

1 VECTORS AND KINEMATICS- A FEW MATHEMATICAL PRELIMINARIES

1.1 Introduction

The goal of this book is to help you acquire a deep understanding of the principles of mechanics. The subject of mechanics is at the very heart of physics; its concepts are essential for understanding the everyday physical world as well as phenomena on the atomic and cosmic scales. The concepts of mechanics, such as momentum, angular momentum, and energy, play a vital role in practically every area of physics.

We shall use mathematics frequently in our discussion of physical principles, since mathematics lets us express complicated ideas quickly and transparently, and it often points the way to new insights. Furthermore, the interplay of theory and experiment in physics is based on quantitative prediction and measurement. For these reasons, we shall devote this chapter to developing some necessary mathematical tools and postpone our discussion of the principles of mechanics until Chap. 2.

1.2 Vectors

The study of vectors provides a good introduction to the role of mathematics in physics. By using vector notation, physical laws can often be written in compact and simple form. (As a matter of fact, modern vector notation was invented by a physicist, Willard Gibbs of Yale University, primarily to simplify the appearance of equations.) For example, here is how Newton's second law (which we shall discuss in the next chapter) appears in nineteenth century notation:

$$F_x = ma_x$$

$$F_y = ma_y$$

$$F_z = ma_z.$$

In vector notation, one simply writes

$$\mathbf{F} = m\mathbf{a}.$$

Our principal motivation for introducing vectors is to simplify the form of equations. However, as we shall see in the last chapter of the book, vectors have a much deeper significance. Vectors are closely related to the fundamental ideas of symmetry and their use can lead to valuable insights into the possible forms of unknown laws.

Definition of a Vector

Vectors can be approached from three points of view—geometric, analytic, and axiomatic. Although all three points of view are useful, we shall need only the geometric and analytic approaches in our discussion of mechanics.

From the geometric point of view, a vector is a *directed line segment*. In writing, we can represent a vector by an arrow and label it with a letter capped by a symbolic arrow. In print, bold-faced letters are traditionally used.

In order to describe a vector we must specify both its length and its direction. Unless indicated otherwise, we shall assume that parallel translation does not change a vector. Thus the arrows at left all represent the same vector.

If two vectors have the same length and the same direction they are equal. The vectors **B** and **C** are equal:

$$\mathbf{B} = \mathbf{C}.$$

The length of a vector is called its *magnitude*. The magnitude of a vector is indicated by vertical bars or, if no confusion will occur, by using italics. For example, the magnitude of **A** is written $|\mathbf{A}|$, or simply A . If the length of **A** is $\sqrt{2}$, then $|\mathbf{A}| = A = \sqrt{2}$.

If the length of a vector is one unit, we call it a *unit vector*. A unit vector is labeled by a caret; the vector of unit length parallel to **A** is $\hat{\mathbf{A}}$. It follows that

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|},$$

and conversely

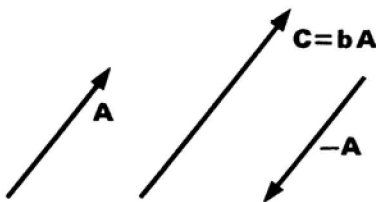
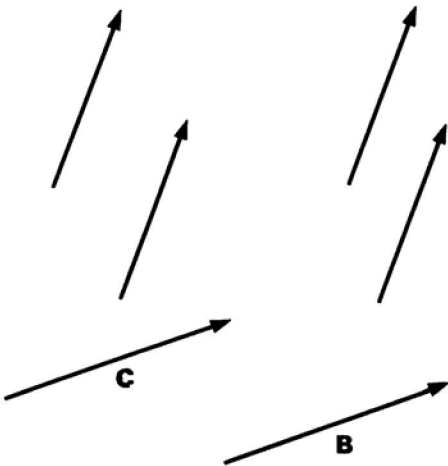
$$\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}.$$

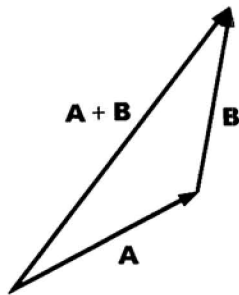
The Algebra of Vectors

Multiplication of a Vector by a Scalar If we multiply **A** by a positive scalar b , the result is a new vector $\mathbf{C} = b\mathbf{A}$. The vector **C** is parallel to **A**, and its length is b times greater. Thus $\hat{\mathbf{C}} = \hat{\mathbf{A}}$, and $|\mathbf{C}| = b|\mathbf{A}|$.

The result of multiplying a vector by -1 is a new vector opposite in direction (antiparallel) to the original vector.

Multiplication of a vector by a negative scalar evidently can change both the magnitude and the direction sense.

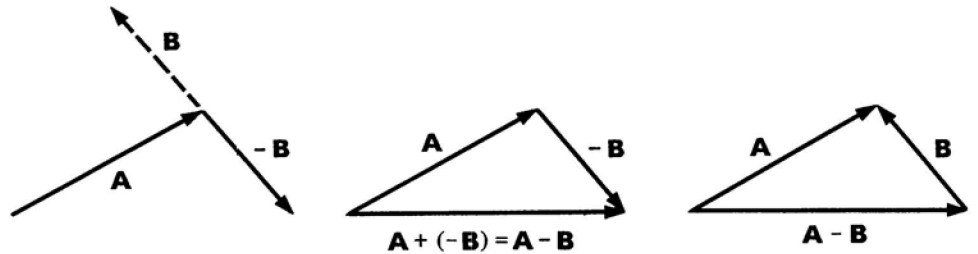




Addition of Two Vectors Addition of vectors has the simple geometrical interpretation shown by the drawing.

The rule is: To add \mathbf{B} to \mathbf{A} , place the tail of \mathbf{B} at the head of \mathbf{A} . The sum is a vector from the tail of \mathbf{A} to the head of \mathbf{B} .

Subtraction of Two Vectors Since $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, in order to subtract \mathbf{B} from \mathbf{A} we can simply multiply it by -1 and then add. The sketches below show how.

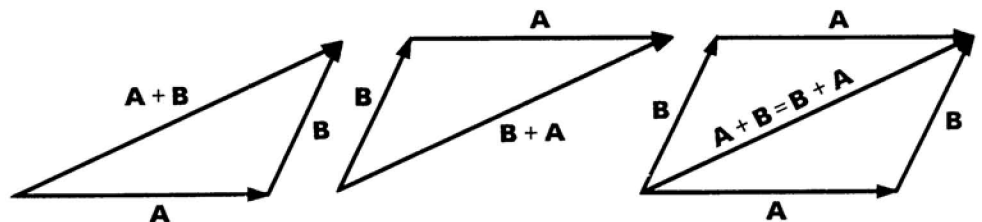


An equivalent way to construct $\mathbf{A} - \mathbf{B}$ is to place the *head* of \mathbf{B} at the *head* of \mathbf{A} . Then $\mathbf{A} - \mathbf{B}$ extends from the *tail* of \mathbf{A} to the *tail* of \mathbf{B} , as shown in the right hand drawing above.

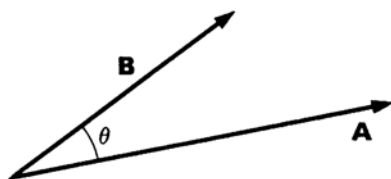
It is not difficult to prove the following laws. We give a geometrical proof of the commutative law; try to cook up your own proofs of the others.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	Associative law
$c(d\mathbf{A}) = (cd)\mathbf{A}$	
$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$	Distributive law
$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$	

Proof of the Commutative law of vector addition



Although there is no great mystery to addition, subtraction, and multiplication of a vector by a scalar, the result of “multiplying” one vector by another is somewhat less apparent. Does multiplication yield a vector, a scalar, or some other quantity? The choice is up to us, and we shall define two types of products which are useful in our applications to physics.



Scalar Product ("Dot" Product) The first type of product is called the *scalar* product, since it represents a way of combining two vectors to form a scalar. The scalar product of **A** and **B** is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is often called the dot product. $\mathbf{A} \cdot \mathbf{B}$ is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \cos \theta.$$

Here θ is the angle between **A** and **B** when they are drawn tail to tail.

Since $|\mathbf{B}| \cos \theta$ is the projection of **B** along the direction of **A**, $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \times (\text{projection of } \mathbf{B} \text{ on } \mathbf{A})$.

Similarly,

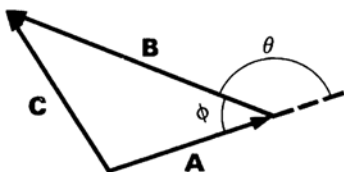
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{B}| \times (\text{projection of } \mathbf{A} \text{ on } \mathbf{B}).$$

If $\mathbf{A} \cdot \mathbf{B} = 0$, then $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, or **A** is perpendicular to **B** (that is, $\cos \theta = 0$). Scalar multiplication is unusual in that the dot product of two nonzero vectors can be 0.

Note that $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$.

By way of demonstrating the usefulness of the dot product, here is an almost trivial proof of the law of cosines.

Example 1.1 Law of Cosines



$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$$

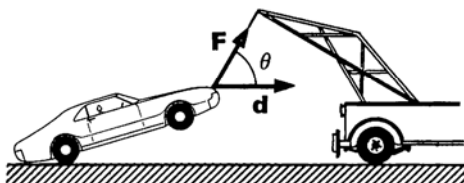
$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2|\mathbf{A}| |\mathbf{B}| \cos \theta$$

This result is generally expressed in terms of the angle ϕ :

$$C^2 = A^2 + B^2 - 2AB \cos \phi.$$

(We have used $\cos \theta = \cos (\pi - \phi) = -\cos \phi$.)

Example 1.2 Work and the Dot Product



The dot product finds its most important application in the discussion of work and energy in Chap. 4. As you may already know, the work W done by a force F on an object is the displacement d of the object times the component of F along the direction of d . If the force is applied at an angle θ to the displacement,

$$W = (F \cos \theta)d.$$

Granting for the time being that force and displacement are vectors,

$$W = \mathbf{F} \cdot \mathbf{d}.$$

Vector Product (“Cross” Product) The second type of product we need is the vector product. In this case, two vectors **A** and **B** are combined to form a third vector **C**. The symbol for vector product is a cross:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

An alternative name is the *cross product*.

The vector product is more complicated than the scalar product because we have to specify both the magnitude and direction of $\mathbf{A} \times \mathbf{B}$. The magnitude is defined as follows: if

$$\mathbf{C} = \mathbf{A} \times \mathbf{B},$$

then

$$|\mathbf{C}| = |\mathbf{A}| |\mathbf{B}| \sin \theta,$$

where θ is the angle between **A** and **B** when they are drawn tail to tail. (To eliminate ambiguity, θ is always taken as the angle smaller than π .) Note that the vector product is zero when $\theta = 0$ or π , even if $|\mathbf{A}|$ and $|\mathbf{B}|$ are not zero.

When we draw **A** and **B** tail to tail, they determine a plane. We define the direction of **C** to be perpendicular to the plane of **A** and **B**. **A**, **B**, and **C** form what is called a *right hand triple*. Imagine a right hand coordinate system with **A** and **B** in the *xy* plane as shown in the sketch. **A** lies on the *x* axis and **B** lies toward the *y* axis. If **A**, **B**, and **C** form a right hand triple, then **C** lies on the *z* axis. We shall always use right hand coordinate systems such as the one shown at left. Here is another way to determine the direction of the cross product. Think of a right hand screw with the axis perpendicular to **A** and **B**. Rotate it in the direction which swings **A** into **B**. **C** lies in the direction the screw advances. (Warning: Be sure not to use a left hand screw. Fortunately, they are rare. Hot water faucets are among the chief offenders; your honest everyday wood screw is right handed.)

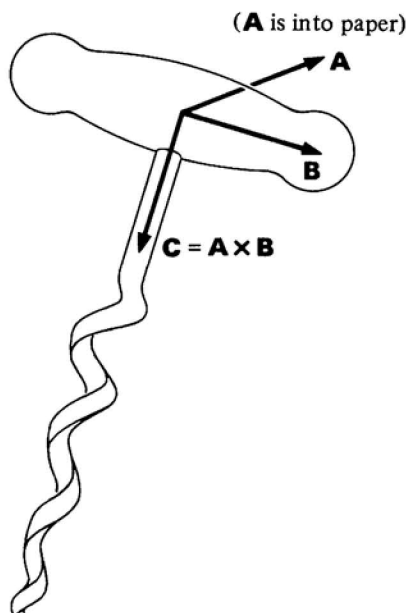
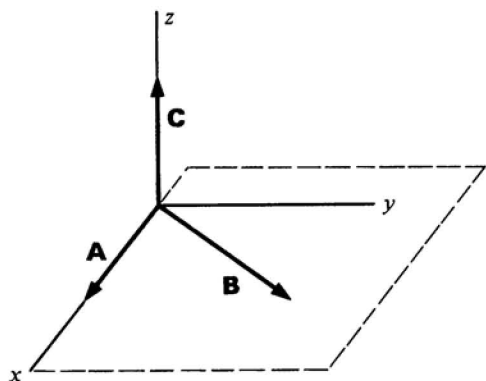
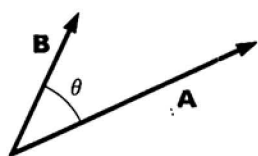
A result of our definition of the cross product is that

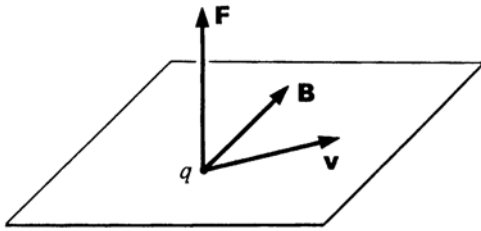
$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}.$$

Here we have a case in which the order of multiplication is important. The vector product is *not* commutative. (In fact, since reversing the order reverses the sign, it is anticommutative.) We see that

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

for any vector **A**.



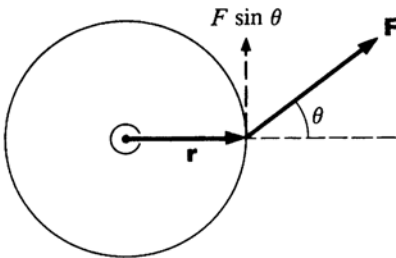
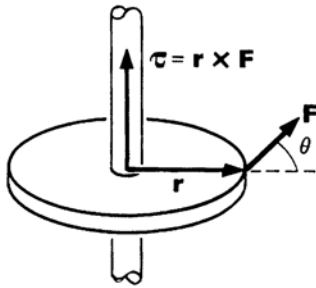
Example 1.3 Examples of the Vector Product in Physics

The vector product has a multitude of applications in physics. For instance, if you have learned about the interaction of a charged particle with a magnetic field, you know that the force is proportional to the charge q , the magnetic field B , and the velocity of the particle v . The force varies as the sine of the angle between v and B , and is perpendicular to the plane formed by v and B , in the direction indicated. A simpler way to give all these rules is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Another application is the definition of torque. We shall develop this idea later. For now we simply mention in passing that the torque $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$



Top view

where \mathbf{r} is a vector from the axis about which the torque is evaluated to the point of application of the force \mathbf{F} . This definition is consistent with the familiar idea that torque is a measure of the ability of an applied force to produce a twist. Note that a large force directed parallel to \mathbf{r} produces no twist; it merely pulls. Only $F \sin \theta$, the component of force perpendicular to \mathbf{r} , produces a torque. The torque increases as the lever arm gets larger. As you will see in Chap. 6, it is extremely useful to associate a direction with torque. The natural direction is along the axis of rotation which the torque tends to produce. All these ideas are summarized in a nutshell by the simple equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

Example 1.4 Area as a Vector

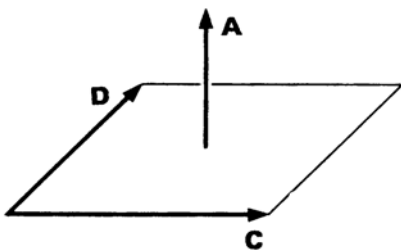
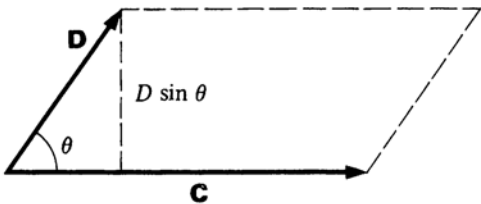
We can use the cross product to describe an area. Usually one thinks of area in terms of magnitude only. However, many applications in physics require that we also specify the orientation of the area. For example, if we wish to calculate the rate at which water in a stream flows through a wire loop of given area, it obviously makes a difference whether the plane of the loop is perpendicular or parallel to the flow. (In the latter case the flow through the loop is zero.) Here is how the vector product accomplishes this:

Consider the area of a quadrilateral formed by two vectors, \mathbf{C} and \mathbf{D} . The area of the parallelogram A is given by

$$\begin{aligned} A &= \text{base} \times \text{height} \\ &= CD \sin \theta \\ &= |\mathbf{C} \times \mathbf{D}|. \end{aligned}$$

If we think of A as a vector, we have

$$\mathbf{A} = \mathbf{C} \times \mathbf{D}.$$



We have already shown that the magnitude of \mathbf{A} is the area of the parallelogram, and the vector product defines the convention for assigning a direction to the area. The direction is defined to be perpendicular to the plane of the area; that is, the direction is parallel to a *normal* to the surface. The sign of the direction is to some extent arbitrary; we could just as well have defined the area by $\mathbf{A} = \mathbf{D} \times \mathbf{C}$. However, once the sign is chosen, it is unique.

