CHAPTER 7

WORK AND ENERGY

A fundamental problem of particle dynamics is to find how a particle will move, given the forces that act on it. By "how a particle will move" we mean how its position varies with time. In the previous two chapters we solved this problem for the special case of a constant force, in which case the formulas for constant acceleration can be used to find $\mathbf{r}(t)$*, completing the solution of the problem.*

The problem is more difficult, however, when the force acting on a particle and thus its acceleration are not constant. We can solve such problems by integration methods, as illustrated in Sections 6-5 and 6-7, respectively, for forces depending on time and velocity. In this chapter, we extend the analysis to forces that depend on the position of the particle, such as the gravitational force exerted by the Earth on any nearby object and the force exerted by a stretched spring on a body to which it is attached. This analysis leads us to the concepts of work *and* kinetic energy *and to the development of the* work–energy theorem, *which is the central feature of this chapter. In Chapter 8 we consider a broader view of energy, embodied in the law of conservation of energy, a concept that has played a major role in the development of physics.*

7-1 WORK DONE BY A CONSTANT FORCE

Consider a particle acted on by a constant force **F**, and assume the simplest case in which the motion takes place in a straight line in the direction of the force. In such a situation we define the *work W done by the force on the particle* as the product of the magnitude of the force F and the magnitude of the displacement s through which the force acts. We write this as

$$W = Fs. \tag{1}$$

In a more general case, the constant force acting on a particle may not act in the direction in which the particle moves. In this case we define the work done by the force on the particle as the product of the component of the force along the line of motion and the magnitude of the displacement s. In Fig. 1, a particle experiences a constant force **F** that makes an angle ϕ with the direction of the displacement **s** of the particle. The work W done by **F** during this displacement is, according to our definition,

$$W = (F\cos\phi)s. \tag{2}$$

Of course, other forces may also act on the particle. Equation 2 refers only to the work done on the particle by one particular force **F**. The work done on the particle by the other forces must be calculated separately. To find the total work done on the particle, we add the values of the work done by all the separate forces. (Alternatively, as we discuss in Section 7-4, we can first find the net force on the particle and then calculate the work that would be done by a single force equal to the net force. The two methods of finding the work done on a particle are equivalent, and they always yield the same result for the work done on the particle.)

When ϕ is zero, the work done by **F** is simply Fs, in agreement with Eq. 1. Thus, when a horizontal force moves a body horizontally, or when a vertical force lifts a body vertically, the work done by the force is the product of the magnitude of the force and the distance moved.

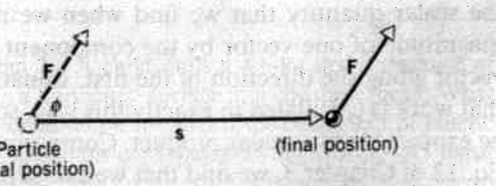

Figure 1 A force **F** acts on a particle as it undergoes a displacement **s**. The component of **F** that does work on the particle is $F\cos\phi$. The work done by the force **F** on the particle is $Fs\cos\phi$, which we can also write as $\mathbf{F}\cdot\mathbf{s}$.

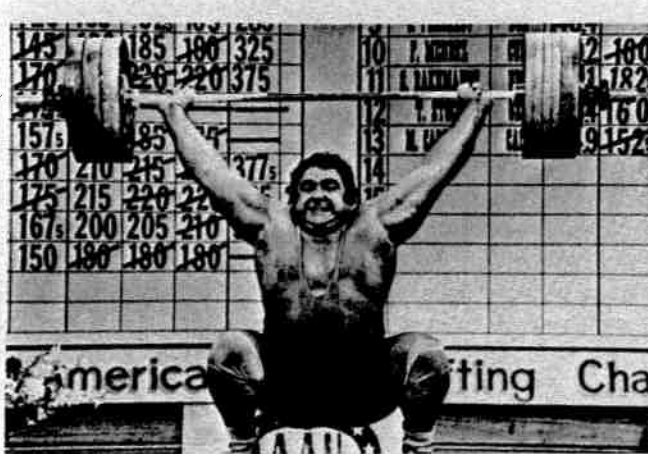

Figure 2 The weightlifter is exerting a great force on the weights, but at the instant shown he is doing no work because he is holding them in place. There is a force but no displacement. Of course, he probably has already done some work to have lifted them off the floor to that height.

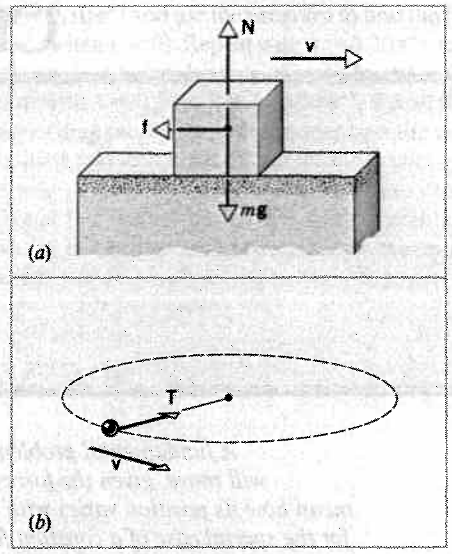

Figure 3 Work is not necessarily done by all the forces applied to a body, even if the body is in motion. In (*a*), the weight and the normal force do no work, because they are perpendicular to the displacement (which is in the direction of the velocity **v**). Work is done by the frictional force. In (*b*), which shows a body attached to a cord and revolving in a horizontal circle, the tension **T** in the cord does no work on the body, because it has no component in the direction of the displacement.

When ϕ is 90°, the force has no component in the direction of motion. That force then does no work on the body. For instance, a weightlifter (Fig. 2) does work in lifting the weights off the ground, but he does no work in holding them up (because there is no displacement). If he were to carry the weights above his head while walking, he would again (according to our definition of work) do no work on them, assuming there to be no vertical displacement, because the vertical force he exerts would be perpendicular to the horizontal displacement. Figure 3 shows other examples of forces applied to a body that do no work on the body.

Notice that we can write Eq. 2 either as $(F \cos \phi)s$ or $F(s \cos \phi)$. This suggests that the work can be calculated in two different ways, which give the same result: either we multiply the magnitude of the displacement by the component of the force in the direction of the displacement, or we multiply the magnitude of the force by the component of the displacement in the direction of the force. Each way reminds us of an important part of the definition of work: there must be a component of **s** in the direction of **F**, and there must be a component of **F** in the direction of **s** (Fig. 4).

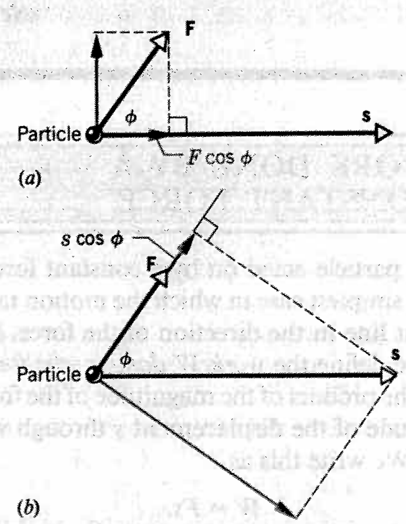

Figure 4 (*a*) The work W interpreted as $W = (s)(F \cos \phi)$. (*b*) The work W interpreted as $W = (F)(s \cos \phi)$.

