2 NEWTON'S LAWS–THE FOUNDATIONS OF NEWTONIAN MECHANICS

2.1 Introduction

Our aim in this chapter is to understand Newton's laws of motion. From one point of view this is a modest task: Newton's laws are simple to state and involve little mathematical complexity. Their simplicity is deceptive, however. As we shall see, they combine definitions, observations from nature, partly intuitive concepts, and some unexamined assumptions on the properties of space and time. Newton's statement of the laws of motion left many of these points unclear. It was not until two hundred years after Newton that the foundations of classical mechanics were carefully examined, principally by Ernst Mach,[1] and our treatment is very much in the spirit of Mach.

Newton's laws of motion are by no means self-evident. In Aristotle's system of mechanics, a force was thought to be needed to maintain a body in uniform motion. Aristotelian mechanics was accepted for thousands of years because, superficially, it seemed intuitively correct. Careful reasoning from observation and a real effort of thought was needed to break out of the aristotelian mold. Most of us are still not accustomed to thinking in newtonian terms, and it takes both effort and practice to learn to analyze situations from the newtonian point of view. We shall spend a good deal of time in this chapter looking at applications of Newton's laws, for only in this way can we really come to understand them. However, in addition to deepening our understanding of dynamics, there is an immediate reward—we shall be able to analyze quantitatively physical phenomena which at first sight may seem incomprehensible.

Although Newton's laws provide a direct introduction to classical mechanics, it should be pointed out that there are a number of other approaches. Among these are the formulations of Lagrange and Hamilton, which take energy rather than force as the fundamental concept. However, these methods are physically equivalent to the newtonian approach, and even though we could use one of them as our point of departure, a deep understanding of Newton's laws is an invaluable asset to understanding any systematic treatment of mechanics.

A word about the validity of newtonian mechanics: possibly you already know something about modern physics—the development early in this century of relativity and quantum mechanics. If so,

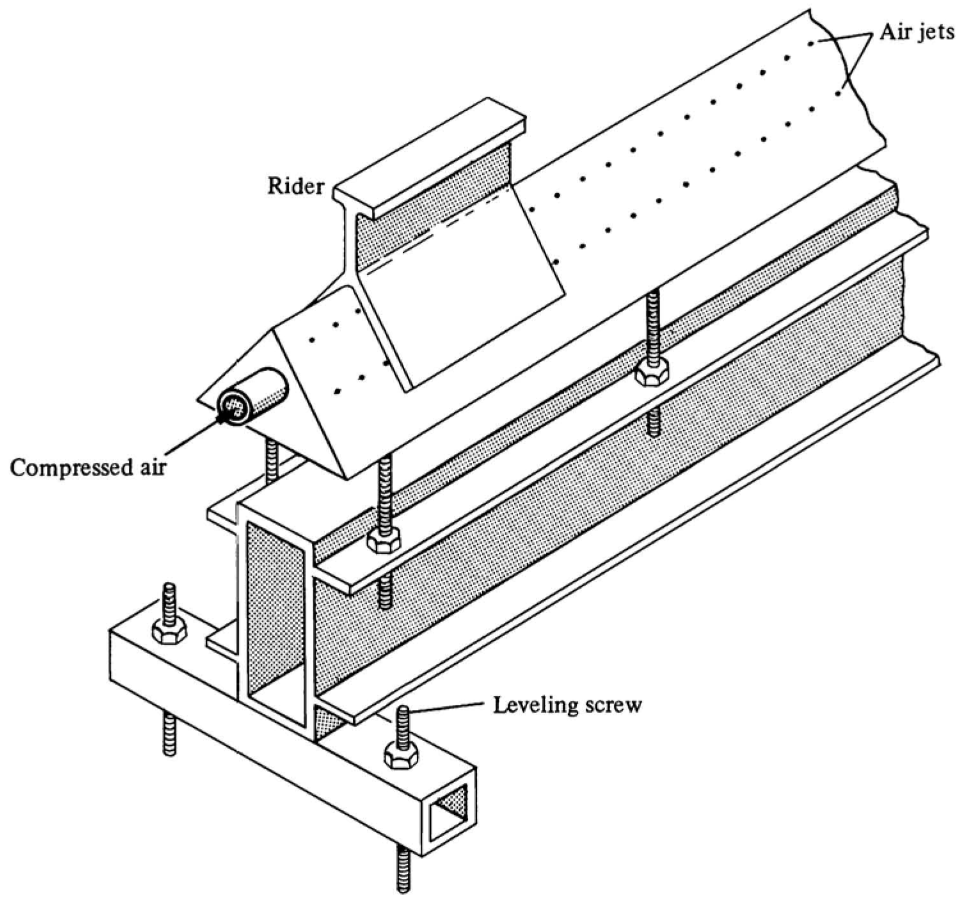
[1] Mach's text, "The Science of Mechanics" (1883), translated the arguments from Newton's "Principia" into a more logically satisfying form. His analysis of the assumptions of newtonian mechanics played a major role in the development of Einstein's special theory of relativity, as we shall see in Chap. 10.