1.3 Components of a Vector

The fact that we have discussed vectors without introducing a particular coordinate system shows why vectors are so useful; vector operations are defined without reference to coordinate systems. However, eventually we have to translate our results from the abstract to the concrete, and at this point we have to choose a coordinate system in which to work.

For simplicity, let us restrict ourselves to a two-dimensional system, the familiar xy plane. The diagram shows a vector \mathbf{A} in the xy plane. The projections of \mathbf{A} along the two coordinate axes are called the components of \mathbf{A} . The components of \mathbf{A} along the x and y axes are, respectively, A_x and A_y . The magnitude of \mathbf{A} is $|\mathbf{A}| = (A_x^2 + A_y^2)^{1/2}$, and the direction of \mathbf{A} is such that it makes an angle $\theta = \arctan(A_y/A_x)$ with the x axis.

Since the components of a vector define it, we can specify a vector entirely by its components. Thus

$$\mathbf{A} = (A_x, A_y)$$

or, more generally, in three dimensions,

$$\mathbf{A} = (A_x, A_y, A_z).$$

Prove for yourself that $|\mathbf{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2}$. The vector \mathbf{A} has a meaning independent of any coordinate system. However, the components of \mathbf{A} depend on the coordinate system being used. To illustrate this, here is a vector \mathbf{A} drawn in two different coordinate systems. In the first case,

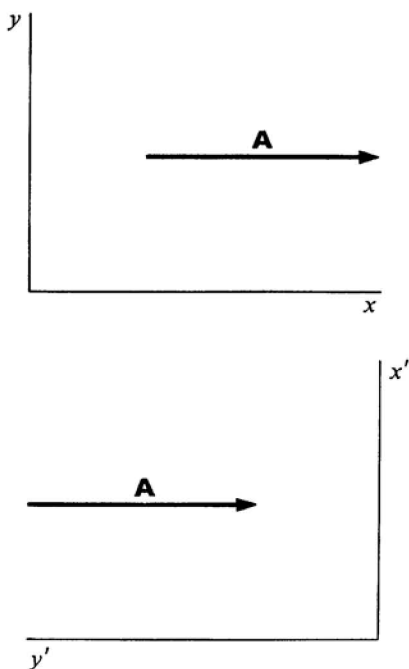
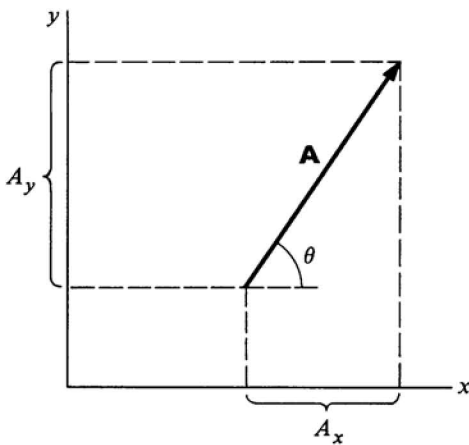
$$\mathbf{A} = (A, 0) \quad (x, y \text{ system}),$$

while in the second

$$\mathbf{A} = (0, -A) \quad (x', y' \text{ system}).$$

Unless noted otherwise, we shall restrict ourselves to a single coordinate system, so that if

$$\mathbf{A} = \mathbf{B},$$



then

$$A_x = B_x \quad A_y = B_y \quad A_z = B_z.$$

The single vector equation $\mathbf{A} = \mathbf{B}$ symbolically represents three scalar equations.

All vector operations can be written as equations for components. For instance, multiplication by a scalar gives

$$c\mathbf{A} = (cA_x, cA_y).$$

The law for vector addition is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x, A_y + B_y, A_z + B_z).$$

By writing \mathbf{A} and \mathbf{B} as the sums of vectors along each of the coordinate axes, you can verify that

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

We shall defer evaluating the cross product until the next section.

Example 1.5 Vector Algebra

Let

$$\mathbf{A} = (3, 5, -7)$$

$$\mathbf{B} = (2, 7, 1).$$

Find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A} \cdot \mathbf{B}$, and the cosine of the angle between \mathbf{A} and \mathbf{B} .

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (3 + 2, 5 + 7, -7 + 1) \\ &= (5, 12, -6) \end{aligned}$$

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= (3 - 2, 5 - 7, -7 - 1) \\ &= (1, -2, -8) \end{aligned}$$

$$\begin{aligned} |\mathbf{A}| &= (3^2 + 5^2 + 7^2)^{\frac{1}{2}} \\ &= \sqrt{83} \\ &= 9.11 \end{aligned}$$

$$\begin{aligned} |\mathbf{B}| &= (2^2 + 7^2 + 1^2)^{\frac{1}{2}} \\ &= \sqrt{54} \\ &= 7.35 \end{aligned}$$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= 3 \times 2 + 5 \times 7 - 7 \times 1 \\ &= 34 \end{aligned}$$

$$\cos(\mathbf{A}, \mathbf{B}) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \frac{34}{(9.11)(7.35)} = 0.507$$

Example 1.6 Construction of a Perpendicular Vector

Find a unit vector in the xy plane which is perpendicular to $\mathbf{A} = (3, 5, 1)$.

We denote the vector by $\mathbf{B} = (B_x, B_y, B_z)$. Since \mathbf{B} is in the xy plane, $B_z = 0$. For \mathbf{B} to be perpendicular to \mathbf{A} , we have $\mathbf{A} \cdot \mathbf{B} = 0$.

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= 3B_x + 5B_y \\ &= 0\end{aligned}$$

Hence $B_y = -\frac{3}{5}B_x$. However, \mathbf{B} is a unit vector, which means that $B_x^2 + B_y^2 = 1$. Combining these gives $B_x^2 + \frac{9}{25}B_x^2 = 1$, or $B_x = \sqrt{\frac{25}{34}} = \pm 0.857$ and $B_y = -\frac{3}{5}B_x = \mp 0.514$.

The ambiguity in sign of B_x and B_y indicates that \mathbf{B} can point along a line perpendicular to \mathbf{A} in either of two directions.

1.4 Base Vectors

Base vectors are a set of orthogonal (perpendicular) unit vectors, one for each dimension. For example, if we are dealing with the familiar cartesian coordinate system of three dimensions, the base vectors lie along the x , y , and z axes. The x unit vector is denoted by $\hat{\mathbf{i}}$, the y unit vector by $\hat{\mathbf{j}}$, and the z unit vector by $\hat{\mathbf{k}}$.

The base vectors have the following properties, as you can readily verify:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}.$$

We can write any vector in terms of the base vectors.

$$\mathbf{A} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$$

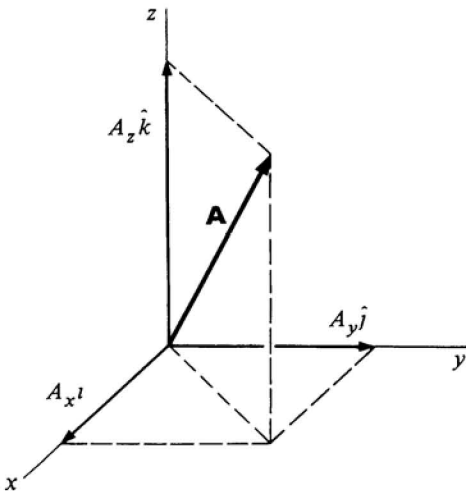
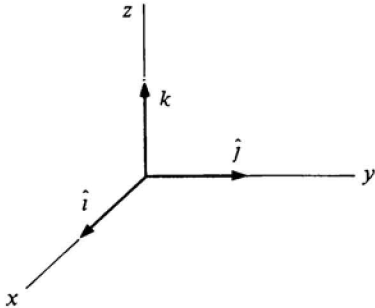
The sketch illustrates these two representations of a vector.

To find the component of a vector in any direction, take the dot product with a unit vector in that direction. For instance,

$$A_z = \mathbf{A} \cdot \hat{\mathbf{k}}.$$

It is easy to evaluate the vector product $\mathbf{A} \times \mathbf{B}$ with the aid of the base vectors.

$$\mathbf{A} \times \mathbf{B} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})$$



Consider the first term:

$$A_x \hat{\mathbf{i}} \times \mathbf{B} = A_x B_x (\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_x B_y (\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + A_x B_z (\hat{\mathbf{i}} \times \hat{\mathbf{k}}).$$

(We have assumed the associative law here.) Since $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, and $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, we find

$$A_x \hat{\mathbf{i}} \times \mathbf{B} = A_x (B_y \hat{\mathbf{k}} - B_z \hat{\mathbf{j}}).$$

The same argument applied to the y and z components gives

$$A_y \hat{\mathbf{j}} \times \mathbf{B} = A_y (B_z \hat{\mathbf{i}} - B_x \hat{\mathbf{k}})$$

$$A_z \hat{\mathbf{k}} \times \mathbf{B} = A_z (B_x \hat{\mathbf{j}} - B_y \hat{\mathbf{i}}).$$

A quick way to derive these relations is to work out the first and then to obtain the others by cyclically permuting x , y , z , and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ (that is, $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$, and $\hat{\mathbf{i}} \rightarrow \hat{\mathbf{j}}$, $\hat{\mathbf{j}} \rightarrow \hat{\mathbf{k}}$, $\hat{\mathbf{k}} \rightarrow \hat{\mathbf{i}}$.) A simple way to remember the result is to use the following device: write the base vectors and the components of \mathbf{A} and \mathbf{B} as three rows of a determinant,¹ like this

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{\mathbf{i}}(A_y B_z - A_z B_y) - \hat{\mathbf{j}}(A_x B_z - A_z B_x) + \hat{\mathbf{k}}(A_x B_y - A_y B_x). \end{aligned}$$

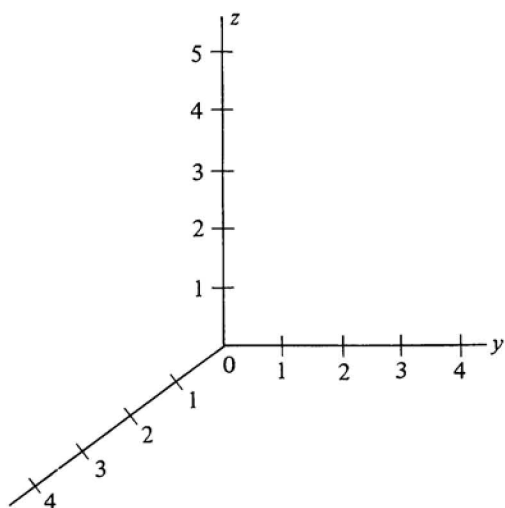
For instance, if $\mathbf{A} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{B} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{vmatrix} \\ &= 10\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 11\hat{\mathbf{k}}. \end{aligned}$$

1.5 Displacement and the Position Vector

So far we have discussed only abstract vectors. However, the reason for introducing vectors here is concrete—they are just right for describing kinematical laws, the laws governing the geometrical properties of motion, which we need to begin our discussion of mechanics. Our first application of vectors will be to the description of position and motion in familiar three dimensional space. Although our first application of vectors is to the motion of a point in space, don't conclude that this is the only

¹ If you are unfamiliar with simple determinants, most of the books listed at the end of the chapter discuss determinants.



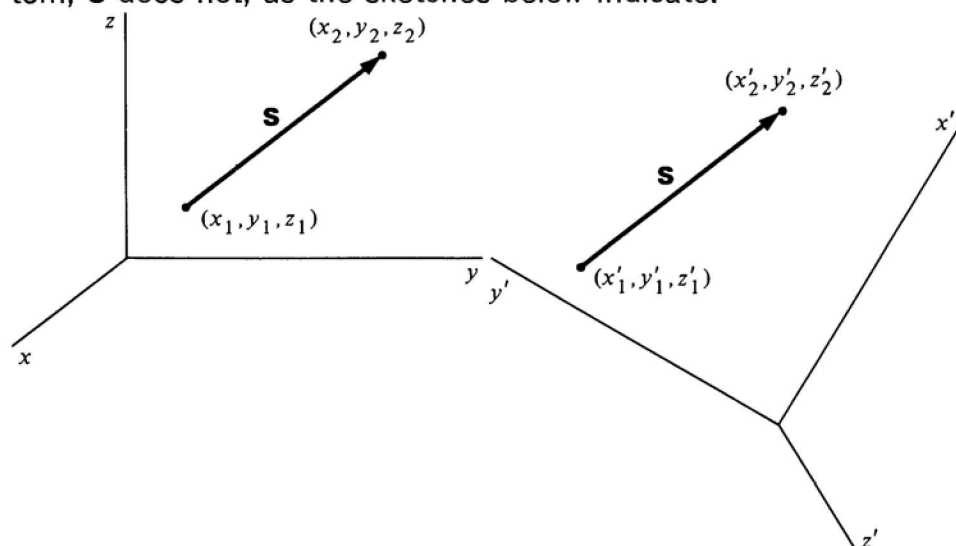
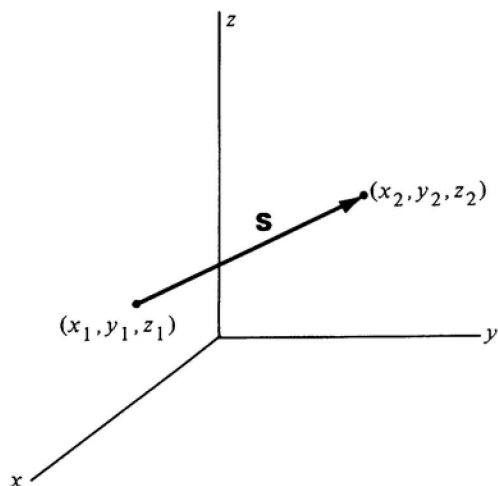
application, or even an unusually important one. Many physical quantities besides displacements are vectors. Among these are velocity, force, momentum, and gravitational and electric fields.

To locate the position of a point in space, we start by setting up a coordinate system. For convenience we choose a three dimensional cartesian system with axes x , y , and z , as shown.

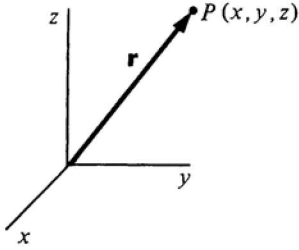
In order to measure position, the axes must be marked off in some convenient unit of length—meters, for instance.

The position of the point of interest is given by listing the values of its three coordinates, x_1, y_1, z_1 . These numbers do *not* represent the components of a vector according to our previous discussion. (They specify a position, not a magnitude and direction.) However, if we move the point to some new position, x_2, y_2, z_2 , then the *displacement* defines a vector \mathbf{S} with coordinates $S_x = x_2 - x_1, S_y = y_2 - y_1, S_z = z_2 - z_1$.

\mathbf{S} is a vector from the initial position to the final position—it defines the displacement of a point of interest. Note, however, that \mathbf{S} contains no information about the initial and final positions separately—only about the *relative* position of each. Thus, $S_z = z_2 - z_1$ depends on the *difference* between the final and initial values of the z coordinates; it does not specify z_2 or z_1 separately. \mathbf{S} is a true vector; although the values of the coordinates of the initial and final points depend on the coordinate system, \mathbf{S} does not, as the sketches below indicate.



One way in which our displacement vector differs from a mathematician's vector is that his vectors are usually pure quantities, with components given by absolute numbers, whereas \mathbf{S} has the physical dimension of length associated with it. We will use the convention that the magnitude of a vector has dimensions



so that a unit vector is dimensionless. Thus, a displacement of 8 m (8 meters) in the x direction is $\mathbf{S} = (8 \text{ m}, 0, 0)$. $|\mathbf{S}| = 8 \text{ m}$, and $\hat{\mathbf{S}} = \mathbf{S}/|\mathbf{S}| = \hat{\mathbf{i}}$.