Work is a *scalar*, although the two quantities involved in its definition, force and displacement, are vectors. In Section 3-5 we defined the *scalar product* of two vectors as the scalar quantity that we find when we multiply the magnitude of one vector by the component of a second vector along the direction of the first. Equation 2 shows that work is calculated in exactly this way, so work must be expressible as a scalar product. Comparing Eq. 2 with Eq. 13 of Chapter 3, we find that we can express work as

$$W = \mathbf{F} \cdot \mathbf{s}, \tag{3}$$

where the dot indicates a scalar (or dot) product.

Work can be either positive or negative. If a force has a component opposite to the direction of the motion, the work done by that force is negative. This corresponds to an obtuse angle between the force and displacement vectors. For example, when you lower an object to the floor, the work done on the object by the upward force of your hand holding the object is negative. In this case ϕ is 180°, for **F** points up and **s** points down. (The gravitational force in this case does positive work as the object moves down.)

Although the force **F** is an invariant, independent in both magnitude and direction of our choice of inertial frames, the displacement **s** is *not*. Depending on the inertial frame from which the measurement is made, an observer could measure essentially any magnitude and direction for the displacement **s**. Thus observers in different inertial frames, who will agree on the forces that act on a body, will disagree in their evaluation of the work done by the forces acting on the body. Different observers might find the work to be positive, negative, or even zero. We explore this point later in Section 7-6.

Work as we have defined it (Eq. 3) proves to be a very useful concept in physics. Our special definition of the word "work" does not correspond to the colloquial usage of the term. This may be confusing. A person holding a heavy weight at rest in the air may be working hard in the physiological sense, but from the point of view of physics that person is not doing any work on the weight. We say this because the applied force causes no displacement of the weight.

If, on the other hand, we consider the weightlifter to be a system of particles (which we treat in Chapter 9), we find that microscopically work is indeed being done. A muscle is not a solid support and cannot sustain a load in a static manner. The individual muscle fibers repeatedly relax and contract, and if we analyze the situation in this manner we would find that work is done in each contraction. That is why the weightlifter becomes tired in supporting the weight. In this chapter we do not consider this "internal" work. The word *work* is used only in the strict sense of Eq. 3, so that it does indeed vanish in the case of no displacement of the particle on which the force acts.

The unit of work is determined from the work done by a unit force in moving a body a unit distance in the direction of the force. The SI unit of work is 1 *newton-meter*, called 1 *joule* (abbreviation J). In the British system the unit of work is the foot-pound. In cgs systems the unit of work is 1 dyne-centimeter, called 1 erg. Using the relations between the newton, dyne, and pound, and between the meter, centimeter, and foot, we obtain 1 joule = 10^7 ergs = 0.7376 ft·lb.

A convenient unit of work when dealing with atomic or subatomic particles is the *electron-volt* (abbreviation eV), where 1 eV = 1.60×10^{-19} J. The work required to remove an outer electron from an atom has a typical magnitude of several eV. The work required to remove a proton or a neutron from a nucleus has a typical magnitude of several MeV (10^6 eV).

Sample Problem 1 A block of mass $m = 11.7$ kg is to be pushed a distance of $s = 4.65$ m along an incline so that it is raised a distance of $h = 2.86$ m in the process (Fig. 5*a*). Assuming frictionless surfaces, calculate how much work you would do if you applied a force parallel to the incline to push the block up at constant speed.

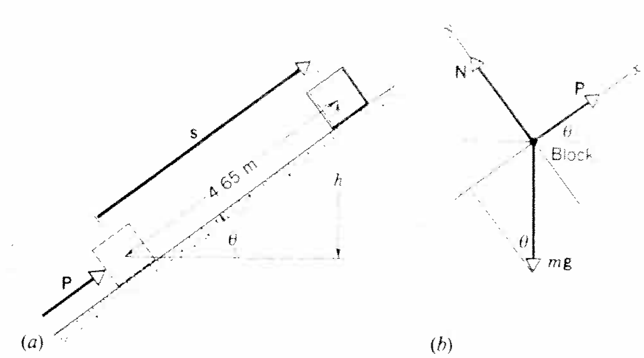

Figure 5 Sample Problem 1. (*a*) A force **P** moves a block up a plane through a displacement **s**. (*b*) A free-body diagram for the block.

Solution A free-body diagram of the block is given in Fig. 5*b*. We must first find *P*, the magnitude of the force pushing the block up the incline. Because the motion is not accelerated (we are given that the speed is constant), the net force parallel to the plane must be zero. If we choose our *x* axis parallel to the plane, with its positive direction up the plane, we have, from Newton's second law,

$$x \text{ component:} \qquad P - mg \sin \theta = 0,$$

or

$$P = mg \sin \theta = (11.7 \text{ kg})(9.80 \text{ m/s}^2)\left(\frac{2.86 \text{ m}}{4.65 \text{ m}}\right) = 70.5 \text{ N}.$$

Then the work done by **P**, from Eq. 3 with $\phi = 0°$, is

$$W = \mathbf{P} \cdot \mathbf{s} = Ps \cos 0° = Ps = (70.5 \text{ N})(4.65 \text{ m}) = 328 \text{ J}.$$

Note that the angle ϕ (=0°) used in this expression is the angle between the applied force and the displacement of the block, both of which are parallel to the incline. The angle ϕ must not be confused with the angle θ of the incline.

If you were to raise the block vertically at constant speed without using the incline, the work you do would be the vertical force, which is equal to mg, times the vertical distance h, or

$$W = mgh = (11.7 \text{ kg})(9.80 \text{ m/s}^2)(2.86 \text{ m}) = 328 \text{ J},$$

the same as before. The only difference is that the incline permits a smaller force ($P = 70.5$ N) to raise the block than would be required without the incline ($mg = 115$ N). On the other hand, you must push the block a greater distance (4.65 m) up the incline than you would if you raised it directly (2.86 m).

Sample Problem 2 A child pulls a 5.6-kg sled a distance of $s = 12$ m along a horizontal surface at a constant speed. What work does the child do on the sled if the coefficient of kinetic friction μ_k is 0.20 and the cord makes an angle of $\phi = 45°$ with the horizontal?

