least massive block receiving the largest share of the available kinetic energy. If, for example, $m_2 = 10m_1$, then

$$f_1 = \frac{10m_1}{m_1 + 10m_1} = 0.91 \quad \text{and} \quad f_2 = \frac{m_1}{m_1 + 10m_1} = 0.09.$$

In this case, the lighter block (m_1) gets 91% of the available kinetic energy and the heavier block (m_2) gets the remaining 9%. In the limit $m_2 \gg m_1$, the lighter block gets essentially all the kinetic energy.

The expressions for f_1 and f_2 apply equally well to a stone falling in the gravitational field of the Earth. Let m_2 represent the mass of the Earth and m_1 the mass of the stone. In the reference frame of their center of mass, the stone takes nearly all the kinetic energy ($f_1 \approx 1$) and the Earth takes very little ($f_2 \approx 0$). The magnitudes of the linear momenta of the stone and Earth

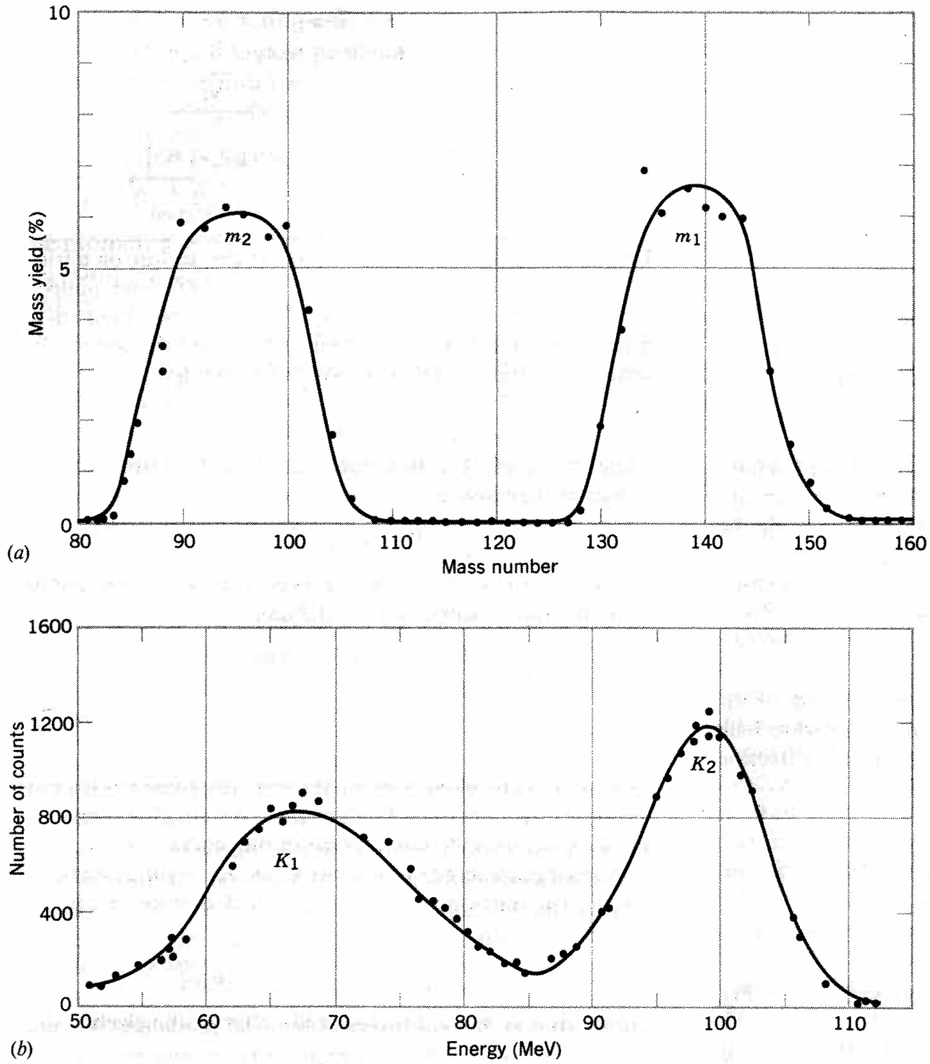

Figure 16 (*a*) The mass distribution of the fragments emitted in nuclear fission. The vertical scale gives the fraction of fissions that result in a fragment with the mass number given by the horizontal scale. (*b*) The energy distribution for fragments emitted in fission.

are equal, but the small velocity of the Earth is compensated by its enormous mass. This argument justifies neglecting the kinetic energy of the Earth when we used the conservation of energy in Chapter 8 to analyze objects falling in the Earth's gravity.

Another practical example of this effect occurs in the case of nuclear fission, in which a heavy nucleus such as ^{235}U splits into two lighter fragments. The fragments are driven apart by their mutual electrical repulsion from an initial position in which they are very close together and nearly at rest. From Eq. 29, we expect the ratio of the kinetic energies to be

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2} = \left(\frac{m_1}{m_2}\right)\left(\frac{v_1}{v_2}\right)^2 = \frac{m_2}{m_1}.$$

That is, the heavier fragment gets the smaller kinetic energy.

Fission is a statistical process, in which there is a distribution of possible masses of the fragments and a corresponding distribution in the fragment kinetic energies. Figure 16*a* shows the mass distribution and Fig. 16*b* shows the kinetic energy distribution. Note that fission into fragments of equal mass is very rare; one fragment usually has a mass number of about 138 and the other about 94. A typical mass ratio m_2/m_1 is thus about $94/138 = 0.68$. A typical kinetic energy ratio K_1/K_2 is about 67 MeV/99 MeV = 0.68, equal to the typical mass ratio, as expected. Thus the sharing of kinetic energy between the fission fragments is done according to the restriction that momentum is conserved.

9-7 WORK AND ENERGY IN A SYSTEM OF PARTICLES *(Optional)*

Figure 17 shows a skater pushing away from a railing, gaining kinetic energy in the process. If you ask the skater where this kinetic energy comes from, he will probably tell you that, judging by his muscular exertions, the required energy must come from his own store of internal energy. Let us try to verify this claim by applying conservation of energy to the system consisting of the skater alone.

From Eq. 28 of Chapter 8 we have

$$\Delta U + \Delta K_{cm} + \Delta E_{int} = W. \quad (30)$$

In deriving Eq. 33 of Chapter 8, we divided the kinetic energy of a system into two terms: ΔK_{int}, which represented the internal