Although vectors define displacements rather than positions, it is in fact possible to describe the position of a point with respect to the origin of a given coordinate system by a special vector, known as the *position vector*, which extends from the origin to the point of interest. We shall use the symbol \mathbf{r} to denote the position vector. The position of an arbitrary point P at (x, y, z) is written as

$$\mathbf{r} = (x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

Unlike ordinary vectors, \mathbf{r} depends on the coordinate system. The sketch to the left shows position vectors \mathbf{r} and \mathbf{r}' indicating the position of the same point in space but drawn in different coordinate systems. If \mathbf{R} is the vector from the origin of the unprimed coordinate system to the origin of the primed coordinate system, we have

$$\mathbf{r}' = \mathbf{r} - \mathbf{R}.$$

In contrast, a true vector, such as a displacement \mathbf{S} , is independent of coordinate system. As the bottom sketch indicates,

$$\begin{aligned} \mathbf{S} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (\mathbf{r}'_2 + \mathbf{R}) - (\mathbf{r}'_1 + \mathbf{R}) \\ &= \mathbf{r}'_2 - \mathbf{r}'_1. \end{aligned}$$

1.6 Velocity and Acceleration

Motion in One Dimension

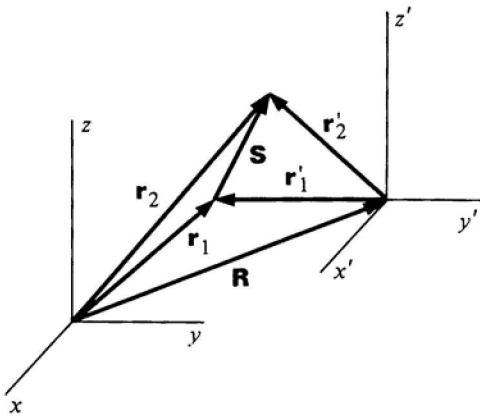
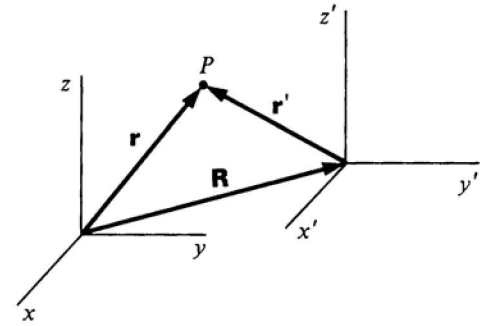
Before applying vectors to velocity and acceleration in three dimensions, it may be helpful to review briefly the case of one dimension, motion along a straight line.

Let x be the value of the coordinate of a particle moving along a line. x is measured in some convenient unit, such as meters, and we assume that we have a continuous record of position versus time.

The *average velocity* \bar{v} of the point between two times, t_1 and t_2 , is defined by

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

(We shall often use a bar to indicate an average of a quantity.)



The *instantaneous velocity* v is the limit of the average velocity as the time interval approaches zero.

$$v = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

The limit we have introduced in defining v is precisely that involved in the definition of a derivative. In fact, we have¹

$$v = \frac{dx}{dt}.$$

In a similar fashion, the *instantaneous acceleration* is

$$\begin{aligned} a &= \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \\ &= \frac{dv}{dt}. \end{aligned}$$

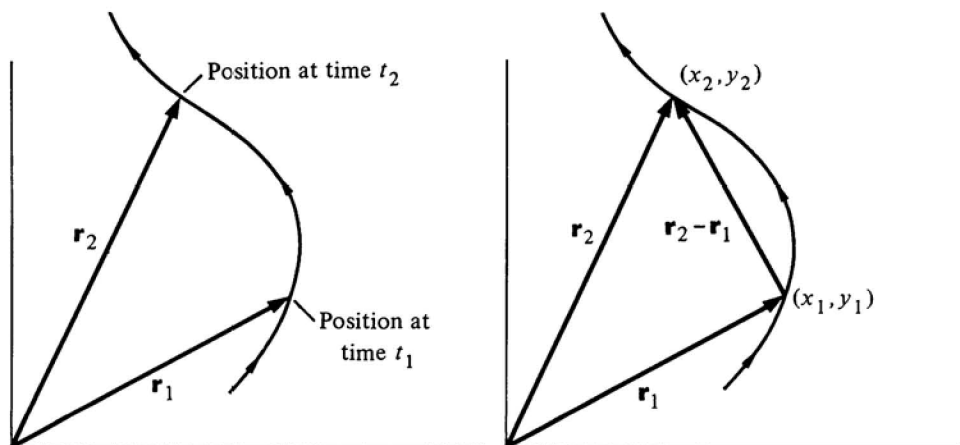
The concept of speed is sometimes useful. Speed s is simply the magnitude of the velocity: $s = |\mathbf{v}|$.

Motion in Several Dimensions

Our task now is to extend the ideas of velocity and acceleration to several dimensions. Consider a particle moving in a plane. As time goes on, the particle traces out a path, and we suppose that we know the particle's coordinates as a function of time. The instantaneous position of the particle at some time t_1 is

$$\mathbf{r}(t_1) = [x(t_1), y(t_1)] \quad \text{or} \quad \mathbf{r}_1 = (x_1, y_1),$$

¹ Physicists generally use the Leibnitz notation dx/dt , since this is a handy form for using differentials (see Note 1.1). Starting in Sec. 1.9 we shall use Newton's notation \dot{x} , but only to denote derivatives with respect to time.



where x_1 is the value of x at $t = t_1$, and so forth. At time t_2 the position is

$$\mathbf{r}_2 = (x_2, y_2).$$

The displacement of the particle between times t_1 and t_2 is

$$\mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1, y_2 - y_1).$$

We can generalize our example by considering the position at some time t , and at some later time $t + \Delta t$.† The displacement of the particle between these times is

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

This vector equation is equivalent to the two scalar equations

$$\Delta x = x(t + \Delta t) - x(t)$$

$$\Delta y = y(t + \Delta t) - y(t).$$

The *velocity* \mathbf{v} of the particle as it moves along the path is defined to be

$$\begin{aligned} \mathbf{v} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} \\ &= \frac{d\mathbf{r}}{dt}. \end{aligned}$$

which is equivalent to the two scalar equations

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

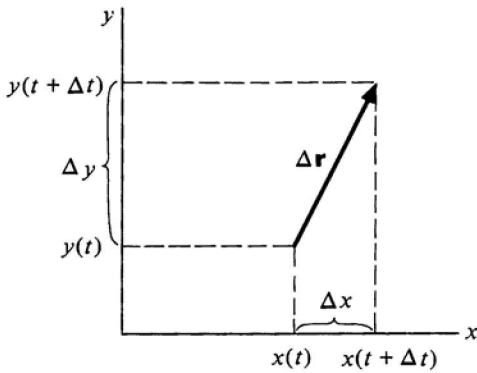
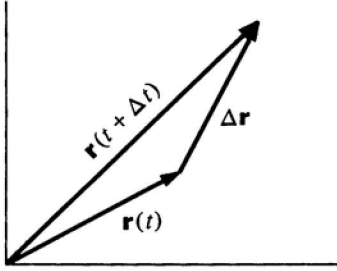
$$v_y = \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

Extension of the argument to three dimensions is trivial. The third component of velocity is

$$v_z = \lim_{\Delta t \rightarrow 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} = \frac{dz}{dt}.$$

Our definition of velocity as a vector is a straightforward generalization of the familiar concept of motion in a straight line. Vector notation allows us to describe motion in three dimensions with a single equation, a great economy compared with the three equations we would need otherwise. The equation $\mathbf{v} = d\mathbf{r}/dt$ expresses the results we have just found.

† We will often use the quantity Δ to denote a difference or change, as in the case here of $\Delta \mathbf{r}$ and Δt . However, this implies nothing about the size of the quantity, which may be large or small, as we please.



Alternatively, since $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we obtain by simple differentiation¹

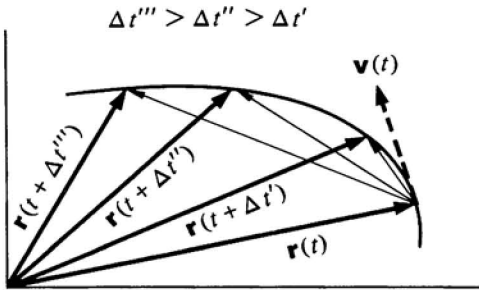
$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

as before.

Let the particle undergo a displacement $\Delta\mathbf{r}$ in time Δt . In the limit $\Delta t \rightarrow 0$, $\Delta\mathbf{r}$ becomes tangent to the trajectory, as the sketch indicates. However, the relation

$$\begin{aligned}\Delta\mathbf{r} &\approx \frac{d\mathbf{r}}{dt} \Delta t \\ &= \mathbf{v} \Delta t,\end{aligned}$$

which becomes exact in the limit $\Delta t \rightarrow 0$, shows that \mathbf{v} is parallel to $\Delta\mathbf{r}$; the instantaneous velocity \mathbf{v} of a particle is everywhere tangent to the trajectory.



Example 1.7 Finding \mathbf{v} from \mathbf{r}

The position of a particle is given by

$$\mathbf{r} = A(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}),$$

where α is a constant. Find the velocity, and sketch the trajectory.

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= A(\alpha e^{\alpha t}\hat{\mathbf{i}} - \alpha e^{-\alpha t}\hat{\mathbf{j}})\end{aligned}$$

or

$$\begin{aligned}v_x &= A\alpha e^{\alpha t} \\ v_y &= -A\alpha e^{-\alpha t}.\end{aligned}$$

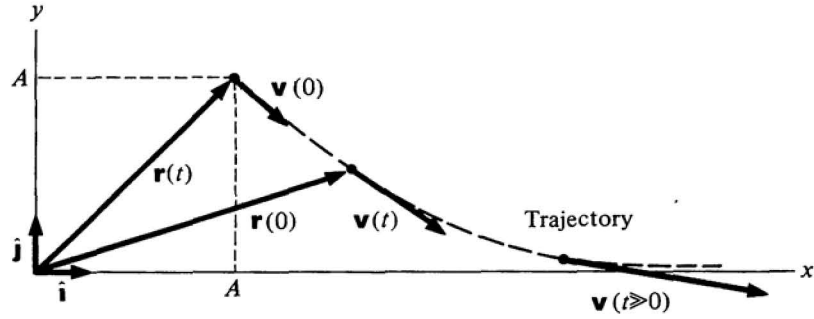
The magnitude of \mathbf{v} is

$$\begin{aligned}v &= (v_x^2 + v_y^2)^{\frac{1}{2}} \\ &= A\alpha(e^{2\alpha t} + e^{-2\alpha t})^{\frac{1}{2}}.\end{aligned}$$

In sketching the motion of a point, it is usually helpful to look at limiting cases. At $t = 0$, we have

$$\begin{aligned}\mathbf{r}(0) &= A(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \\ \mathbf{v}(0) &= \alpha A(\hat{\mathbf{i}} - \hat{\mathbf{j}}).\end{aligned}$$

¹ Caution: We can neglect the cartesian unit vectors when we differentiate, since their directions are fixed. Later we shall encounter unit vectors which can change direction, and then differentiation is more elaborate.



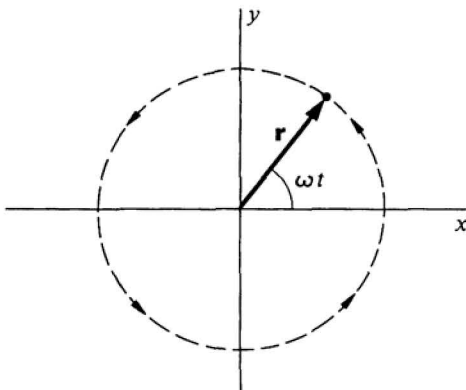
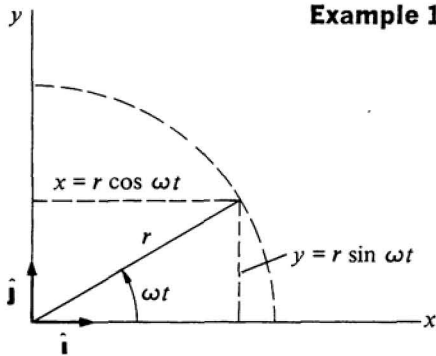
As $t \rightarrow \infty$, $e^{\alpha t} \rightarrow \infty$ and $e^{-\alpha t} \rightarrow 0$. In this limit $\mathbf{r} \rightarrow Ae^{\alpha t}\mathbf{i}$, which is a vector along the x axis, and $\mathbf{v} \rightarrow \alpha Ae^{\alpha t}\mathbf{i}$; the speed increases without limit.

Similarly, the acceleration \mathbf{a} is defined by

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} \\ &= \frac{d^2\mathbf{r}}{dt^2}.\end{aligned}$$

We could continue to form new vectors by taking higher derivatives of \mathbf{r} , but we shall see in our study of dynamics that \mathbf{r} , \mathbf{v} , and \mathbf{a} are of chief interest.

Example 1.8 Uniform Circular Motion



Circular motion plays an important role in physics. Here we look at the simplest and most important case—*uniform* circular motion, which is circular motion at constant speed.

Consider a particle moving in the xy plane according to $\mathbf{r} = r(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j})$, where r and ω are constants. Find the trajectory, the velocity, and the acceleration.

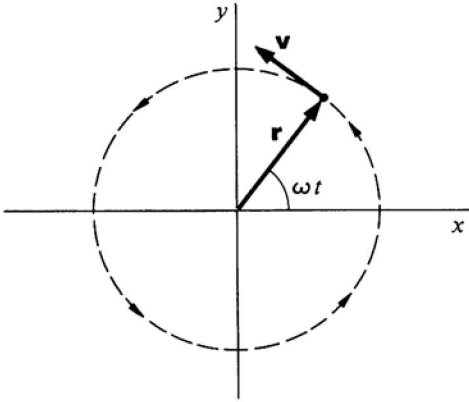
$$|\mathbf{r}| = [r^2 \cos^2 \omega t + r^2 \sin^2 \omega t]^{\frac{1}{2}}$$

Using the familiar identity $\sin^2 \theta + \cos^2 \theta = 1$,

$$\begin{aligned}|\mathbf{r}| &= [r^2(\cos^2 \omega t + \sin^2 \omega t)]^{\frac{1}{2}} \\ &= r = \text{constant}.\end{aligned}$$

The trajectory is a circle.

The particle moves counterclockwise around the circle, starting from $(r, 0)$ at $t = 0$. It traverses the circle in a time T such that $\omega T = 2\pi$. ω is called the *angular velocity* of the motion and is measured in radians



per second. T , the time required to execute one complete cycle, is called the *period*.

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= r\omega(-\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}})\end{aligned}$$

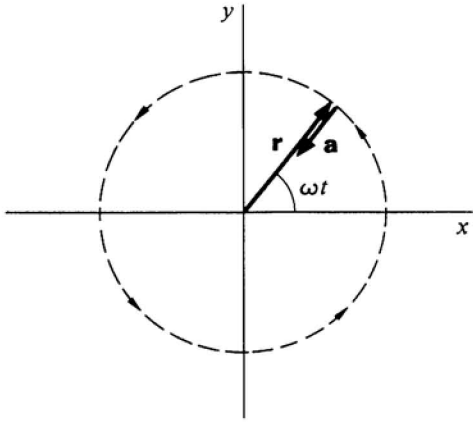
We can show that \mathbf{v} is tangent to the trajectory by calculating $\mathbf{v} \cdot \mathbf{r}$:

$$\begin{aligned}\mathbf{v} \cdot \mathbf{r} &= r^2\omega(-\sin \omega t \cos \omega t + \cos \omega t \sin \omega t) \\ &= 0.\end{aligned}$$

Since \mathbf{v} is perpendicular to \mathbf{r} , it is tangent to the circle as we expect. Incidentally, it is easy to show that $|\mathbf{v}| = r\omega = \text{constant}$.

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= r\omega^2[-\cos \omega t \hat{\mathbf{i}} - \sin \omega t \hat{\mathbf{j}}] \\ &= -\omega^2\mathbf{r}\end{aligned}$$

The acceleration is directed radially inward, and is known as the *centripetal acceleration*. We shall have more to say about it shortly.



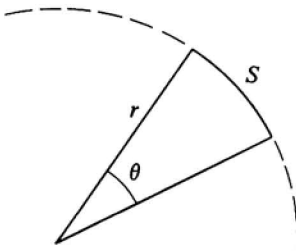
A Word about Dimension and Units

Physicists call the fundamental physical units in which a quantity is measured the *dimension* of the quantity. For example, the dimension of velocity is distance/time and the dimension of acceleration is velocity/time or (distance/time)/time = distance/time². As we shall discuss in Chap. 2, mass, distance, and time are the fundamental physical units used in mechanics.

To introduce a system of units, we specify the standards of measurement for mass, distance, and time. Ordinarily we measure distance in meters and time in seconds. The units of velocity are then meters per second (m/s) and the units of acceleration are meters per second² (m/s²).

The natural unit for measuring angle is the *radian* (rad). The angle θ in radians is S/r , where S is the arc subtended by θ in a circle of radius r :

$$\theta = \frac{S}{r}.$$



$2\pi \text{ rad} = 360^\circ$. We shall always use the radian as the unit of angle, unless otherwise stated. For example, in $\sin \omega t$, ωt is in radians. ω therefore has the dimensions 1/time and the units

radians per second. (The radian is dimensionless, since it is the ratio of two lengths.)