Solution The situation is shown in Fig. 6*a* and the forces acting on the sled are shown in the free-body diagram of Fig. 6*b*. **P** is the

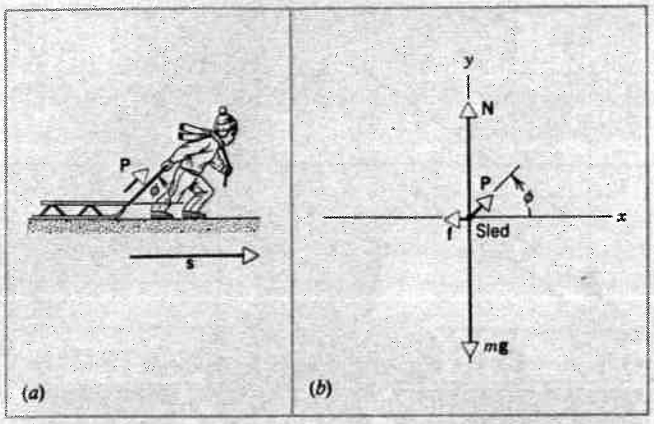

Figure 6 Sample Problem 2. (*a*) A child displaces a sled an amount **s** by pulling with a force **P** on a rope that makes an angle ϕ with the horizontal. (*b*) A free-body diagram for the sled.

child's pull, mg the sled's weight, **f** the frictional force, and **N** the normal force exerted by the surface on the sled. The work done by the child on the sled is

$$W = \mathbf{P} \cdot \mathbf{s} = Ps\cos\phi.$$

To evaluate this we first must determine P, whose value has not been given. To obtain P we refer to the free-body diagram of Fig. 6*b*.

The sled is unaccelerated, so that from the second law of motion we obtain the following:

$$x \text{ component:} \quad P\cos\phi - f = 0,$$

$$y \text{ component:} \quad P\sin\phi + N - mg = 0.$$

We know that f and N are related by

$$f = \mu_k N.$$

These three equations contain three unknown quantities: P, f, and N. To find P we eliminate f and N from these equations and solve the remaining equation for P. You should verify that

$$P = \frac{\mu_k mg}{\cos\phi + \mu_k \sin\phi}.$$

With $\mu_k = 0.20$, $mg = (5.6 \text{ kg})(9.8 \text{ m/s}^2) = 55 \text{ N}$, and $\phi = 45°$ we obtain

$$P = \frac{(0.20)(55 \text{ N})}{\cos 45° + (0.20)(\sin 45°)} = 13 \text{ N}.$$

Then with $s = 12$ m, the work done by the child on the sled is

$$W = Ps\cos\phi = (13 \text{ N})(12 \text{ m})(\cos 45°) = 110 \text{ J}.$$

The vertical component of the pull **P** does no work on the sled. Note, however, that it reduces the normal force between the sled and the surface ($N = mg - P\sin\phi$) and thereby reduces the magnitude of the force of friction ($f = \mu_k N$).

Would the child do more work, less work, or the same amount of work on the sled if P were applied horizontally instead of at 45° from the horizontal? Do any of the other forces acting on the sled do work on it?

7-2 WORK DONE BY A VARIABLE FORCE: ONE-DIMENSIONAL CASE

We now consider the work done by a force that is not constant. Let the force act only in one direction, which we take to be the x direction, and let it vary in magnitude with x according to the function $F(x)$. Suppose a body that moves in the x direction is acted on by this force. What is the work done by this variable force when the body moves from an initial position x_i to a final position x_f?

In Fig. 7 we plot F versus x. Let us divide the total displacement into a number N of small intervals of equal width δx (Fig. 7*a*). Consider the first interval, in which there is a small displacement δx from x_i to $x_i + \delta x$. During this small displacement the force $F(x)$ has a nearly constant value F_1, and the small amount of work δW_1 it does in that interval is approximately

$$\delta W_1 = F_1\,\delta x. \tag{4}$$

Likewise, during the second interval, there is a small displacement from $x_i + \delta x$ to $x_i + 2\delta x$, and the force $F(x)$

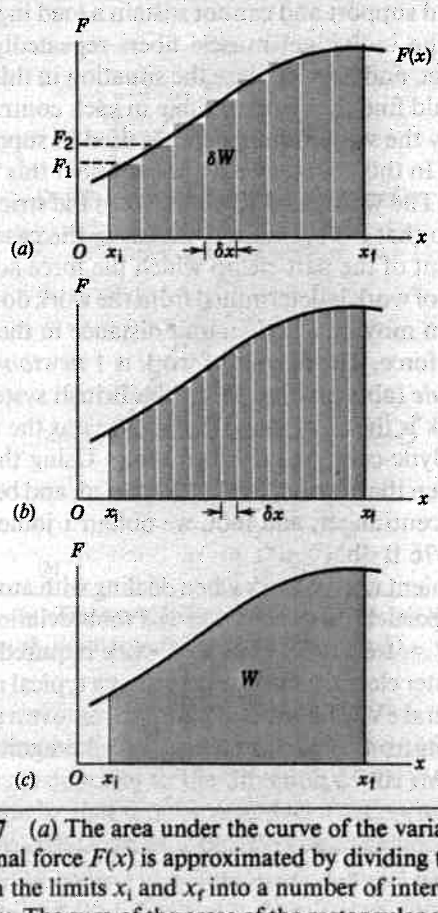

Figure 7 (*a*) The area under the curve of the variable one-dimensional force $F(x)$ is approximated by dividing the region between the limits x_i and x_f into a number of intervals of width δx. The sum of the areas of the rectangular strips is approximately equal to the area under the curve. (*b*) A better approximation is obtained using a larger number of narrower strips. (*c*) In the limit $\delta x \to 0$, the actual area is obtained.

has a nearly constant value F_2. The work done by the force in the second interval is approximately $\delta W_2 = F_2\,\delta x$. The total work W done by $F(x)$ in displacing the body from x_i to x_f is approximately the sum of a large number of terms like that of Eq. 4, in which F has a different value for each term. Hence

$$W = \delta W_1 + \delta W_2 + \delta W_3 + \cdots$$
$$= F_1\,\delta x + F_2\,\delta x + F_3\,\delta x + \cdots$$

or

$$W = \sum_{n=1}^{N} F_n\,\delta x, \tag{5}$$

where the Greek letter sigma (Σ) stands for the sum over all N intervals from x_i to x_f.

To make a better approximation we can divide the total displacement from x_i to x_f into a larger number of intervals, as in Fig. 7*b*, so that δx is smaller and the value of F_n in each interval is more typical of the force within the interval. It is clear that we can obtain better and better approximations by taking δx smaller and smaller so as to have a larger and larger number of intervals. We can obtain an exact result for the work done by F if we let δx go to zero and the number of intervals N go to infinity. Hence the exact result is

$$W = \lim_{\delta x \to 0} \sum_{n=1}^{N} F_n\,\delta x. \tag{6}$$

The relation

$$\lim_{\delta x \to 0} \sum F_n\,\delta x = \int_{x_i}^{x_f} F(x)\,dx,$$

as you may have learned in your calculus course, defines the integral of F with respect to x from x_i to x_f. Numerically, this quantity is exactly equal to the area between the force curve and the x axis between the limits x_i and x_f (Fig. 7*c*). Hence, an integral can be interpreted graphically as an area. The symbol $\int$ is a distorted S (for sum) and symbolizes the integration process. We can write the total work done by F in displacing a body from x_i to x_f as

$$W = \int_{x_i}^{x_f} F(x)\,dx. \tag{7}$$

Because we have eliminated the vector notation from this one-dimensional equation, we must take care explicitly to put in the sign of F, positive if F is in the direction of increasing x and negative if F is in the direction of decreasing x.

As an example of a variable force, we consider a spring that acts on a particle of mass m (Fig. 8). The particle moves in the horizontal direction, which we take to be the x direction, with the origin ($x = 0$) representing the position of the particle when the spring is relaxed (Fig. 8*a*). An external force F_{ext} acts on the particle in a direction opposite to the spring force. We assume that the external force is always approximately equal to the spring force, so that the particle is nearly in equilibrium at all times ($a = 0$).

Let the particle be displaced a distance x from its original position at $x = 0$ (Fig. 8*b*). As the agent exerts a force

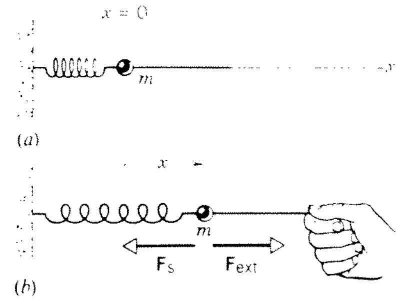

Figure 8 (*a*) A particle of mass m is attached to a spring, which is in its relaxed position. (*b*) The particle is displaced a distance x, where it is acted on by two forces, the restoring force of the spring and a pull from an external agent.

F_{ext} on the particle, the spring exerts an opposing force F_s. This force is given to a good approximation by

$$F_s = -kx, \tag{8}$$

where k is a positive constant called the *force constant* of the spring. The constant k is a measure of the force necessary to produce a given stretching of the spring: stiffer springs have larger values of k. Equation 8 is the *force law* for springs and is known as *Hooke's law*. The minus sign in Eq. 8 reminds us that the direction of the force exerted by the spring is always opposite to the direction of the displacement of the particle. When the spring is stretched, $x > 0$ and F_s is negative; when the spring is compressed, $x < 0$ and F_s is positive. The force exerted by the spring is a *restoring force:* it always tends to *restore* the particle to its position at $x = 0$. Most real springs will obey Eq. 8 reasonably well if we do not stretch them beyond a limited range.

Let us first consider the work done *on* the particle *by* the spring when the particle moves from initial position x_i to final position x_f. We use Eq. 7 with the force F_s:

$$W_s = \int_{x_i}^{x_f} F_s(x)\,dx = \int_{x_i}^{x_f} (-kx)\,dx$$
$$= \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2. \tag{9}$$

The sign of the work done by the spring on the particle is positive if $x_i^2 > x_f^2$ (that is, if the magnitude of the initial displacement of the particle is greater than that of its final displacement). Note that the spring does *positive* work when it acts to restore the particle to its position at $x = 0$. If the magnitude of the initial displacement is smaller than that of the final displacement, the spring does *negative* work on the particle.

If we are interested in knowing the work done by the spring on the particle when the particle moves from its original position at $x = 0$ through a displacement x, we let $x_i = 0$ and $x_f = x$ and obtain

$$W_s = \int_0^x (-kx)\,dx = -\tfrac{1}{2}kx^2. \tag{10}$$

Note that the work done by the spring in compression through a displacement x is the same as that done in extension through x, because the displacement x is squared in Eq. 10; either sign for x gives a positive value for x^2 and a negative value for W_s.

How much work does the *external agent* do when the particle moves from $x_i = 0$ to $x_f = x$? To keep the particle in equilibrium, the external force F_{ext} must be equal in magnitude to the spring force but opposite in sign, so $F_{ext} = +kx$. Repeating the calculation as in Eq. 10 for the work done by the external agent gives

$$W_{ext} = +\tfrac{1}{2}kx^2. \qquad (11)$$

Note that this is exactly the negative of Eq. 10.

We can also find W_s and W_{ext} by computing the area between the appropriate force–displacement curve and the x axis from $x = 0$ to any arbitrary value x. In Fig. 9 the two straight sloping lines passing through the origin are plots of the external force against displacement ($F_{ext} = +kx$) and of the spring force against displacement ($F_s = -kx$). The right-hand side of the plot ($x > 0$) corresponds to stretching the spring and the left-hand side ($x < 0$) to compressing it.

In *stretching* the spring, the work done by the external force is positive and is represented by the upper triangle on the right of Fig. 9 labeled W_{ext}. The base of this triangle is $+x$ and its altitude is $+kx$; its area is therefore

$$\tfrac{1}{2}(+x)(+kx) = +\tfrac{1}{2}kx^2$$

in agreement with Eq. 11. When the spring is stretched, the work done by the spring force is negative and is represented by the lower triangle labeled W_s on the right side of Fig. 9; this triangle can be shown by a similar geometrical argument to have an area of $-\tfrac{1}{2}kx^2$, in agreement with Eq. 10.

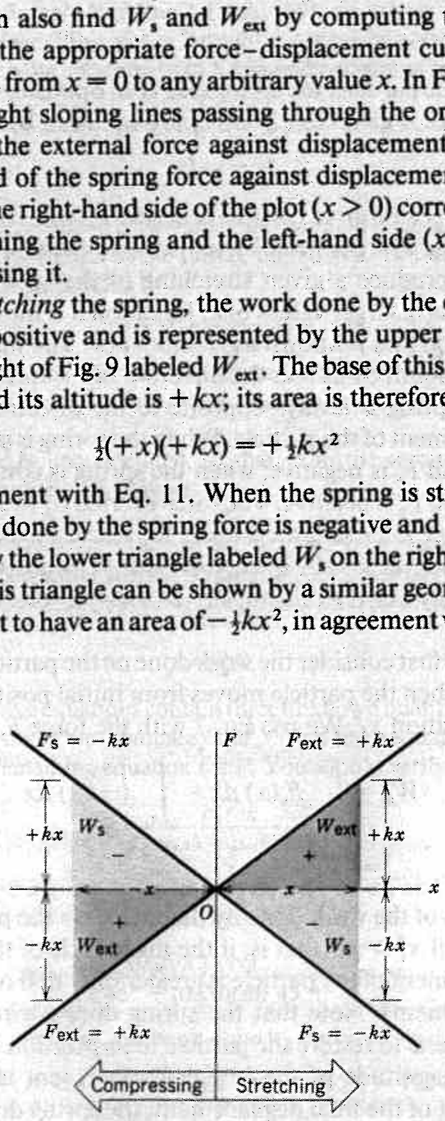

Figure 9 The work W_s done by the spring force is represented by the negative areas (shown with gray shading), and the work W_{ext} done by the external force, which is in equilibrium with the spring force, is represented by the positive areas (shown with colored shading). Whether the spring is stretched ($x > 0$) or compressed ($x < 0$), W_s is negative and W_{ext} is positive.