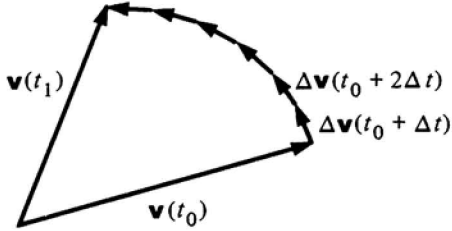
To avoid gross errors, it is a good idea to check to see that both sides of an equation have the same dimensions or units. For example, the equation $v = \alpha r e^{\alpha t}$ is dimensionally correct; since exponentials and their arguments are always dimensionless, α has the units $1/s$, and the right hand side has the correct units, meters per second.

1.7 Formal Solution of Kinematical Equations

Dynamics, which we shall take up in the next chapter, enables us to find the acceleration of a body directly. Once we know the acceleration, finding the velocity and position is a simple matter of integration. Here is the formal integration procedure.

If the acceleration is known as a function of time, the velocity can be found from the defining equation

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t)$$



by integration with respect to time. Suppose we want to find $\mathbf{v}(t_1)$ given the initial velocity $\mathbf{v}(t_0)$ and the acceleration $\mathbf{a}(t)$. Dividing the time interval $t_1 - t_0$ into n parts $\Delta t = (t_1 - t_0)/n$,

$$\begin{aligned} \mathbf{v}(t_1) &\approx \mathbf{v}(t_0) + \Delta\mathbf{v}(t_0 + \Delta t) + \Delta\mathbf{v}(t_0 + 2\Delta t) + \cdots + \Delta\mathbf{v}(t_1) \\ &\approx \mathbf{v}(t_0) + \mathbf{a}(t_0 + \Delta t) \Delta t + \mathbf{a}(t_0 + 2\Delta t) \Delta t + \cdots + \mathbf{a}(t_1) \Delta t, \end{aligned}$$

since $\Delta\mathbf{v}(t) \approx \mathbf{a}(t) \Delta t$. Taking the x component,

$$v_x(t_1) \approx v_x(t_0) + a_x(t_0 + \Delta t) \Delta t + \cdots + a_x(t_1) \Delta t.$$

The approximation becomes exact in the limit $n \rightarrow \infty$ ($\Delta t \rightarrow 0$), and the sum becomes an integral:

$$v_x(t_1) = v_x(t_0) + \int_{t_0}^{t_1} a_x(t) dt.$$

The y and z components can be treated similarly. Combining the results,

$$\begin{aligned} v_x(t_1)\hat{\mathbf{i}} + v_y(t_1)\hat{\mathbf{j}} + v_z(t_1)\hat{\mathbf{k}} &= v_x(t_0)\hat{\mathbf{i}} + \int_{t_0}^{t_1} a_x(t) dt \hat{\mathbf{i}} \\ &\quad + v_y(t_0)\hat{\mathbf{j}} + \int_{t_0}^{t_1} a_y(t) dt \hat{\mathbf{j}} + v_z(t_0)\hat{\mathbf{k}} + \int_{t_0}^{t_1} a_z(t) dt \hat{\mathbf{k}} \end{aligned}$$

or

$$\mathbf{v}(t_1) = \mathbf{v}(t_0) + \int_{t_0}^{t_1} \mathbf{a}(t) dt.$$

This result is the same as the formal integration of $d\mathbf{v} = \mathbf{a} dt$.

$$\int_{t_0}^{t_1} d\mathbf{v} = \int_{t_0}^{t_1} \mathbf{a}(t) dt$$

$$\mathbf{v}(t_1) - \mathbf{v}(t_0) = \int_{t_0}^{t_1} \mathbf{a}(t) dt$$

Sometimes we need an expression for the velocity at an arbitrary time t , in which case we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^t \mathbf{a}(t') dt'.$$

The dummy variable of integration has been changed from t to t' to avoid confusion with the upper limit t . We have designated the initial velocity $\mathbf{v}(t_0)$ by \mathbf{v}_0 to make the notation more compact. When $t = t_0$, $\mathbf{v}(t)$ reduces to \mathbf{v}_0 , as we expect.

Example 1.9 Finding Velocity from Acceleration

A Ping-Pong ball is released near the surface of the moon with velocity $\mathbf{v}_0 = (0, 5, -3)$ m/s. It accelerates (downward) with acceleration $\mathbf{a} = (0, 0, -2)$ m/s². Find its velocity after 5 s.

The equation

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^t \mathbf{a}(t') dt'$$

is equivalent to the three component equations

$$v_x(t) = v_{0x} + \int_0^t a_x(t') dt'$$

$$v_y(t) = v_{0y} + \int_0^t a_y(t') dt'$$

$$v_z(t) = v_{0z} + \int_0^t a_z(t') dt'.$$

Taking these equations in turn with the given values of \mathbf{v}_0 and \mathbf{a} , we obtain at $t = 5$ s:

$$v_x = 0 \text{ m/s}$$

$$v_y = 5 \text{ m/s}$$

$$v_z = -3 + \int_0^5 (-2) dt' = -13 \text{ m/s}.$$

Position is found by a second integration. Starting with

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t),$$

we find, by an argument identical to the above,

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t') dt'.$$

A particularly important case is that of *uniform acceleration*. If we take $\mathbf{a} = \text{constant}$ and $t_0 = 0$, we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t$$

and

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t (\mathbf{v}_0 + \mathbf{a}t') dt'$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2.$$

Quite likely you are already familiar with this in its one dimensional form. For instance, the x component of this equation is

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

where v_{0x} is the x component of \mathbf{v}_0 . This expression is so familiar that you may inadvertently apply it to the general case of varying acceleration. Don't! It only holds for *uniform* acceleration. In general, the full procedure described above must be used.

Example 1.10 Motion in a Uniform Gravitational Field

Suppose that an object moves freely under the influence of gravity so that it has a constant downward acceleration g . Choosing the z axis vertically upward, we have

$$\mathbf{a} = -g\hat{\mathbf{k}}.$$

If the object is released at $t = 0$ with initial velocity \mathbf{v}_0 , we have

$$x = x_0 + v_{0x}t$$

$$y = y_0 + v_{0y}t$$

$$z = z_0 + v_{0z}t - \frac{1}{2}gt^2.$$

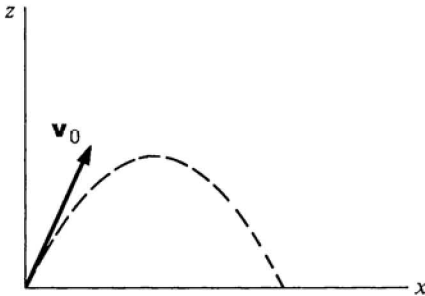
Without loss of generality, we can let $\mathbf{r}_0 = 0$, and assume that $v_{0y} = 0$. (The latter assumption simply means that we choose the coordinate system so that the initial velocity is in the xz plane.) Then

$$x = v_{0x}t$$

$$z = v_{0z}t - \frac{1}{2}gt^2.$$

The path of the object is shown in the sketch. We can eliminate time from the two equations for x and z to obtain the *trajectory*.

$$z = \frac{v_{0z}}{v_{0x}} x - \frac{g}{2v_{0x}^2} x^2$$



This is the well-known parabola of free fall projectile motion. However, as mentioned above, uniform acceleration is not the most general case.

Example 1.11 Nonuniform Acceleration—The Effect of a Radio Wave on an Ionospheric Electron

The ionosphere is a region of electrically neutral gas, composed of positively charged ions and negatively charged electrons, which surrounds the earth at a height of approximately 200 km (120 mi). If a radio wave passes through the ionosphere, its electric field accelerates the charged particle. Because the electric field oscillates in time, the charged particles tend to jiggle back and forth. The problem is to find the motion of an electron of charge $-e$ and mass m which is initially at rest, and which is suddenly subjected to an electric field $\mathbf{E} = \mathbf{E}_0 \sin \omega t$ (ω is the frequency of oscillation in radians per second).

The law of force for the charge in the electric field is $\mathbf{F} = -e\mathbf{E}$, and by Newton's second law we have $\mathbf{a} = \mathbf{F}/m = -e\mathbf{E}/m$. (If the reasoning behind this is a mystery to you, ignore it for now. It will be clear later. This example is meant to be a mathematical exercise—the physics is an added dividend.) We have

$$\begin{aligned}\mathbf{a} &= \frac{-e\mathbf{E}}{m} \\ &= \frac{-e\mathbf{E}_0}{m} \sin \omega t.\end{aligned}$$

\mathbf{E}_0 is a constant vector and we shall choose our coordinate system so that the x axis lies along it. Since there is no acceleration in the y or z directions, we need consider only the x motion. With this understanding, we can drop subscripts and write a for a_x .

$$a(t) = \frac{-eE_0}{m} \sin \omega t = a_0 \sin \omega t$$

where

$$a_0 = \frac{-eE_0}{m}.$$

Then

$$\begin{aligned}v(t) &= v_0 + \int_0^t a(t') dt' \\ &= v_0 + \int_0^t a_0 \sin \omega t' dt' \\ &= v_0 - \frac{a_0}{\omega} \cos \omega t' \Big|_0^t = v_0 - \frac{a_0}{\omega} (\cos \omega t - 1)\end{aligned}$$

and

$$\begin{aligned}
 x &= x_0 + \int_0^t v(t') dt' \\
 &= x_0 + \int_0^t \left[v_0 - \frac{a_0}{\omega} (\cos \omega t' - 1) \right] dt' \\
 &= x_0 + \left(v_0 + \frac{a_0}{\omega} \right) t - \frac{a_0}{\omega^2} \sin \omega t.
 \end{aligned}$$

We are given that $x_0 = v_0 = 0$, so we have

$$x = \frac{a_0}{\omega} t - \frac{a_0}{\omega^2} \sin \omega t.$$

The result is interesting: the second term oscillates and corresponds to the jiggling motion of the electron, which we predicted. The first term, however, corresponds to motion with uniform velocity, so in addition to the jiggling motion the electron starts to drift away. Can you see why?

1.8 More about the Derivative of a Vector

In Sec. 1.6 we demonstrated how to describe velocity and acceleration by vectors. In particular, we showed how to differentiate the vector \mathbf{r} to obtain a new vector $\mathbf{v} = d\mathbf{r}/dt$. We will want to differentiate other vectors with respect to time on occasion, and so it is worthwhile generalizing our discussion.

Consider some vector $\mathbf{A}(t)$ which is a function of time. The change in \mathbf{A} during the interval from t to $t + \Delta t$ is

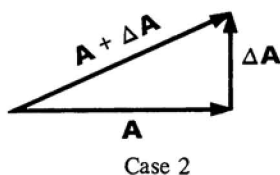
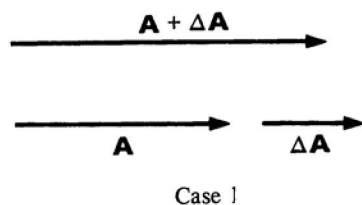
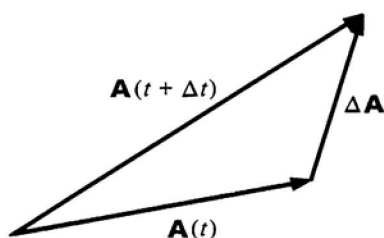
$$\Delta \mathbf{A} = \mathbf{A}(t + \Delta t) - \mathbf{A}(t).$$

In complete analogy to the procedure we followed in differentiating \mathbf{r} in Sec. 1.6, we define the time derivative of \mathbf{A} by

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t}.$$

It is important to appreciate that $d\mathbf{A}/dt$ is a new vector which can be large or small, and can point in any direction, depending on the behavior of \mathbf{A} .

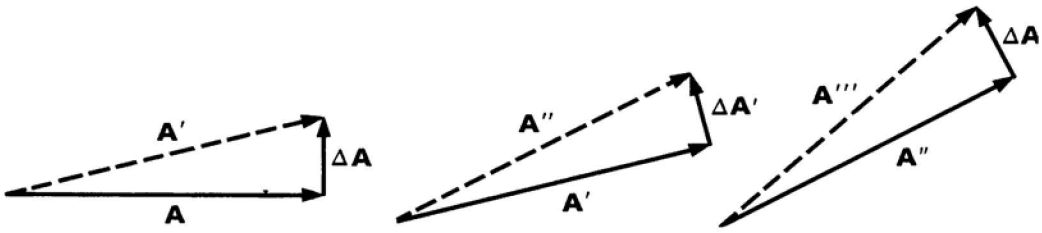
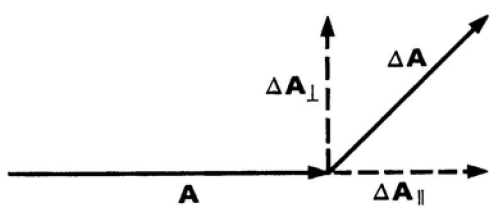
There is one important respect in which $d\mathbf{A}/dt$ differs from the derivative of a simple scalar function. \mathbf{A} can change in both *magnitude* and *direction*—a scalar function can change only in magnitude. This difference is important. The figure illustrates the addition of a small increment $\Delta \mathbf{A}$ to \mathbf{A} . In the first case $\Delta \mathbf{A}$ is parallel to \mathbf{A} ; this leaves the direction unaltered but changes the magnitude to $|\mathbf{A}| + |\Delta \mathbf{A}|$. In the second, $\Delta \mathbf{A}$ is perpendicular



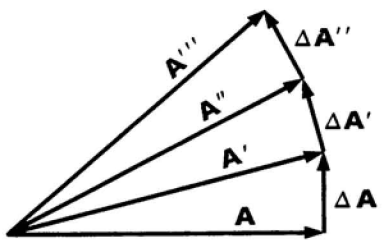
to \mathbf{A} . This causes a change of *direction* but leaves the magnitude practically unaltered.

In general, \mathbf{A} will change in both magnitude and direction. Even so, it is useful to visualize both types of change taking place simultaneously. In the sketch to the left we show a small increment $\Delta\mathbf{A}$ resolved into a component vector $\Delta\mathbf{A}_{\parallel}$ parallel to \mathbf{A} and a component vector $\Delta\mathbf{A}_{\perp}$ perpendicular to \mathbf{A} . In the limit where we take the derivative, $\Delta\mathbf{A}_{\parallel}$ changes the magnitude of \mathbf{A} but not its direction, while $\Delta\mathbf{A}_{\perp}$ changes the direction of \mathbf{A} but not its magnitude.

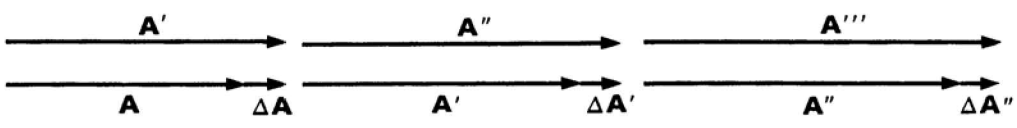
Students who do not have a clear understanding of the two ways a vector can change sometimes make an error by neglecting one of them. For instance, if $d\mathbf{A}/dt$ is always perpendicular to \mathbf{A} , \mathbf{A} must *rotate*, since its magnitude cannot change; its time dependence arises solely from change in direction. The illustrations below show how rotation occurs when $\Delta\mathbf{A}$ is always perpendicular to \mathbf{A} . The rotational motion is made more apparent by drawing



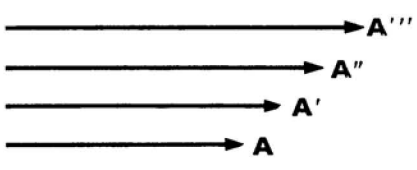
the successive vectors at a common origin.



Contrast this with the case where $\Delta\mathbf{A}$ is always parallel to \mathbf{A} .



Drawn from a common origin, the vectors look like this:



The following example relates the idea of rotating vectors to circular motion.

Example 1.12 Circular Motion and Rotating Vectors

In Example 1.8 we discussed the motion given by

$$\mathbf{r} = r(\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}).$$

The velocity is

$$\mathbf{v} = r\omega(-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}).$$

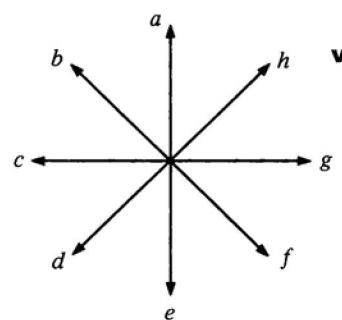
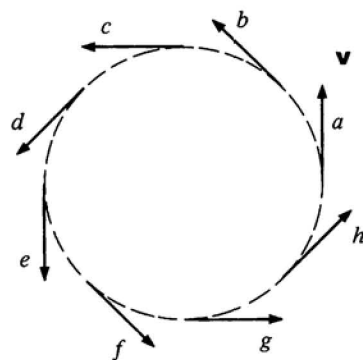
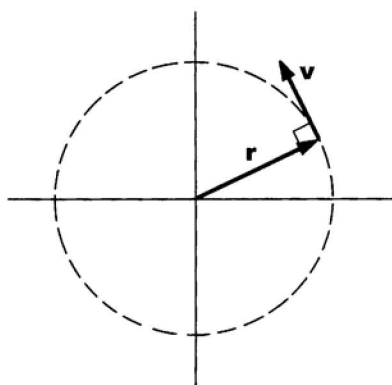
Since

$$\begin{aligned} \mathbf{r} \cdot \mathbf{v} &= r^2\omega(-\cos \omega t \sin \omega t + \sin \omega t \cos \omega t) \\ &= 0, \end{aligned}$$

we see that $d\mathbf{r}/dt$ is perpendicular to \mathbf{r} . We conclude that the magnitude of \mathbf{r} is constant, so that the only possible change in \mathbf{r} is due to rotation. Since the trajectory is a circle, this is precisely the case: \mathbf{r} rotates about the origin.