In *compressing* the spring, as the left side of Fig. 9 shows, the work W_{ext} done by the external agent is still positive, and the work W_s done by the spring is still negative, just as we expect from the signs of the forces and the displacement.

Sample Problem 3 A spring hangs vertically in equilibrium. A block of mass $m = 6.40$ kg is attached to the spring, but the block is held in place so that at first the spring does not stretch. Now the hand holding the block is slowly lowered, allowing the block to descend at constant speed until equilibrium is reached, at which point the hand is removed. A measurement shows that the spring has been stretched by a distance $s = 0.124$ m over its previous equilibrium length. Find the work done on the block in this process by (*a*) gravity, (*b*) the spring, and (*c*) the hand.

Solution We are not given the force constant of the spring, but we can find it because we know that at the stretched position the block is in equilibrium between the upward spring force and the downward force of gravity:

$$\sum F = mg - ks = 0.$$

We have chosen the downward direction to be positive here. Solving for k, we find

$$k = mg/s = (6.40\text{ kg})(9.80\text{ m/s}^2)/(0.124\text{ m}) = 506\text{ N/m}.$$

To find the work done by gravity, W_g, we note that gravity is a constant force, and the force and the displacement are parallel, so we can use Eq. 1:

$$W_g = Fs = mgs = (6.40\text{ kg})(9.80\text{ m/s}^2)(0.124\text{ m}) = +7.78\text{ J}.$$

This is positive, because the force and displacement are in the same direction. To find the work W_s done by the spring, we use Eq. 10 with $x = s$:

$$W_s = -\tfrac{1}{2}ks^2 = -\tfrac{1}{2}(506\text{ N/m})(0.124\text{ m})^2 = -3.89\text{ J}.$$

This is negative, because the force and displacement are in opposite directions.

One way to find the work W_h done by the hand is to find the force exerted by the hand as the block is lowered. If the block is in equilibrium during the entire process, then the upward force F_h exerted by the hand can be found from Newton's second law with $a = 0$:

$$\sum F = -kx - F_h + mg = 0,$$

or

$$F_h = mg - kx.$$

The work can be found from an integral of the form of Eq. 7, with a negative sign introduced to indicate that the force is opposite to the displacement:

$$W_h = -\int_0^s F_h\,dx = -\int_0^s (mg - kx)\,dx = -mgs + \tfrac{1}{2}ks^2$$
$$= -mgs + \frac{1}{2}\left(\frac{mg}{s}\right)s^2 = -\tfrac{1}{2}mgs = -3.89\text{ J}.$$

A simpler way to obtain this result is to recognize that if the block (which we treat as a particle) is lowered slowly and uniformly, then the net force is zero, and the total work done by all the forces acting on the particle must therefore be zero:

$$W_{net} = W_s + W_g + W_h = 0,$$

$$W_h = -W_s - W_g = -(-3.89\text{ J}) - 7.78\text{ J} = -3.89\text{ J}.$$

Note that the work done by the hand is equal to the work done by the spring.

7-3 WORK DONE BY A VARIABLE FORCE: TWO-DIMENSIONAL CASE *(Optional)*

The force **F** acting on a particle may vary in direction as well as in magnitude, and the particle may move along a curved path. To compute the work in this general case we divide the path into a large number of small displacements $\delta\mathbf{s}$, each pointing tangent to the path in the direction of motion. Figure 10 shows two selected displacements for a particular situation; it also shows the force **F** and the angle ϕ between **F** and $\delta\mathbf{s}$ at each location. We can find the amount of work δW done on the particle during a displacement $\delta\mathbf{s}$ from

$$\delta W = \mathbf{F}\cdot\delta\mathbf{s} = F\cos\phi\,\delta s. \tag{12}$$

Here **F** is the force at the point where we take $\delta\mathbf{s}$. The work done by the variable force **F** on the particle as the particle moves from i to f in Fig. 10 is found approximately by adding up (summing) the elements of work done over each of the line segments that make up the path from i to f. If the line segments $\delta\mathbf{s}$ become infinitesimally small, they may be replaced by differentials $d\mathbf{s}$ and the sum over the line segments may be replaced by an integral, as in Eq. 7. The work is then found from

$$W = \int_i^f \mathbf{F}\cdot d\mathbf{s} = \int_i^f F\cos\phi\,ds. \tag{13}$$

We cannot evaluate this integral until we are able to say how F and ϕ in Eq. 13 vary from point to point along the path; both are functions of the x and y coordinates of the particle in Fig. 10.

We can obtain an expression equivalent to Eq. 13 by writing **F** and $d\mathbf{s}$ in terms of their components. Thus $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$ and $d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j}$, so that $\mathbf{F}\cdot d\mathbf{s} = F_x\,dx + F_y\,dy$. In this evaluation recall that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$ (see Eq. 14, Chapter 3). Substituting this result into Eq. 13, we obtain

$$W = \int_i^f (F_x\,dx + F_y\,dy). \tag{14}$$

Integrals such as those in Eq. 13 and 14 are called *line integrals;* to evaluate them we must know how $F\cos\phi$ or F_x and F_y vary as the particle moves along a particular line (or curve). The extension of Eq. 14 to three dimensions is straightforward.

Sample Problem 4 A small object of mass m is suspended from a string of length L. The object is pulled sideways by a force P that is always horizontal, until the string finally makes an angle ϕ_m with the vertical (Fig. 11*a*). The displacement is accomplished so slowly that we may regard the system as being in equilibrium during the process. Find the work done by all the forces that act on the object.

Solution The motion is along an arc of radius L, and the displacement $d\mathbf{s}$ is always along the arc. At an intermediate point in the motion, the cord makes an angle ϕ with the vertical, and from the free-body diagram of Fig. 11*b* we see by applying Newton's second law that

$$x\text{ component:}\quad P - T\sin\phi = 0,$$

$$y\text{ component:}\quad T\cos\phi - mg = 0.$$

Combining these two equations to eliminate T, we find

$$P = mg\tan\phi.$$

Since P acts only in the x direction, we can use Eq. 14 with $F_x = P$ and $F_y = 0$ to find the work done by P. Thus

$$W_P = \int P\,dx = \int_0^{\phi_m} mg\tan\phi\,dx.$$

To carry out the integral over ϕ, we must have a single integration variable; we choose to define x in terms of ϕ. At an arbitrary intermediate position, when the horizontal coordinate is x, we

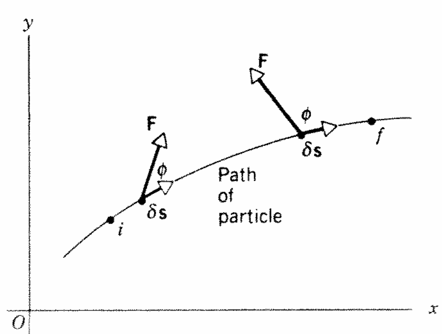

Figure 10 A particle moves from point i to point f along the path shown. During its motion it is acted on by a force **F** that varies in both magnitude and direction. As $\delta s \to 0$, we replace the interval by ds, which is in the direction of the instantaneous velocity and therefore tangent to the path.

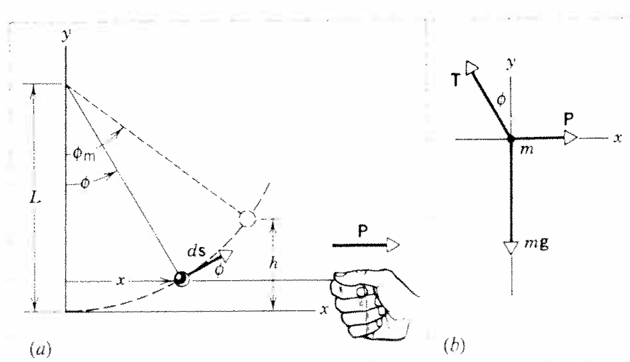

Figure 11 Sample Problem 4. (*a*) A particle is suspended from a string of length L and is pulled aside by a horizontal force P. The maximum angle reached is ϕ_m. (*b*) A free-body diagram for the particle.

see that $x = L \sin \phi$ and thus $dx = L \cos \phi \, d\phi$. Substituting for dx, we can now carry out the integration:

$$W_P = \int_0^{\phi_m} mg \tan \phi \,(L \cos \phi \, d\phi)$$

$$= mgL \int_0^{\phi_m} \sin \phi \, d\phi = mgL(-\cos \phi)\Big|_0^{\phi_m}$$

$$= mgL(1 - \cos \phi_m).$$

From Fig. 11a, we can see that $h = L(1 - \cos \phi_m)$, and thus

$$W_P = mgh.$$

The work W_g done by the (constant) gravitational force mg can be evaluated using a similar technique based on Eq. 14 (taking $F_x = 0$, $F_y = -mg$) to give $W_g = -mgh$ (see Problem 16). The minus sign enters because the direction of the vertical displacement is opposite to the direction of the gravitational force. The work W_T done by the tension in the string is zero, because **T** is perpendicular to the displacement $d\mathbf{s}$ at every point of the motion. Now you can see that the total work is zero: $W_{net} = W_P + W_g + W_T = mgh - mgh + 0 = 0$, consistent with the net force on the particle being zero at all times during its motion.

Note that in this problem the (positive) work done by the horizontal force **P** in effect cancels the (negative) work done by the vertical force $m\mathbf{g}$. This can occur because work is a *scalar:* it has no direction or components. The motion of the particle depends on the *total* work done on it, which is the scalar sum of the values of the work associated with each of the individual forces. ■

7-4 KINETIC ENERGY AND THE WORK–ENERGY THEOREM

In this section we consider the effect of work on the motion of a particle. An unbalanced force applied to a particle will certainly change the particle's state of motion. Newton's second law provides us with one way to analyze this change of motion. We now consider a different approach that ultimately gives the same result as Newton's laws but is often simpler to apply. It also leads us into one of the many important *conservation laws* that play such an important role in our interpretation of physical processes.

In this discussion, we consider not the work done on a particle by a single force, but the net work W_{net} done by all the forces that act on the particle. There are two ways to find the net work. The first is to find the net force, that is, the vector sum of all the forces that act on the particle,

$$\mathbf{F}_{net} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots, \qquad (15)$$

and then treat this net force as a single force in calculating the work according to Eq. 7 in one dimension or Eq. 13 in more than one dimension. In the second approach, we calculate the work done by each of the forces that act on the particle,

$$W_1 = \int \mathbf{F}_1 \cdot d\mathbf{s}, \qquad W_2 = \int \mathbf{F}_2 \cdot d\mathbf{s},$$
$$W_3 = \int \mathbf{F}_3 \cdot d\mathbf{s}, \cdots,$$

and then, since work is a scalar, we can add the work done by each of the individual forces to find the net work:

$$W_{net} = W_1 + W_2 + W_3 + \cdots. \qquad (16)$$

The two methods give equal results, and the choice between them is merely a matter of convenience.

We know that a net unbalanced force applied to a particle will change its state of motion by accelerating it, let us say from initial velocity v_i to final velocity v_f. What is the effect of the work done on the particle by this net unbalanced force?

We first look at the answer to this question in the case of the constant force in one dimension. Under the influence of this force, the particle moves from x_i to x_f, and it accelerates uniformly from v_i to v_f. The work done is

$$W_{net} = F_{net}(x_f - x_i) = ma(x_f - x_i).$$

Because the acceleration a is constant, we can use Eq. 20 of Chapter 2, written $v_f^2 = v_i^2 + 2a(x_f - x_i)$, to obtain

$$W_{net} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2. \qquad (17)$$

That is, the result of the net work on the particle has been to bring about a change in the value of the quantity $\frac{1}{2}mv^2$ from point i to point f. This quantity is called the *kinetic energy* K of the particle, with the definition

$$K = \tfrac{1}{2}mv^2. \qquad (18)$$

In terms of the kinetic energy K, we can rewrite Eq. 17 as

$$W_{net} = K_f - K_i = \Delta K. \qquad (19)$$

Equation 19 is the mathematical representation of an important result called the *work–energy theorem,* which in words can be stated as follows:

The net work done by the forces acting on a particle is equal to the change in the kinetic energy of the particle.

Although we have derived it in the case of a constant resultant force, the work–energy theorem holds in general for nonconstant forces as well. Later in this section we give a general proof for nonconstant forces.

Like work, kinetic energy is a scalar quantity; unlike work, kinetic energy is never negative. We have already mentioned that work depends on the choice of reference frame, and it therefore should not be surprising that kinetic energy does also. Of course, we already know that observers in different inertial frames will differ in their measurements of velocity, and they will therefore differ in assigning kinetic energies to particles. Although the observers disagree on the numbers to be assigned to work and to kinetic energy, they nevertheless find the same relation to hold between these quantities, namely, $W_{net} = \Delta K$.

For Eq. 19 to be dimensionally consistent, kinetic energy must have the same units as work, namely, joules, ergs, foot-pounds, electron-volts, and so on.

When the magnitude of the velocity of a particle is constant, there is no change in kinetic energy, and therefore the resultant force does no work. In uniform circular motion, for example, the resultant force acts toward the center of the circle and is always at right angles to the direction of motion. Such a force does no work on the particle: it changes the direction of the velocity of the particle but not its magnitude. Only when the resultant force has a component in the direction of motion does it do work on the particle and change its kinetic energy.

The work-energy theorem does *not* represent a new, independent law of classical mechanics. We have simply *defined* work (Eq. 7, for instance) and kinetic energy (Eq. 18) and *derived* the relation between them from Newton's second law. The work-energy theorem is useful, however, for solving problems in which the net work done on a particle by external forces is easily computed and in which we are interested in finding the particle's speed at certain positions. Of even more significance is the work-energy theorem as a starting point for a broad generalization of the concept of energy and how energy can be stored or shared among the parts of a complex system. The principle of conservation of energy is the subject of the next chapter.