We showed earlier that $\mathbf{a} = -\omega^2\mathbf{r}$. Since $\mathbf{r} \cdot \mathbf{v} = 0$, it follows that $\mathbf{a} \cdot \mathbf{v} = -\omega^2\mathbf{r} \cdot \mathbf{v} = 0$ and $d\mathbf{v}/dt$ is perpendicular to \mathbf{v} . This means that the velocity vector has constant magnitude, so that it too must rotate if it is to change in time.

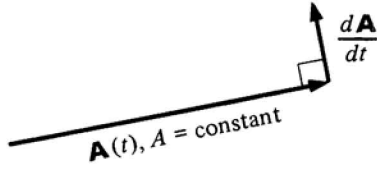
That \mathbf{v} indeed rotates is readily seen from the sketch, which shows \mathbf{v} at various positions along the trajectory. In the second sketch the same



velocity vectors are drawn from a common origin. It is apparent that each time the particle completes a traversal, the velocity vector has swung around through a full circle.

Perhaps you can show that the acceleration vector also undergoes uniform rotation.

Suppose a vector $\mathbf{A}(t)$ has constant magnitude A . The only way $\mathbf{A}(t)$ can change in time is by rotating, and we shall now develop a useful expression for the time derivative $d\mathbf{A}/dt$ of such a



rotating vector. The direction of $d\mathbf{A}/dt$ is always perpendicular to \mathbf{A} . The magnitude of $d\mathbf{A}/dt$ can be found by the following geometrical argument.

The change in \mathbf{A} in the time interval t to $t + \Delta t$ is

$$\Delta\mathbf{A} = \mathbf{A}(t + \Delta t) - \mathbf{A}(t).$$

Using the angle $\Delta\theta$ defined in the sketch,

$$|\Delta\mathbf{A}| = 2A \sin \frac{\Delta\theta}{2}.$$

For $\Delta\theta \ll 1$, $\sin \Delta\theta/2 \approx \Delta\theta/2$, as discussed in Note 1.1. We have

$$\begin{aligned} |\Delta\mathbf{A}| &\approx 2A \frac{\Delta\theta}{2} \\ &= A \Delta\theta \end{aligned}$$

and

$$\left| \frac{\Delta\mathbf{A}}{\Delta t} \right| = A \frac{\Delta\theta}{\Delta t}.$$

Taking the limit $\Delta t \rightarrow 0$,

$$\left| \frac{d\mathbf{A}}{dt} \right| = A \frac{d\theta}{dt}.$$

$d\theta/dt$ is called the *angular velocity* of \mathbf{A} .

For a simple application of this result, let \mathbf{A} be the rotating vector \mathbf{r} discussed in Examples 1.8 and 1.12. Then $\theta = \omega t$ and

$$\left| \frac{d\mathbf{r}}{dt} \right| = r \frac{d}{dt}(\omega t) = r\omega \quad \text{or} \quad v = r\omega.$$

Returning now to the general case, a change in \mathbf{A} is the result of a rotation *and* a change in magnitude.

$$\Delta\mathbf{A} = \Delta\mathbf{A}_\perp + \Delta\mathbf{A}_\parallel.$$

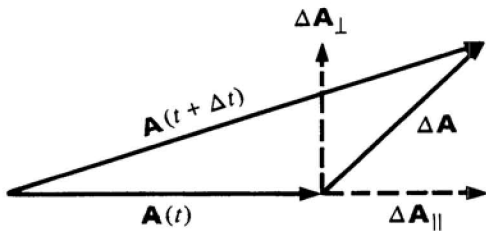
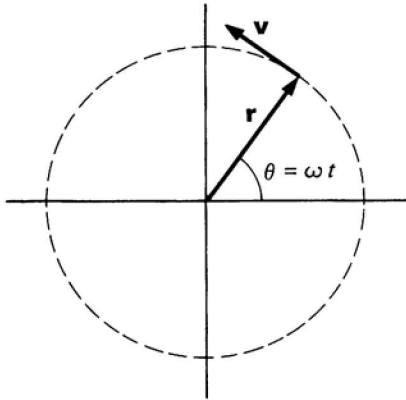
For $\Delta\theta$ sufficiently small,

$$|\Delta\mathbf{A}_\perp| = A \Delta\theta$$

$$|\Delta\mathbf{A}_\parallel| = \Delta A$$

and, dividing by Δt and taking the limit,

$$\begin{aligned} \left| \frac{d\mathbf{A}_\perp}{dt} \right| &= A \frac{d\theta}{dt} \\ \left| \frac{d\mathbf{A}_\parallel}{dt} \right| &= \frac{dA}{dt}. \end{aligned}$$



$d\mathbf{A}_\perp/dt$ is zero if \mathbf{A} does not rotate ($d\theta/dt = 0$), and $d\mathbf{A}_\parallel/dt$ is zero if \mathbf{A} is constant in magnitude.

We conclude this section by stating some formal identities in vector differentiation. Their proofs are left as exercises. Let the scalar c and the vectors \mathbf{A} and \mathbf{B} be functions of time. Then

$$\frac{d}{dt}(c\mathbf{A}) = \frac{dc}{dt}\mathbf{A} + c\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}.$$

In the second relation, let $\mathbf{A} = \mathbf{B}$. Then

$$\frac{d}{dt}(A^2) = 2\mathbf{A} \cdot \frac{d\mathbf{A}}{dt},$$

and we see again that if $d\mathbf{A}/dt$ is perpendicular to \mathbf{A} , the magnitude of \mathbf{A} is constant.

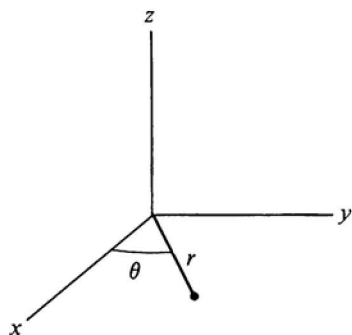
1.9 Motion in Plane Polar Coordinates

Polar Coordinates

Rectangular, or cartesian, coordinates are well suited to describing motion in a straight line. For instance, if we orient the coordinate system so that one axis lies in the direction of motion, then only a single coordinate changes as the point moves. However, rectangular coordinates are not so useful for describing circular motion, and since circular motion plays a prominent role in physics, it is worth introducing a coordinate system more natural to it.

We should mention that although we can use any coordinate system we like, the proper choice of a coordinate system can vastly simplify a problem, so that the material in this section is very much in the spirit of more advanced physics. Quite likely some of this material will be entirely new to you. Be patient if it seems strange or even difficult at first. Once you have studied the examples and worked a few problems, it will seem much more natural.

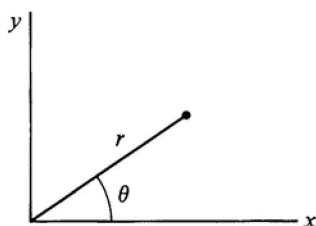
Our new coordinate system is based on the cylindrical coordinate system. The z axis of the cylindrical system is identical to that of the cartesian system. However, position in the xy plane is



described by distance r from the z axis and the angle θ that r makes with the x axis. These coordinates are shown in the sketch. We see that

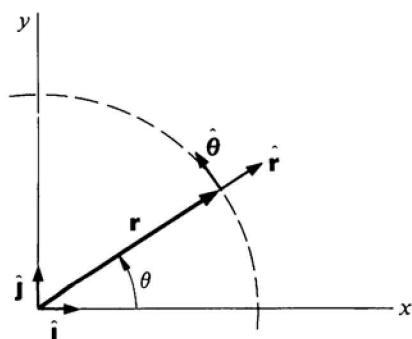
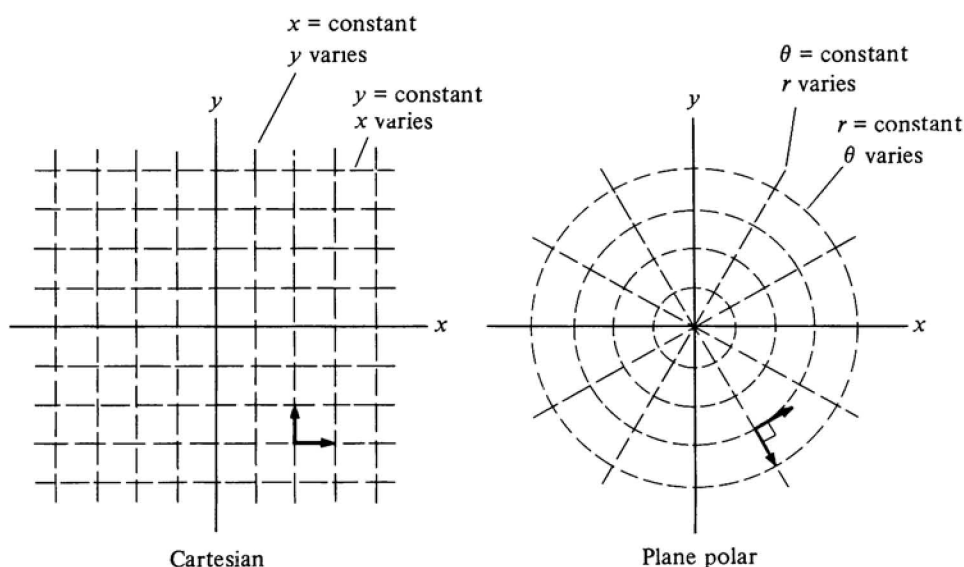
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$



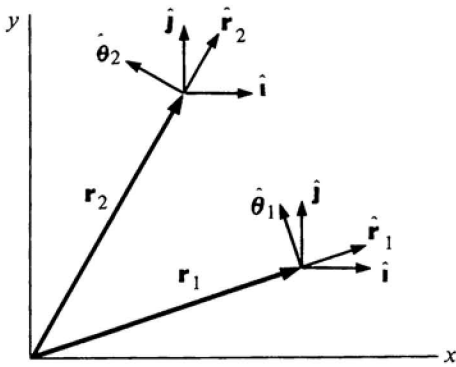
Since we shall be concerned primarily with motion in a plane, we neglect the z axis and restrict our discussion to two dimensions. The coordinates r and θ are called *plane polar* coordinates. In the following sections we shall learn to describe position, velocity, and acceleration in plane polar coordinates.

The contrast between cartesian and plane polar coordinates is readily seen by comparing drawings of constant coordinate lines for the two systems.



The lines of constant x and of constant y are straight and perpendicular to each other. Lines of constant θ are also straight, directed radially outward from the origin. In contrast, lines of constant r are circles concentric to the origin. Note, however, that the lines of constant θ and constant r are perpendicular wherever they intersect.

In Sec. 1.4 we introduced the base vectors \hat{i} and \hat{j} which point in the direction of increasing x and increasing y , respectively. In a similar fashion we now introduce two new unit vectors, \hat{r} and $\hat{\theta}$, which point in the direction of increasing r and increasing θ . There is an important difference between these base vectors and the



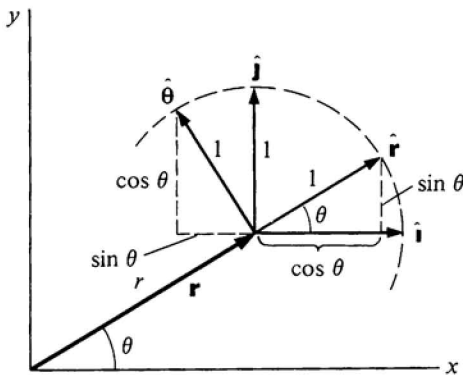
cartesian base vectors: the directions of \hat{r} and $\hat{\theta}$ vary with position, whereas \hat{i} and \hat{j} have fixed directions. The drawing shows this by illustrating both sets of base vectors at two points in space. Because \hat{r} and $\hat{\theta}$ vary with position, kinematical formulas can look more complicated in polar coordinates than in the cartesian system. (It is not that polar coordinates are complicated, it is simply that cartesian coordinates are simpler than they have a right to be. Cartesian coordinates are the only coordinates whose base vectors have fixed directions.)

Although \hat{r} and $\hat{\theta}$ vary with position, note that they depend on θ only, not on r . We can think of \hat{r} and $\hat{\theta}$ as being functionally dependent on θ .

The drawing shows the unit vectors \hat{i} , \hat{j} and \hat{r} , $\hat{\theta}$ at a point in the xy plane. We see that

$$\hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta$$

$$\hat{\theta} = -\hat{i} \sin \theta + \hat{j} \cos \theta.$$



Before proceeding, convince yourself that these expressions are reasonable by checking them at a few particularly simple points, such as $\theta = 0$, and $\pi/2$. Also verify that \hat{r} and $\hat{\theta}$ are orthogonal (i.e., perpendicular) by showing that $\hat{r} \cdot \hat{\theta} = 0$.

It is easy to verify that we indeed have the same vector \mathbf{r} no matter whether we describe it by cartesian or polar coordinates. In cartesian coordinates we have

$$\mathbf{r} = x\hat{i} + y\hat{j},$$

and in polar coordinates we have

$$\mathbf{r} = r\hat{r}.$$

If we insert the above expression for \hat{r} , we obtain

$$x\hat{i} + y\hat{j} = r(\hat{i} \cos \theta + \hat{j} \sin \theta).$$

We can separately equate the coefficients of \hat{i} and \hat{j} to obtain

$$x = r \cos \theta \quad y = r \sin \theta,$$

as we expect.

The relation

$$\mathbf{r} = r\hat{r}$$

is sometimes confusing, because the equation as written seems to make no reference to the angle θ . We know that two parameters

are needed to specify a position in two dimensional space (in cartesian coordinates they are x and y), but the equation $\mathbf{r} = r\hat{\mathbf{r}}$ seems to contain only the quantity r . The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is oriented as well as the value of r to tell how far we are from the origin. Although θ does not occur explicitly in $r\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r} = r\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ . However, by common convention $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

The orthogonality of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ plus the fact that they are unit vectors, $|\hat{\mathbf{r}}| = 1$, $|\hat{\boldsymbol{\theta}}| = 1$, means that we can continue to evaluate scalar products in the simple way we are accustomed to. If

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} \quad \text{and} \quad \mathbf{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}},$$

then

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta.$$

Of course, the $\hat{\mathbf{r}}$'s and the $\hat{\boldsymbol{\theta}}$'s must refer to the same point in space for this simple rule to hold.

Velocity in Polar Coordinates

Now let us turn our attention to describing velocity with polar coordinates. Recall that in cartesian coordinates we have

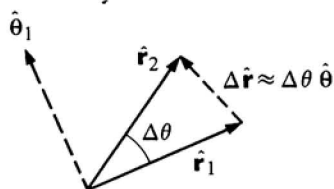
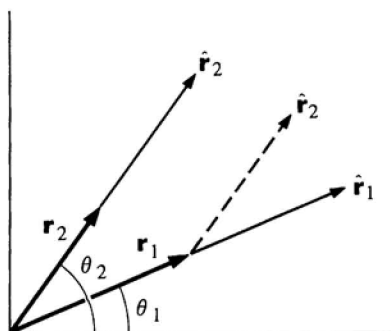
$$\begin{aligned} \mathbf{v} &= \frac{d}{dt}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \\ &= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}. \end{aligned}$$

(Remember that \dot{x} stands for dx/dt .)

The same vector, \mathbf{v} , expressed in polar coordinates is given by

$$\begin{aligned} \mathbf{v} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}. \end{aligned}$$

The first term on the right is obviously the component of the velocity directed radially outward. We suspect that the second term is the component of velocity in the tangential ($\hat{\boldsymbol{\theta}}$) direction. This is indeed the case. However to prove it we must evaluate $d\hat{\mathbf{r}}/dt$. Since this step is slightly tricky, we shall do it three different ways. Take your pick!

**Evaluating $d\hat{\mathbf{r}}/dt$**

Method 1 We can invoke the ideas of the last section to find $d\hat{\mathbf{r}}/dt$. Since $\hat{\mathbf{r}}$ is a unit vector, its magnitude is constant and $d\hat{\mathbf{r}}/dt$ is perpendicular to $\hat{\mathbf{r}}$; as θ increases, $\hat{\mathbf{r}}$ rotates.

$$|\Delta \hat{\mathbf{r}}| \approx |\hat{\mathbf{r}}| \Delta \theta = \Delta \theta,$$

$$\frac{|\Delta \hat{\mathbf{r}}|}{\Delta t} \approx \frac{\Delta \theta}{\Delta t},$$

and, taking the limit, we obtain

$$\left| \frac{d\hat{\mathbf{r}}}{dt} \right| = \frac{d\theta}{dt}.$$

As the sketch shows, as θ increases, $\hat{\mathbf{r}}$ swings in the $\hat{\boldsymbol{\theta}}$ direction, hence

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta} \hat{\boldsymbol{\theta}}.$$

If this method is too casual for your taste, you may find methods 2 or 3 more appealing.