General Proof of the Work-Energy Theorem

The following calculation gives a proof of Eq. 19 in the case of nonconstant forces in one dimension. The equivalent calculation in two or three dimensions is left as an exercise (see Problem 34). We let F_{net} represent the net force acting on the particle. The net work done by all the external forces that act on the particle is just $W_{net} = \int F_{net}\,dx$. With a bit of mathematical manipulation we can accomplish a change of integration variable and put this in a more useful form:

$$F_{net} = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v = mv\frac{dv}{dx}.$$

Thus

$$W_{net} = \int F_{net}\,dx = \int mv\frac{dv}{dx}\,dx = \int mv\,dv.$$

The variable of integration is now the velocity v. Let us integrate from initial velocity v_i to final velocity v_f:

$$W_{net} = \int_{v_i}^{v_f} mv\,dv = m\int_{v_i}^{v_f} v\,dv = \tfrac{1}{2}m(v_f^2 - v_i^2)$$
$$= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2.$$

This is identical with Eq. 19 and shows that the work-energy theorem holds even for nonconstant forces.

Sample Problem 5 One method of determining the kinetic energy of neutrons in a beam, such as from a nuclear reactor, is to measure how long it takes a particle in the beam to pass two fixed points a known distance apart. This technique is known as the *time-of-flight* method. Suppose a neutron travels a distance of $d = 6.2$ m in a time of $t = 160\ \mu$s. What is its kinetic energy? The mass of a neutron is 1.67×10^{-27} kg.

Solution We find the speed from

$$v = \frac{d}{t} = \frac{6.2\text{ m}}{160 \times 10^{-6}\text{ s}} = 3.88 \times 10^4\text{ m/s}.$$

From Eq. 18, the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1.67 \times 10^{-27}\text{ kg})(3.88 \times 10^4\text{ m/s})^2$$
$$= 1.26 \times 10^{-18}\text{ J} = 7.9\text{ eV}.$$

In nuclear reactors, neutrons are produced in nuclear fission with typical kinetic energies of a few MeV. Negative work has been done on the neutrons in this example by an external agent (called a moderator), thereby reducing their kinetic energies by a considerable factor from a few MeV to a few eV.

Sample Problem 6 A body of mass $m = 4.5$ g is dropped from rest at a height $h = 10.5$ m above the Earth's surface. What will its speed be just before it strikes the ground?

Solution We assume that the body can be treated as a particle. We could solve this problem using a method based on Newton's laws, such as we considered in Chapter 5. We choose instead to solve it here using the work-energy theorem. The gain in kinetic energy is equal to the work done by the resultant force, which here is the force of gravity. This force is constant and directed along the line of motion, so that the work done by gravity is

$$W = \mathbf{F}\cdot\mathbf{s} = mgh.$$

Initially, the body has a speed $v_0 = 0$ and finally a speed v. The gain in kinetic energy of the body is

$$\Delta K = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 - 0.$$

According to the work-energy theorem, $W = \Delta K$ and so

$$mgh = \tfrac{1}{2}mv^2.$$

The speed of the body is then

$$v = \sqrt{2gh} = \sqrt{2(9.80\text{ m/s}^2)(10.5\text{ m})} = 14.3\text{ m/s}.$$

Note that this result is independent of the mass of the object, as we have previously deduced using Newton's laws.

Sample Problem 7 A block of mass $m = 3.63$ kg slides on a horizontal frictionless table with a speed of $v = 1.22$ m/s. It is brought to rest in compressing a spring in its path. By how much is the spring compressed if its force constant k is 135 N/m?

Solution The change in kinetic energy of the block is

$$\Delta K = K_f - K_i = 0 - \tfrac{1}{2}mv^2.$$

The work W done by the spring on the block when the spring is compressed from its relaxed length through a distance d is, according to Eq. 10,

$$W = -\tfrac{1}{2}kd^2.$$

Using the work-energy theorem, $W = \Delta K$, we obtain

$$-\tfrac{1}{2}kd^2 = -\tfrac{1}{2}mv^2$$

or

$$d = v\sqrt{\frac{m}{k}} = (1.22\ \text{m/s})\sqrt{\frac{3.63\ \text{kg}}{135\ \text{N/m}}} = 0.200\ \text{m}.$$

Limitation of the Work-Energy Theorem

We derived the work-energy theorem, Eq. 19, directly from Newton's second law, which, in the form in which we have stated it, applies *only to particles*. Hence the work-energy theorem, as we have presented it so far, likewise applies only to particles. We can apply this important theorem to real objects only if those objects behave like particles. Previously, we considered an object to behave like a particle if all parts of the object move in exactly the same way. In the use of the work-energy theorem, we can treat an extended object as a particle if the only kind of energy it has is directed kinetic energy.

Consider, for example, a test car that is crashed head-on into a heavy, rigid concrete barrier. The directed kinetic energy of the car certainly decreases as the car hits the barrier, crumples up, and comes to rest. However, there are forms of energy other than directed kinetic energy that enter into this situation. There is internal energy associated with the bending and crumpling of the body of the car; some of this internal energy may appear, for instance, as an increase in the temperature of the car, and some may be transferred to the surroundings as heat. Note that, even though the barrier may exert a large force on the car during the crash, the force does no work because *the point of application of the force on the car does not move.* (Recall our original definition of work—given by Eq. 1 and illustrated in Fig. 1—the force must act through some distance to do work.) Thus in this case $\Delta K \neq 0$, but $W = 0$; clearly, Eq. 19 does not hold. The car does *not* behave like a particle: every part of it does *not* move in exactly the same way.

For similar reasons, from the work-energy standpoint, we cannot treat a sliding block acted on by a frictional force as a particle (even though we *can* continue to treat it as a particle, as we did in Chapter 6, when analyzing its behavior using Newton's laws). The frictional force, which we represented as a constant force **f**, is in reality quite complicated, involving the making and breaking of many microscopic welds (see Section 6-2), which deform the surfaces and result in changes in internal energy of the surfaces (which may in part be revealed as an increase in the temperature of the surfaces). Because of the difficulty of accounting for these other forms of energy, and because the objects do not behave as particles, it is generally not correct to apply the particle form of the work-energy theorem to objects subject to frictional forces.

In these examples, we must view the crashing car and the sliding block not as particles but as systems containing large numbers of particles. Although it would be correct to apply the work-energy theorem to each individual particle in the system, it would be hopelessly complicated to do so. In Chapter 9, we begin to develop a simpler method for dealing with complex systems of particles, and we show how to extend the work-energy theorem so that we may apply it in such cases.

7-5 POWER

In designing a mechanical system, it is often necessary to consider not only how much work must be done but also how rapidly the work is to be done. The same amount of work is done in raising a given body through a given height whether it takes 1 second or 1 year to do so. However, the *rate at which work is done* is very different in the two cases.

We define *power* as the rate at which work is done. (Here we consider only *mechanical* power, which results from mechanical work. A more general view of power as energy delivered per unit time permits us to broaden the concept of power to include electrical power, solar power, and so on.) The average power $\overline{P}$ delivered by an agent that exerts a particular force on a body is the total work done by that force on the body divided by the total time interval, or

$$\overline{P} = \frac{W}{t}. \tag{20}$$

The instantaneous power P delivered by an agent is

$$P = \frac{dW}{dt}, \tag{21}$$

where dW is the small amount of work done in the infinitesimal time interval dt. If the power is constant in time, then $P = \overline{P}$ and

$$W = Pt. \tag{22}$$

The SI unit of power is the joule per second, which is called 1 *watt* (abbreviation W). This unit is named in honor of James Watt (1736–1819), who made major improvements to the steam engines of his day and pointed the way toward today's more efficient engines. In the British system, the unit of power is 1 ft·lb/s, although a more common practical unit, the *horsepower* (hp), is generally used to describe the power of such devices as electric motors or automobile engines. One horsepower is defined to be 550 ft·lb/s, which is equivalent to about 746 W.

Work can also be expressed in units of power × time. This is the origin of the term *kilowatt-hour,* which the electric company uses to measure how much work (in the form of electrical energy) it has delivered to your house. One kilowatt-hour is the work done in 1 hour by an agent working at a constant rate of 1 kW.

We can also express the power delivered to a body in terms of the velocity of the body and the force that acts on it. In general, we can rewrite Eq. 21 as

$$P=\frac{dW}{dt}=\frac{\mathbf{F}\cdot d\mathbf{s}}{dt}=\mathbf{F}\cdot\frac{d\mathbf{s}}{dt}$$

which becomes, after substituting the velocity $\mathbf{v}$ for $d\mathbf{s}/dt$,

$$P=\mathbf{F}\cdot\mathbf{v}. \qquad (23)$$

If $\mathbf{F}$ and $\mathbf{v}$ are parallel to one another, this can be written

$$P=Fv. \qquad (24)$$

Note that the power can be negative if $\mathbf{F}$ and $\mathbf{v}$ are antiparallel. Delivering negative power to a body means doing negative work on it: the force exerted on the body by the external agent is in a direction opposite to its displacement $d\mathbf{s}$ and therefore opposite to $\mathbf{v}$.

Sample Problem 8 An elevator has an empty weight of 5160 N (1160 lb). It is designed to carry a maximum load of 20 passengers from the ground floor to the 25th floor of a building in a time of 18 seconds. Assuming the average weight of a passenger to be 710 N (160 lb) and the distance between floors to be 3.5 m (11 ft), what is the minimum constant power needed for the elevator motor? (Assume that all the work that lifts the elevator comes from the motor and that the elevator has no counterweight.)

Solution The minimum total force that must be exerted is the total weight of the elevator and passengers, $F = 5160\ \text{N} + 20(710)\ \text{N} = 19{,}400\ \text{N}$. The work that must be done is

$$W = Fs = (19{,}400\ \text{N})(25 \times 3.5\ \text{m}) = 1.7 \times 10^6\ \text{J}.$$

The minimum power is therefore

$$P=\frac{W}{t}=\frac{1.7\times10^6\ \text{J}}{18\ \text{s}}=94\ \text{kW}.$$

This is the same as 126 hp, roughly the power delivered by the engine of an automobile. Of course, frictional losses and other inefficiencies will increase the power that the motor must provide to lift the elevator.

In practice, an elevator usually has a counterweight that falls as the elevator cab rises. The motor delivers positive power to the rising cab and negative power to the falling counterweight. Thus the *net* power that the motor must provide is greatly reduced.

7-6 REFERENCE FRAMES *(Optional)*

Newton's laws hold only in inertial reference frames (see Section 6-8), and if they hold in one particular inertial frame then they hold in all reference frames that move at constant velocity relative to that frame. Certain physical quantities, if observed in different inertial frames, always give the same measured result. In Newtonian mechanics, these *invariant* quantities include force, mass, acceleration, and time. Other quantities, such as displacement or velocity, are not invariant when measured from different inertial frames. For example, we discussed in Section 4-6 how to relate velocities measured from two reference frames in relative motion at constant velocity.

Two observers in different inertial frames will measure the same acceleration for a particle, and so they must deduce the same value for the change in its velocity, Δv, but they will in general *not* measure the same change in its kinetic energy. Observers in relative motion will also measure different values for the displacement of a particle, so that (although they measure the same values for the forces acting on the particle, force being an invariant) they will deduce different values for the work done on the particle. In this section we clarify these statements with a specific numerical example that demonstrates the validity of the work–energy theorem from the points of view of observers in different inertial frames.

Consider the following example. A worker on a flatbed railroad car is pushing a crate. The train is moving at a constant speed of 15.0 m/s. The crate has a mass of 12 kg, and in being pushed forward over a distance of 2.4 m its velocity is increased (relative to the car) at constant acceleration from rest to 1.5 m/s. Figure 12*a* shows the starting and finishing positions according

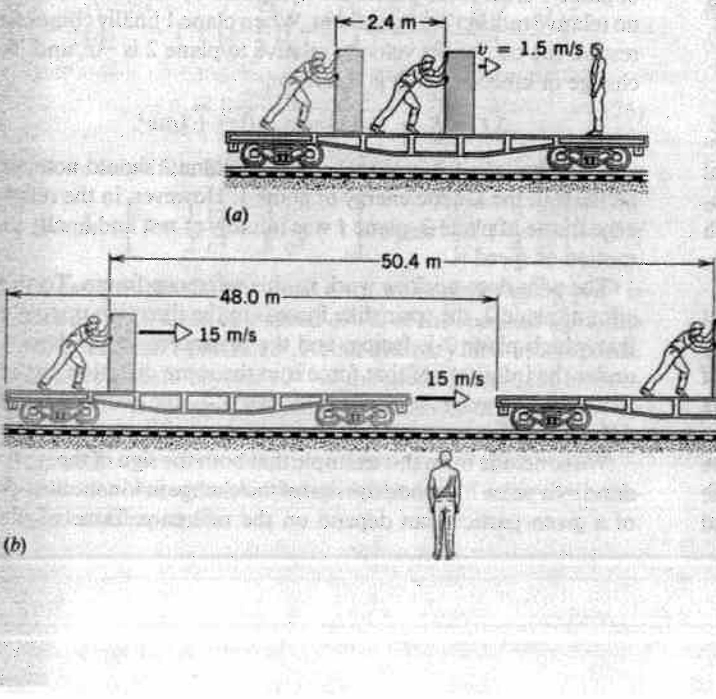

Figure 12 A worker on a flatbed railroad car pushing a crate forward, as viewed by (*a*) an observer on the train and (*b*) an observer on the ground.