Method 2

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \cos \theta + \hat{\mathbf{j}} \sin \theta$$

We note that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are fixed unit vectors, and thus cannot vary in time. θ , on the other hand, does vary as \mathbf{r} changes. Using

$$\begin{aligned} \frac{d}{dt}(\cos \theta) &= \left(\frac{d}{d\theta} \cos \theta \right) \frac{d\theta}{dt} \\ &= -\sin \theta \dot{\theta} \end{aligned}$$

and

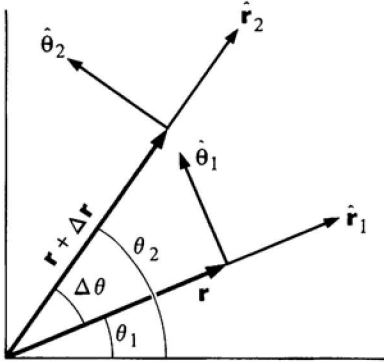
$$\begin{aligned} \frac{d}{dt}(\sin \theta) &= \left(\frac{d}{d\theta} \sin \theta \right) \frac{d\theta}{dt} \\ &= \cos \theta \dot{\theta}, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{d\hat{\mathbf{r}}}{dt} &= \hat{\mathbf{i}} \frac{d}{dt}(\cos \theta) + \hat{\mathbf{j}} \frac{d}{dt}(\sin \theta) \\ &= -\hat{\mathbf{i}} \sin \theta \dot{\theta} + \hat{\mathbf{j}} \cos \theta \dot{\theta} \\ &= (-\hat{\mathbf{i}} \sin \theta + \hat{\mathbf{j}} \cos \theta) \dot{\theta}. \end{aligned}$$

However, recall that $-\hat{i} \sin \theta + \hat{j} \cos \theta = \hat{\theta}$. We obtain

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}.$$



Method 3

The drawing shows \mathbf{r} at two different times, t and $t + \Delta t$. The coordinates are, respectively, (r, θ) and $(r + \Delta r, \theta + \Delta \theta)$. Note that the angle between \hat{r}_1 and \hat{r}_2 is equal to the angle between $\hat{\theta}_1$ and $\hat{\theta}_2$; this angle is $\theta_2 - \theta_1 = \Delta \theta$.

The change in \hat{r} during the time Δt is illustrated by the lower drawing. We see that

$$\Delta \hat{r} = \hat{\theta}_1 \sin \Delta \theta - \hat{r}_1 (1 - \cos \Delta \theta).$$

Hence

$$\begin{aligned} \frac{\Delta \hat{r}}{\Delta t} &= \hat{\theta}_1 \frac{\sin \Delta \theta}{\Delta t} - \hat{r}_1 \frac{(1 - \cos \Delta \theta)}{\Delta t} \\ &= \hat{\theta}_1 \left(\frac{\Delta \theta - \frac{1}{6}(\Delta \theta)^3 + \cdots}{\Delta t} \right) - \hat{r}_1 \left(\frac{\frac{1}{2}(\Delta \theta)^2 - \frac{1}{24}(\Delta \theta)^4 + \cdots}{\Delta t} \right), \end{aligned}$$

where we have used the series expansions discussed in Note 1.1. We need to evaluate

$$\frac{d\hat{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \hat{r}}{\Delta t}.$$

In the limit $\Delta t \rightarrow 0$, $\Delta \theta$ also approaches zero, but $\Delta \theta / \Delta t$ approaches the limit $d\theta/dt$. Therefore

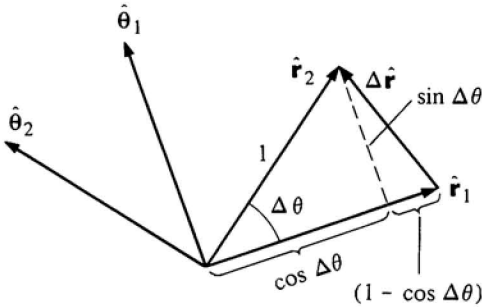
$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} (\Delta \theta)^n = 0 \quad n > 0.$$

The term in \hat{r} entirely vanishes in the limit and we are left with

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta},$$

as before. We also need an expression for $d\hat{\theta}/dt$. You can use any, or all, of the arguments above to prove for yourself that

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}.$$



Since you should be familiar with both results, let's summarize them together:

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\hat{\boldsymbol{\theta}} \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\hat{\mathbf{r}}.\end{aligned}$$

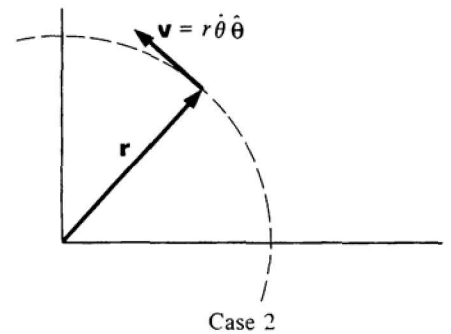
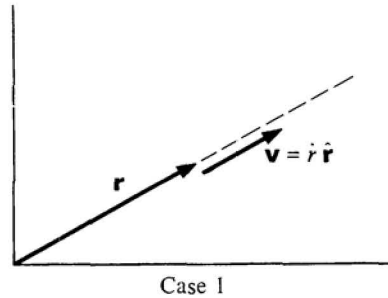
And now, we can return to our problem. On page 30 we showed that

$$\mathbf{v} = \frac{d}{dt} r\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r \frac{d\hat{\mathbf{r}}}{dt}.$$

Using the above results, we can write this as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

As we surmised, the second term is indeed in the tangential (that is, $\hat{\boldsymbol{\theta}}$) direction. We can get more insight into the meaning of each term by considering special cases where only one component varies at a time.



1. $\theta = \text{constant}$, velocity is radial. If θ is a constant, $\dot{\theta} = 0$, and $\mathbf{v} = \dot{r}\hat{\mathbf{r}}$. We have one dimensional motion in a fixed radial direction.
2. $r = \text{constant}$, velocity is tangential. In this case $\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}}$. Since r is fixed, the motion lies on the arc of a circle. The speed of the point on the circle is $r\dot{\theta}$, and it follows that $\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}}$.

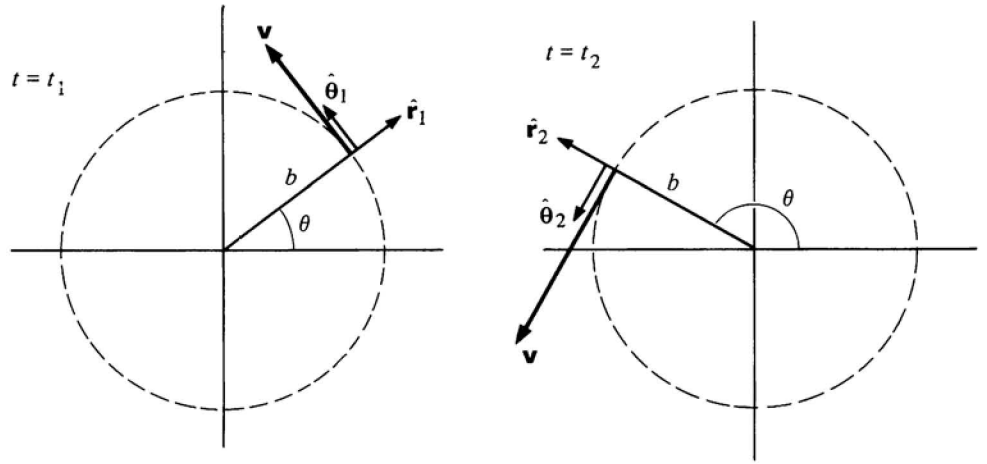
For motion in general, both r and θ change in time.

The next three examples illustrate the use of polar coordinates to describe velocity.

Example 1.13 Circular Motion and Straight Line Motion in Polar Coordinates

A particle moves in a circle of radius b with angular velocity $\dot{\theta} = \alpha t$, where α is a constant. (α has the units radians per second².) Describe the particle's velocity in polar coordinates.

Since $r = b = \text{constant}$, \mathbf{v} is purely tangential and $\mathbf{v} = b\alpha t\hat{\theta}$. The sketches show \hat{r} , $\hat{\theta}$, and \mathbf{v} at a time t_1 and at a later time t_2 .



The particle is located at the position

$$r = b \quad \theta = \theta_0 + \int_0^t \dot{\theta} dt = \theta_0 + \frac{1}{2}\alpha t^2.$$

If the particle is on the x axis at $t = 0$, $\theta_0 = 0$. The particle's position vector is $\mathbf{r} = b\hat{r}$, but as the sketches indicate, θ must be given to specify the direction of \hat{r} .

Consider a particle moving with constant velocity $\mathbf{v} = u\hat{i}$ along the line $y = 2$. Describe \mathbf{v} in polar coordinates.

$$\mathbf{v} = v_r\hat{r} + v_\theta\hat{\theta}.$$

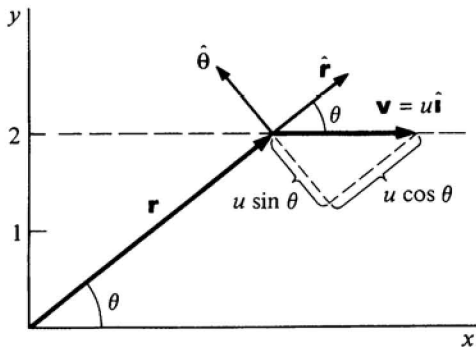
From the sketch,

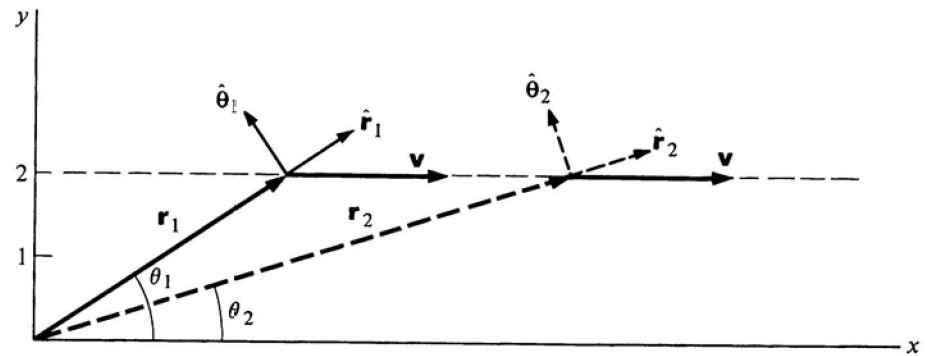
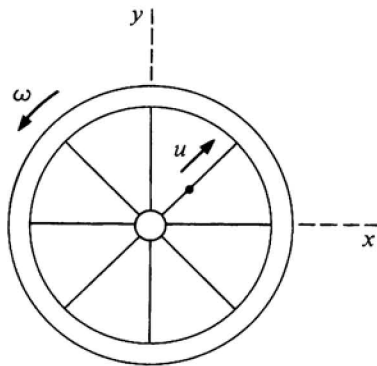
$$v_r = u \cos \theta$$

$$v_\theta = -u \sin \theta$$

$$\mathbf{v} = u \cos \theta \hat{r} - u \sin \theta \hat{\theta}.$$

As the particle moves to the right, θ decreases and \hat{r} and $\hat{\theta}$ change direction. Ordinarily, of course, we try to use coordinates that make the problem as simple as possible; polar coordinates are not well suited here.




Example 1.14 Velocity of a Bead on a Spoke


A bead moves along the spoke of a wheel at constant speed u meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis, and the bead is at the origin. Find the velocity at time t

- In polar coordinates
- In cartesian coordinates.

a. We have $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$. Hence

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

To specify the velocity completely, we need to know the direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. This is obtained from $\mathbf{r} = (r, \theta) = (ut, \omega t)$.

b. In cartesian coordinates, we have

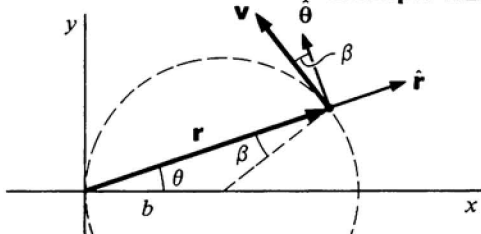
$$v_x = v_r \cos \theta - v_\theta \sin \theta$$

$$v_y = v_r \sin \theta + v_\theta \cos \theta.$$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$, we obtain

$$\mathbf{v} = (u \cos \omega t - ut\omega \sin \omega t)\hat{\mathbf{i}} + (u \sin \omega t + ut\omega \cos \omega t)\hat{\mathbf{j}}.$$

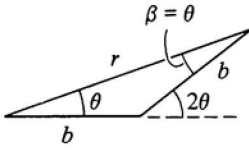
Note how much simpler the result is in plane polar coordinates.

Example 1.15 Off-center Circle


A particle moves with constant speed v around a circle of radius b . Find its velocity vector in polar coordinates using an origin lying on the circle.

With this origin, \mathbf{v} is no longer purely tangential, as the sketch indicates.

$$\begin{aligned} \mathbf{v} &= -v \sin \beta \hat{\mathbf{r}} + v \cos \beta \hat{\boldsymbol{\theta}} \\ &= -v \sin \theta \hat{\mathbf{r}} + v \cos \theta \hat{\boldsymbol{\theta}}. \end{aligned}$$



The last step follows since β and θ are the base angles of an isosceles triangle and are therefore equal. To complete the calculation, we must find θ as a function of time. By geometry, $2\theta = \omega t$ or $\theta = \omega t/2$, where $\omega = v/b$.

Acceleration in Polar Coordinates

Our final task is to find the acceleration. We differentiate \mathbf{v} to obtain

$$\begin{aligned}\mathbf{a} &= \frac{d}{dt} \mathbf{v} \\ &= \frac{d}{dt} (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.\end{aligned}$$

If we substitute the results for $d\hat{\mathbf{r}}/dt$ and $d\hat{\boldsymbol{\theta}}/dt$ from page 33, we obtain

$$\begin{aligned}\mathbf{a} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration which we encountered in Example 1.8. Finally, $2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}$ is the *Coriolis* acceleration. Perhaps you have heard of the Coriolis force, a fictitious force which appears to act in a rotating coordinate system, and which we shall study in Chap. 8. The Coriolis acceleration that we are discussing here is a real acceleration which is present when r and θ both change with time.

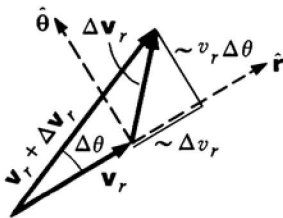
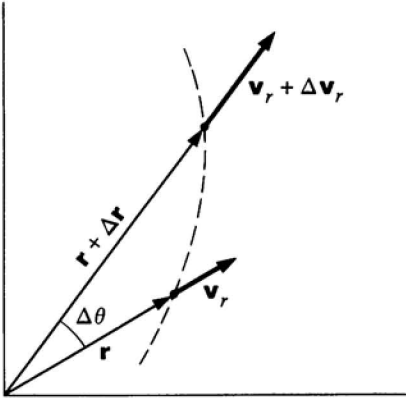
The expression for acceleration in polar coordinates appears complicated. However, by looking at it from the geometric point of view, we can obtain a more intuitive picture.

The instantaneous velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$$

Let us look at the velocity at two different times, treating the radial and tangential terms separately.

The sketch at left shows the radial velocity $\dot{r}\hat{\mathbf{r}} = v_r\hat{\mathbf{r}}$ at two different instants. The change $\Delta\mathbf{v}_r$ has both a radial and a tangential component. As we can see from the sketch (or from the dis-



cussion at the end of Sec. 1.8), the radial component of $\Delta \mathbf{v}_r$ is $\Delta v_r \hat{\mathbf{r}}$ and the tangential component is $v_r \Delta \theta \hat{\boldsymbol{\theta}}$. The radial component contributes

$$\lim_{\Delta t \rightarrow 0} \left(\frac{\Delta v_r}{\Delta t} \hat{\mathbf{r}} \right) = \frac{dv_r}{dt} \hat{\mathbf{r}} = \ddot{r} \hat{\mathbf{r}}$$

to the acceleration. The tangential component contributes

$$\lim_{\Delta t \rightarrow 0} \left(v_r \frac{\Delta \theta}{\Delta t} \hat{\boldsymbol{\theta}} \right) = v_r \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} = \dot{r} \dot{\theta} \hat{\boldsymbol{\theta}},$$

which is one-half the Coriolis acceleration. We see that half the Coriolis acceleration arises from the change of direction of the radial velocity.

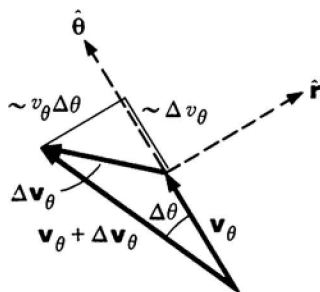
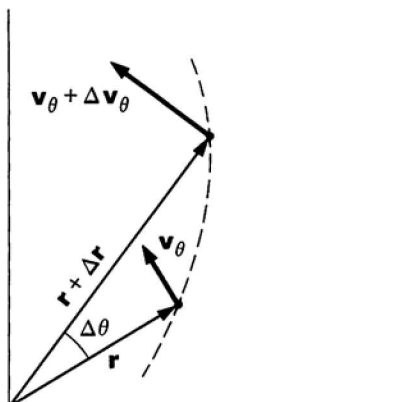
The tangential velocity $r\dot{\theta} \hat{\boldsymbol{\theta}} = v_\theta \hat{\boldsymbol{\theta}}$ can be treated similarly. The change in direction of $\hat{\boldsymbol{\theta}}$ gives $\Delta \mathbf{v}_\theta$ an inward radial component $-v_\theta \Delta \theta \hat{\mathbf{r}}$. This contributes

$$\lim_{\Delta t \rightarrow 0} \left(-v_\theta \frac{\Delta \theta}{\Delta t} \hat{\mathbf{r}} \right) = -v_\theta \dot{\theta} \hat{\mathbf{r}} = -r\dot{\theta}^2 \hat{\mathbf{r}},$$

which we recognize as the centripetal acceleration. Finally, the tangential component of $\Delta \mathbf{v}_\theta$ is $\Delta v_\theta \hat{\boldsymbol{\theta}}$. Since $v_\theta = r\dot{\theta}$, there are two ways the tangential speed can change. If $\dot{\theta}$ increases by $\Delta \dot{\theta}$, v_θ increases by $r \Delta \dot{\theta}$. Second, if r increases by Δr , v_θ increases by $\Delta r \dot{\theta}$. Hence $\Delta v_\theta = r \Delta \dot{\theta} + \Delta r \dot{\theta}$, and the contribution to the acceleration is

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta v_\theta}{\Delta t} \hat{\boldsymbol{\theta}} \right) &= \lim_{\Delta t \rightarrow 0} \left(r \frac{\Delta \dot{\theta}}{\Delta t} + \frac{\Delta r}{\Delta t} \dot{\theta} \right) \hat{\boldsymbol{\theta}} \\ &= (r\ddot{\theta} + \dot{r}\dot{\theta}) \hat{\boldsymbol{\theta}}. \end{aligned}$$

The second term is the remaining half of the Coriolis acceleration; we see that this part arises from the change in tangential speed due to the change in radial distance.



Example 1.16 Acceleration of a Bead on a Spoke

A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the velocity and acceleration.

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r\dot{\theta} \hat{\boldsymbol{\theta}}$$

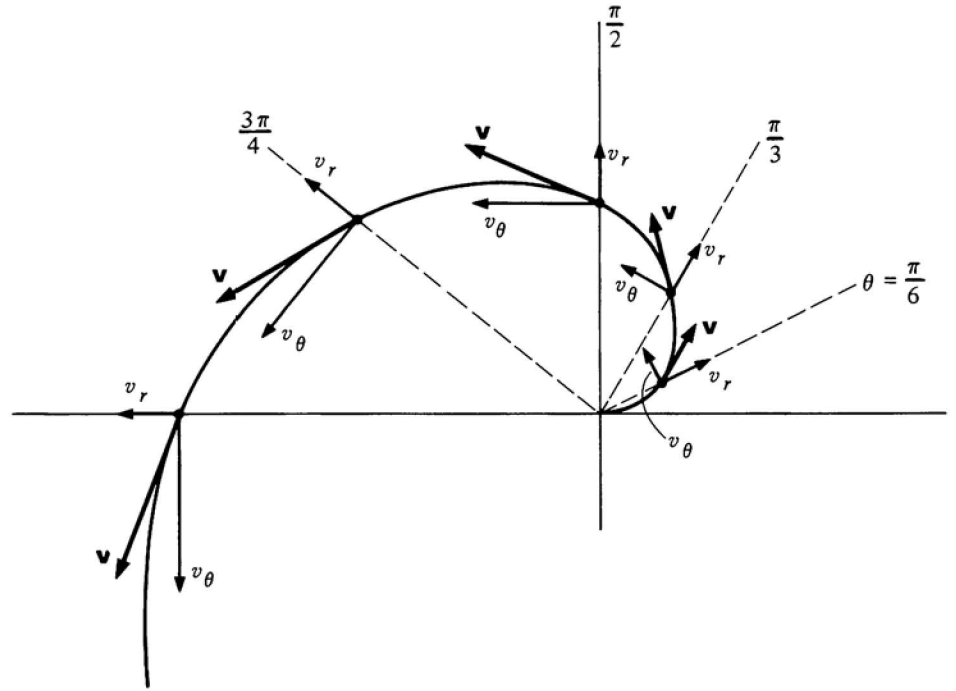
We are given that $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position is given by $r = ut$, and we have

$$\mathbf{v} = u \hat{\mathbf{r}} + ut\omega \hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -u\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.\end{aligned}$$

The velocity is shown in the sketch for several different positions of the wheel. Note that the radial velocity is constant. The tangential acceleration is also constant—can you visualize this?



Example 1.17 Radial Motion without Acceleration

A particle moves with $\dot{\theta} = \omega = \text{constant}$ and $r = r_0 e^{\beta t}$, where r_0 and β are constants. We shall show that for certain values of β , the particle moves with $a_r = 0$.

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t} \omega^2)\hat{\mathbf{r}} + 2\beta r_0 \omega e^{\beta t}\hat{\boldsymbol{\theta}}.\end{aligned}$$

If $\beta = \pm\omega$, the radial part of \mathbf{a} vanishes.

It is very surprising at first that when $r = r_0 e^{\beta t}$ the particle moves with zero radial acceleration. The error is in thinking that \ddot{r} makes the only contribution to a_r ; the term $-r\dot{\theta}^2$ is also part of the radial acceleration, and cannot be neglected.

The paradox is that even though $a_r = 0$, the radial velocity $v_r = \dot{r} = r_0 \omega e^{\beta t}$ is increasing rapidly with time. The answer is that we can be misled by the special case of cartesian coordinates; in polar coordinates,

$$v_r \neq \int a_r(t) dt,$$

because $\int a_r(t) dt$ does not take into account the fact that the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are functions of time.

Note 1.1 Mathematical Approximation Methods

Occasionally in the course of solving a problem in physics you may find that you have become so involved with the mathematics that the physics is totally obscured. In such cases, it is worth stepping back for a moment to see if you cannot sidestep the mathematics by using simple approximate expressions instead of exact but complicated formulas. If you have not yet acquired the knack of using approximations, you may feel that there is something essentially wrong with the procedure of substituting inexact results for exact ones. However, this is not really the case, as the following example illustrates.

Suppose that a physicist is studying the free fall of bodies in vacuum, using a tall vertical evacuated tube. The timing apparatus is turned on when the falling body interrupts a thin horizontal ray of light located a distance L below the initial position. By measuring how long the body takes to pass through the light beam, the physicist hopes to determine the local value of g , the acceleration due to gravity. The falling body in the experiment has a height l .

For a freely falling body starting from rest, the distance s traveled in time t is

$$s = \frac{1}{2}gt^2,$$

which gives

$$t = \sqrt{\frac{2}{g}} \sqrt{s}.$$

The time interval $t_2 - t_1$ required for the body to fall from $s_1 = L$ centimeters to $s_2 = (L + l)$ centimeters is

$$\begin{aligned} t_2 - t_1 &= \sqrt{\frac{2}{g}} (\sqrt{s_2} - \sqrt{s_1}) \\ &= \sqrt{\frac{2}{g}} (\sqrt{L + l} - \sqrt{L}). \end{aligned}$$

If $t_2 - t_1$ is measured experimentally, g is given by

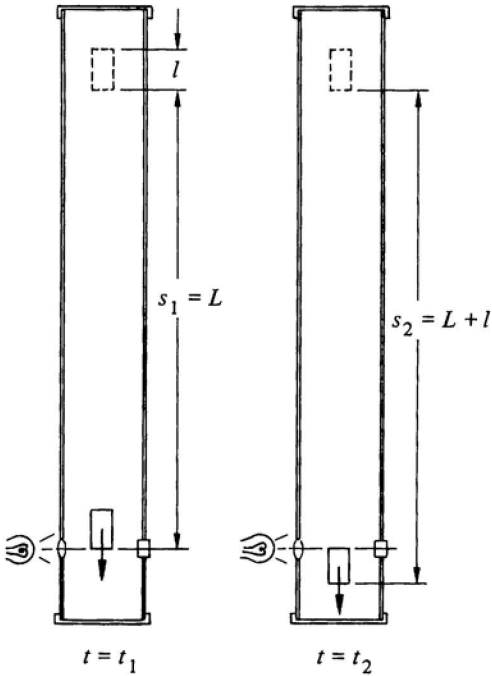
$$g = 2 \left(\frac{\sqrt{L + l} - \sqrt{L}}{t_2 - t_1} \right)^2$$

This formula is exact under the stated conditions, but it may not be the most useful expression for our purposes.

Consider the factor

$$\sqrt{L + l} - \sqrt{L}.$$

In practice, L will be large compared with l (typical values might be $L = 100$ cm, $l = 1$ cm). Our factor is the small difference between two large numbers and is hard to evaluate accurately by using a slide rule or ordinary mathematical tables. Here is a simple approach, known as the method of power series expansion, which enables us to evaluate the factor



to any accuracy we please. As we shall discuss formally later in this Note, the quantity $\sqrt{1+x}$ can be written in the series form

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

for $-1 < x < 1$. Furthermore, if we cut off the series at some point, the error we incur by this approximation is of the order of the first neglected term. We can put the factor in a form suitable for expansion by first extracting \sqrt{L} :

$$\sqrt{L+l} - \sqrt{L} = \sqrt{L} \left(\sqrt{1 + \frac{l}{L}} - 1 \right).$$

The dimensionless ratio l/L plays the part of x in our expansion. Expanding $\sqrt{1 + l/L}$ in the series form gives

$$\begin{aligned} \sqrt{L} \left(\sqrt{1 + \frac{l}{L}} - 1 \right) &= \sqrt{L} \left[1 + \frac{1}{2} \left(\frac{l}{L} \right) - \frac{1}{8} \left(\frac{l}{L} \right)^2 \right. \\ &\quad \left. + \frac{1}{16} \left(\frac{l}{L} \right)^3 + \cdots - 1 \right] \\ &= \sqrt{L} \left[\frac{1}{2} \left(\frac{l}{L} \right) - \frac{1}{8} \left(\frac{l}{L} \right)^2 + \frac{1}{16} \left(\frac{l}{L} \right)^3 + \cdots \right]. \end{aligned}$$

We see that if l/L is much smaller than 1, the successive terms decrease rapidly. The first term in the bracket, $\frac{1}{2}(l/L)$, is the largest term, and extracting it from the bracket yields

$$\begin{aligned} \sqrt{L+l} - \sqrt{L} &= \sqrt{L} \frac{1}{2} \left(\frac{l}{L} \right) \left[1 - \frac{1}{4} \left(\frac{l}{L} \right) + \frac{1}{8} \left(\frac{l}{L} \right)^2 + \cdots \right] \\ &= \frac{l}{2\sqrt{L}} \left[1 - \frac{1}{4} \left(\frac{l}{L} \right) + \frac{1}{8} \left(\frac{l}{L} \right)^2 + \cdots \right]. \end{aligned}$$

Our expansion is now in its final and most useful form. The first factor, $l/(2\sqrt{L})$, gives the dominant behavior and is a useful first approximation. Furthermore, writing the series as we have, with leading term 1, shows clearly the contributions of the successive powers of l/L . For example, if $l/L = 0.01$, the term $\frac{1}{8}(l/L)^2 = 1.2 \times 10^{-5}$ and we make a fractional error of about 1 part in 10^5 by retaining only the preceding terms. In many cases this accuracy is more than enough. For instance, if the time interval $t_2 - t_1$ in the falling body experiment can be measured to only 1 part in 1,000, we gain nothing by evaluating $\sqrt{L+l} - \sqrt{L}$ to greater accuracy than this. On the other hand, if we require greater accuracy, we can easily tell how many terms of the series should be retained.

Practicing physicists make mathematical approximations freely (when justified) and have no compunctions about discarding negligible terms. The ability to do this often makes the difference between being stymied

by impenetrable algebra and arithmetic and successfully solving a problem.

Furthermore, series approximations often allow us to simplify complicated algebraic expressions to bring out the essential physical behavior.

Here are some helpful methods for making mathematical approximations.

1 THE BINOMIAL SERIES

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \\ + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$$

This series is valid for $-1 < x < 1$, and for any value of n . (If n is an integer, the series terminates, the last term being x^n .) The series is exact; the approximation enters when we truncate it. For $n = \frac{1}{2}$, as in our example,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \quad -1 < x < 1.$$

If we need accuracy only to $O(x^2)$ (order of x^2), we have

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3),$$

where the term $O(x^3)$ indicates that terms of order x^3 and higher are not being considered. As a rule of thumb, the error is approximately the size of the first term dropped.

The series can also be applied if $|x| > 1$ as follows:

$$(1+x)^n = x^n \left(1 + \frac{1}{x}\right)^n \\ = x^n \left[1 + n\frac{1}{x} + \frac{n(n-1)}{2!}\left(\frac{1}{x}\right)^2 + \cdots\right].$$

Examples:

$$1. \frac{1}{1+x} = (1+x)^{-1} \\ = 1 - x + x^2 - x^3 + \cdots \quad -1 < x < 1$$

$$2. \frac{1}{1-x} = (1-x)^{-1} \\ = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$$

$$3. (1,001)^{\frac{1}{3}} = (1,000 + 1)^{\frac{1}{3}} = 1,000^{\frac{1}{3}}(1 + 0.001)^{\frac{1}{3}} \\ = 10[1 + 0.001(\frac{1}{3}) + \cdots] \\ \approx 10(1.0003) = 10.003$$

$$4. 2 - \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}}: \text{ for small } x, \text{ this expression is zero to first}$$

approximation. However, this approximation may not be adequate. Using the binomial series, we have

$$\begin{aligned} 2 - \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}} &= 2 - (1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots) \\ &\quad - (1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots) \\ &= -\tfrac{3}{4}x^2. \end{aligned}$$

Notice that the terms linear in x also cancel. To obtain a nonvanishing result we had to go to a high enough order, in this case to order x^2 . It is clear that for a correct result we have to expand all terms to the same order.

2 TAYLOR'S SERIES¹

Analogous to the binomial series, we can try to represent an arbitrary function $f(x)$ by a power series in x :

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

For $x = 0$ we must have

$$f(0) = a_0.$$

Assuming for the moment that it is permissible to differentiate, we have

$$\frac{df}{dx} = f'(x) = a_1 + 2a_2x + \cdots$$

Evaluating at $x = 0$ we have

$$a_1 = f'(x) \Big|_{x=0}.$$

Continuing this process, we find

$$a_k = \frac{1}{k!} f^{(k)}(x) \Big|_{x=0},$$

where $f^{(k)}(x)$ is the k th derivative of $f(x)$. For the sake of a less cumbersome notation, we often write $f^{(k)}(0)$ to stand for $f^{(k)}(x) \Big|_{x=0}$; but bear in mind that $f^{(k)}(0)$ means that we should differentiate $f(x)$ k times and then set x equal to 0.

The power series for $f(x)$, known as a *Taylor series*, can then be expressed formally as

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

This series, if it converges, allows us to find good approximations to $f(x)$ for small values of x (that is, for values of x near zero). Generalizing,

$$f(a+x) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \cdots$$

¹ Taylor's series is discussed in most elementary calculus texts. See the list at the end of the chapter.

gives us the behavior of the function in the neighborhood of the point a . An alternative form for this expression is

$$f(t) = f(a) + f'(a)(t - a) + f''(a) \frac{(t - a)^2}{2!} + \cdots$$

Our formal manipulations are valid only if the series converges. The range of convergence of a Taylor series may be $-\infty < x < \infty$ for some functions (such as e^x) but quite limited for other functions. (The binomial series converges only if $-1 < x < 1$.) The range of convergence is hard to find without considering functions of a complex variable, and we shall avoid these questions by simply assuming that we are dealing with simple functions for which the range of convergence is either infinite or is readily apparent. Here are some examples:

a. The Trigonometric Functions

Let $f(x) = \sin x$, and expand about $x = 0$.

$$f(0) = \sin(0) = 0$$

$$f'(0) = \cos(0) = 1$$

$$f''(0) = -\sin(0) = 0$$

$$f'''(0) = -\cos(0) = -1, \quad \text{etc.}$$

Hence

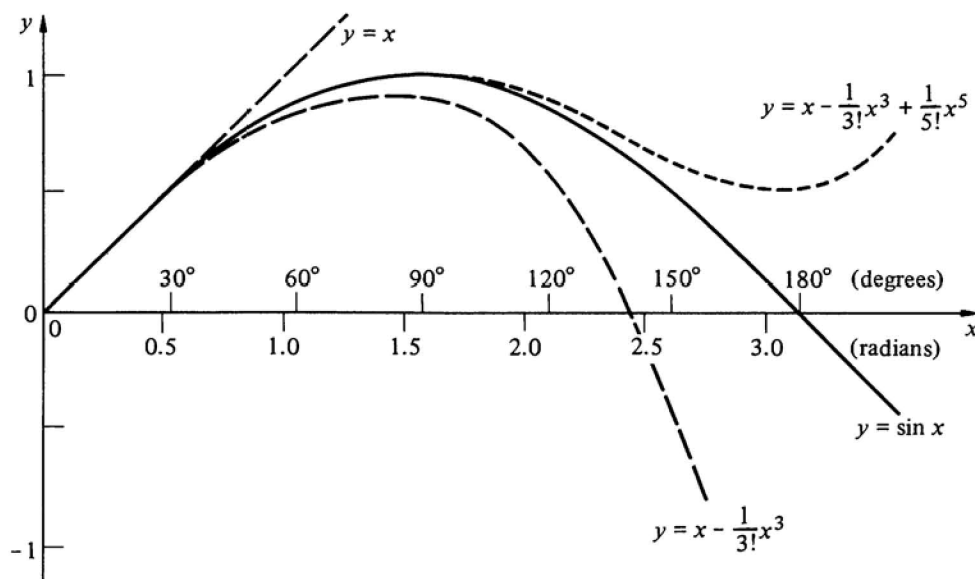
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Similarly

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

These expansions converge for all values of x but are particularly useful for small values of x . To $O(x^2)$, $\sin x = x$, $\cos x = 1 - x^2/2$.

The figure below compares the exact value for $\sin x$ with a Taylor series in which successively higher terms are included. Note how each



term increases the range over which the series is accurate. If an infinite number of terms are included, the Taylor series represents the function accurately everywhere.

b. The Binomial Series

We can derive the binomial series introduced in the last section by letting

$$f(x) = (1 + x)^n.$$

Then

$$f(0) = 1$$

$$f'(0) = n(1 + 0)^{n-1} = n$$

$$f''(0) = n(n-1)$$

$$f^{(k)}(0) = n(n-1)(n-2) \cdots (n-k+1)$$

$$\begin{aligned} (1+x)^n &= 1 + nx + \frac{1}{2!} n(n-1)x^2 + \cdots \\ &\quad + \cdots + \frac{n(n-1) \cdots (n-k+1)}{k!} x^k + \cdots \end{aligned}$$

c. The Exponential Function

If we let $f(x) = e^x$, we have $f'(x) = f(x)$, by the definition of the exponential function. Similarly $f^{(k)}(x) = f(x)$. Since $f(0) = e^0 = 1$, we have

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

This series converges for all values of x .

A useful result from the theory of the Taylor series is that if the series converges at all, it represents the function so well that we are allowed to differentiate or integrate the series any number of times. For example,

$$\begin{aligned} \frac{d}{dx} (\sin x) &= \frac{d}{dx} \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right) \\ &= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \\ &= \cos x. \end{aligned}$$

Furthermore, the Taylor series for the product of two functions is the product of the individual series:

$$\begin{aligned} \sin x \cos x &= \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right) \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \right) \\ &= x - \left(\frac{1}{3!} + \frac{1}{2!} \right) x^3 + \left(\frac{1}{4!} + \frac{1}{3!2!} + \frac{1}{5!} \right) x^5 + \cdots \end{aligned}$$

$$\begin{aligned}
&= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} + \cdots \\
&= \frac{1}{2} \left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots \right] \\
&= \frac{1}{2} [\sin(2x)].
\end{aligned}$$

The Taylor series sometimes comes in handy in the evaluation of integrals. To estimate

$$\int_1^{1.1} \frac{e^z}{z} dz,$$

let $z = 1 + x$. We then have

$$\begin{aligned}
\int_1^{1.1} \frac{e^z}{z} dz &= \int_0^{0.1} \frac{e^{(1+x)}}{1+x} dx \\
&= (e) \int_0^{0.1} \frac{e^x}{1+x} dx \\
&\approx (e) \int_0^{0.1} \frac{(1+x)}{(1+x)} dx \\
&\approx 0.1e.
\end{aligned}$$

The approximation should be better than 1 part in 100 or so, for x always lies in the interval $0 \leq x \leq 0.1$. In this range, $e^x \approx 1 + x$ is a good approximation to two or three significant figures.

3 DIFFERENTIALS

Consider $f(x)$, a function of the independent variable x . Often we need to have a simple approximation for the change in $f(x)$ when x is changed to $x + \Delta x$. Let us denote the change by $\Delta f = f(x + \Delta x) - f(x)$. It is natural to turn to the Taylor series. Expanding the Taylor series for $f(x)$ about the point x gives

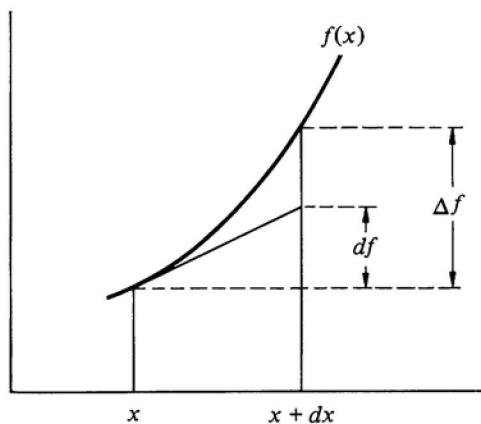
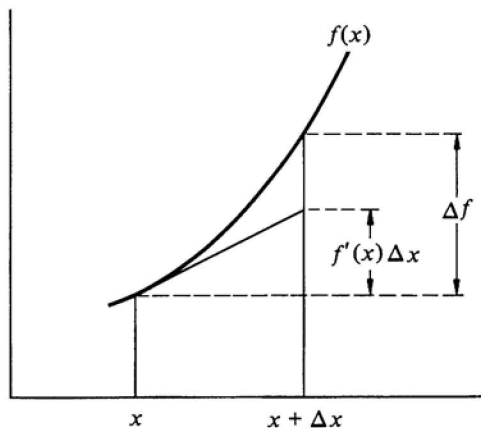
$$f(x + \Delta x) = f(x) + f'(x) \Delta x + \frac{1}{2!} f''(x) \Delta x^2 + \cdots,$$

where, for example, $f'(x)$ stands for df/dx evaluated at the point x . Omitting terms of order $(\Delta x)^2$ and higher yields the simple linear approximation

$$\Delta f = f(x + \Delta x) - f(x) \approx f'(x) \Delta x.$$

This approximation becomes increasingly accurate the smaller the size of Δx . However, for finite values of Δx , the expression

$$\Delta f \approx f'(x) \Delta x$$



has to be considered to be an approximation. The graph at left shows a comparison of $\Delta f \equiv f(x + \Delta x) - f(x)$ with the linear extrapolation $f'(x)\Delta x$. It is apparent that Δf , the actual change in $f(x)$ as x is changed, is generally not exactly equal to Δf for finite Δx .

As a matter of notation, we use the symbol dx to stand for Δx , the increment in x . dx is known as the *differential* of x ; it can be as large or small as we please. We define df , the differential of f , by

$$df \equiv f'(x) dx.$$

This notation is illustrated in the lower drawing. Note that dx and Δx are used interchangeably. On the other hand, df and Δf are different quantities. df is a differential defined by $df = f'(x) dx$, whereas Δf is the actual change $f(x + dx) - f(x)$. Nevertheless, when the linear approximation is justified in a problem, we often use df to represent Δf . We can always do this when eventually a limit will be taken. Here are some examples.

1. $d(\sin \theta) = \cos \theta d\theta$.
2. $d(xe^{x^2}) = (e^{x^2} + 2x^2e^{x^2}) dx$.
3. Let V be the volume of a sphere of radius r :

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr.$$

4. What is the fractional increase in the volume of the earth if its average radius, 6.4×10^6 m, increases by 1 m?

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} \\ &= 3 \frac{dr}{r} \\ &= \frac{3}{6.4 \times 10^6} = 4.7 \times 10^{-7}. \end{aligned}$$

One common use of differentials is in changing the variable of integration. For instance, consider the integral

$$\int_a^b xe^{x^2} dx.$$

A useful substitution is $t = x^2$. The procedure is first to solve for x in terms of t ,

$$x = \sqrt{t},$$

and then to take differentials:

$$dx = \frac{1}{2} \frac{1}{\sqrt{t}} dt.$$

This result is exact, since we are effectively taking the limit. The original integral can now be written in terms of t :

$$\begin{aligned}\int_a^b x e^{x^2} dx &= \int_{t_1}^{t_2} \sqrt{t} e^t \left(\frac{1}{2} \frac{1}{\sqrt{t}} dt \right) = \frac{1}{2} \int_{t_1}^{t_2} e^t dt \\ &= \frac{1}{2} (e^{t_2} - e^{t_1}),\end{aligned}$$

where $t_1 = a^2$ and $t_2 = b^2$.

Some References to Calculus Texts

A very popular textbook is G. B. Thomas, Jr., "Calculus and Analytic Geometry," 4th ed., Addison-Wesley Publishing Company, Inc., Reading, Mass.

The following introductory texts in calculus are also widely used:

M. H. Protter and C. B. Morrey, "Calculus with Analytic Geometry," Addison-Wesley Publishing Company, Inc., Reading, Mass.

A. E. Taylor, "Calculus with Analytic Geometry," Prentice-Hall, Inc., Englewood Cliffs, N.J.

R. E. Johnson and E. L. Keokemeister, "Calculus With Analytic Geometry," Allyn and Bacon, Inc., Boston.

A highly regarded advanced calculus text is R. Courant, "Differential and Integral Calculus," Interscience Publishing, Inc., New York.

If you need to review calculus, you may find the following helpful: Daniel Kleppner and Norman Ramsey, "Quick Calculus," John Wiley & Sons, Inc., New York.

Problems

1.1 Given two vectors, $\mathbf{A} = (2\hat{i} - 3\hat{j} + 7\hat{k})$ and $\mathbf{B} = (5\hat{i} + \hat{j} + 2\hat{k})$, find: (a) $\mathbf{A} + \mathbf{B}$; (b) $\mathbf{A} - \mathbf{B}$; (c) $\mathbf{A} \cdot \mathbf{B}$; (d) $\mathbf{A} \times \mathbf{B}$.

Ans. (a) $7\hat{i} - 2\hat{j} + 9\hat{k}$; (c) 21

1.2 Find the cosine of the angle between

$\mathbf{A} = (3\hat{i} + \hat{j} + \hat{k})$ and $\mathbf{B} = (-2\hat{i} - 3\hat{j} - \hat{k})$.

Ans. -0.805

1.3 The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosine of the angles between the vector and the x , y , and z axes are usually called, in turn α , β , and γ . Prove that $\alpha^2 + \beta^2 + \gamma^2 = 1$, using either geometry or vector algebra.

1.4 Show that if $|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$, then \mathbf{A} is perpendicular to \mathbf{B} .

1.5 Prove that the diagonals of an equilateral parallelogram are perpendicular.

1.6 Prove the law of sines using the cross product. It should only take a couple of lines. (Hint: Consider the area of a triangle formed by \mathbf{A} , \mathbf{B} , \mathbf{C} , where $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$.)

1.7 Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the xy plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}$, and using vector algebra prove that

$$\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi.$$

1.8 Find a unit vector perpendicular to

$$\mathbf{A} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) \quad \text{and} \quad \mathbf{B} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}).$$

$$\text{Ans. } \hat{\mathbf{n}} = \pm(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}})/\sqrt{38}$$

1.9 Show that the volume of a parallelepiped with edges \mathbf{A} , \mathbf{B} , and \mathbf{C} is given by $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

1.10 Consider two points located at \mathbf{r}_1 and \mathbf{r}_2 , separated by distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Find a vector \mathbf{A} from the origin to a point on the line between \mathbf{r}_1 and \mathbf{r}_2 at distance xr from the point at \mathbf{r}_1 , where x is some number.

1.11 Let \mathbf{A} be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}}$.

1.12 The acceleration of gravity can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions.

Show that if the time the body takes to pass a horizontal line A in both directions is T_A , and the time to go by a second line B in both directions is T_B , then, assuming that the acceleration is constant, its magnitude is

$$g = \frac{8h}{T_A^2 - T_B^2},$$

where h is the height of line B above line A .

1.13 An elevator ascends from the ground with uniform speed. At time T_1 a boy drops a marble through the floor. The marble falls with uniform acceleration $g = 9.8 \text{ m/s}^2$, and hits the ground T_2 seconds later. Find the height of the elevator at time T_1 .

$$\text{Ans. clue. If } T_1 = T_2 = 4 \text{ s, } h = 39.2 \text{ m}$$

1.14 A drum of radius R rolls down a slope without slipping. Its axis has acceleration a parallel to the slope. What is the drum's angular acceleration α ?

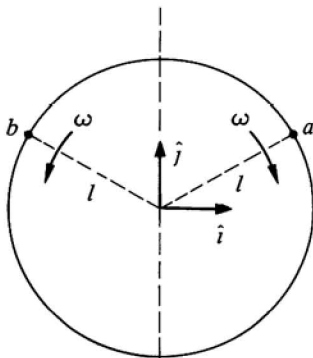
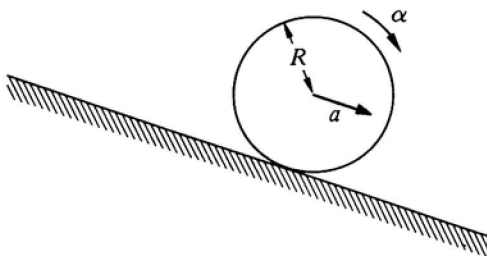
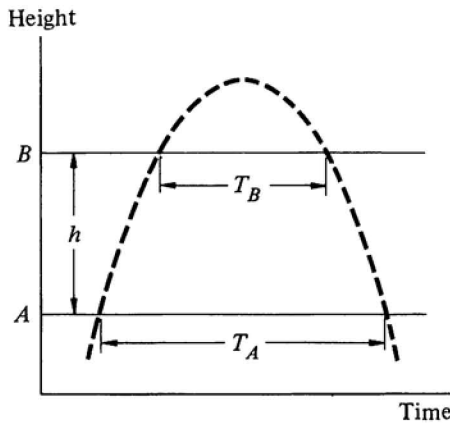
1.15 By *relative velocity* we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.)

a. A point is observed to have velocity \mathbf{v}_A relative to coordinate system A . What is its velocity relative to coordinate system B , which is displaced from system A by distance \mathbf{R} ? (\mathbf{R} can change in time.)

$$\text{Ans. } \mathbf{v}_B = \mathbf{v}_A - d\mathbf{R}/dt$$

b. Particles a and b move in opposite directions around a circle with angular speed ω , as shown. At $t = 0$ they are both at the point $\mathbf{r} = l\hat{\mathbf{j}}$, where l is the radius of the circle.

Find the velocity of a relative to b .



1.16 A sportscar, Fiasco I, can accelerate uniformly to 120 mi/h in 30 s. Its *maximum* braking rate cannot exceed $0.7g$. What is the minimum time required to go $\frac{1}{2}$ mi, assuming it begins and ends at rest? (*Hint:* A graph of velocity vs. time can be helpful.)

1.17 A particle moves in a plane with constant radial velocity $\dot{r} = 4$ m/s. The angular velocity is constant and has magnitude $\dot{\theta} = 2$ rad/s. When the particle is 3 m from the origin, find the magnitude of (a) the velocity and (b) the acceleration.

Ans. (a) $v = \sqrt{52}$ m/s

1.18 The rate of change of acceleration is sometimes known as "jerk." Find the direction and magnitude of jerk for a particle moving in a circle of radius R at angular velocity ω . Draw a vector diagram showing the instantaneous position, velocity, acceleration, and jerk.

1.19 A tire rolls in a straight line without slipping. Its center moves with constant speed V . A small pebble lodged in the tread of the tire touches the road at $t = 0$. Find the pebble's position, velocity, and acceleration as functions of time.

1.20 A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant. $A = (1/\pi)$ m/rad. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

a. Sketch the motion, and indicate the approximate velocity and acceleration at a few points.

b. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

c. At what angles do the radial and tangential accelerations have equal magnitude?

1.21 A boy stands at the peak of a hill which slopes downward uniformly at angle ϕ . At what angle θ from the horizontal should he throw a rock so that it has the greatest range?

Ans. *clue.* If $\phi = 60^\circ$, $\theta = 15^\circ$