you know that there are important areas of physics in which newtonian mechanics fails, while relativity and quantum mechanics succeed. Briefly, newtonian mechanics breaks down for systems moving with a speed comparable to the speed of light, 3×10^8 m/s, and it also fails for systems of atomic dimensions or smaller where quantum effects are significant. The failure arises because of inadequacies in classical concepts of space, time, and the nature of measurement. A natural impulse might be to throw out classical physics and proceed directly to modern physics. We do not accept this point of view for several reasons. In the first place, although the more advanced theories have shown us where classical physics breaks down, they also show us where the simpler methods of classical physics give accurate results. Rather than make a blanket statement that classical physics is right or wrong, we recognize that newtonian mechanics is exceptionally useful in many areas of physics but of limited applicability in other areas. For instance, newtonian physics enables us to predict eclipses centuries in advance, but is useless for predicting the motions of electrons in atoms. It should also be recognized that because classical physics explains so many everyday phenomena, it is an essential tool for all practicing scientists and engineers. Furthermore, most of the important concepts of classical physics are preserved in modern physics, albeit in altered form.

2.2 Newton's Laws

It is important to understand which parts of Newton's laws are based on experiment and which parts are matters of definition. In discussing the laws we must also learn how to apply them, not only because this is the bread and butter of physics but also because this is essential for a real understanding of the underlying concepts.

We start by appealing directly to experiment. Unfortunately, experiments in mechanics are among the hardest in physics because motion in our everyday surroundings is complicated by forces such as gravity and friction. To see the physical essentials, we would like to eliminate all disturbances and examine very simple systems. One way to accomplish this would be to enroll as astronauts, for in the environment of space most of the everyday disturbances are negligible. However, lacking the resources to put ourselves in orbit, we settle for second best, a device known as a *linear air track,* which approximates ideal conditions, but only in one dimension. (Although it is not clear that we can learn anything about three dimensional motion from studying motion in one dimension, happily this turns out to be the case.)



Linear air track

The linear air track is a hollow triangular beam perhaps 2 m long, pierced by many small holes which emit gentle streams of air. A rider rests on the beam, and when the air is turned on, the rider floats on a thin cushion of air. Because of the air suspension, the rider moves with negligible friction. (The reason for this is that the thin film of air has a viscosity typically 5,000 times less than a film of oil.) If the track is leveled carefully, and if we eliminate stray air currents, the rider behaves as if it were isolated in its motion along the track. The rider moves along the track free of gravity, friction, or any other detectable influences.

Now let's observe how the rider behaves. (Try these experiments yourself if possible.) Suppose that we place the rider on the track and carefully release it from rest. As we might expect, the rider stays at rest, at least until a draft hits it or somebody bumps the apparatus. (This isn't too surprising, since we leveled the track until the rider stayed put when left at rest.) Next, we give the rider a slight shove and then let it move freely. The motion seems uncanny, for the rider continues to move along slowly and evenly, neither gaining nor losing speed. This is contrary to our everyday experience that moving bodies stop moving unless we push them. The reason is that in everyday motion, friction usually plays an important role. For instance, the air track rider comes to a grinding halt if we turn off the air and let sliding friction act. Apparently the friction stops the motion. But we are getting ahead of ourselves; let us return to the properly functioning air track and try to generalize from our experience.

It is possible to make a two dimensional air table analogous to the one dimensional air track. (A smooth sheet of glass with a flat piece of dry ice on it does pretty well. The evaporating dry ice provides the gas cushion.) We find again that the undisturbed rider moves with uniform velocity. Three dimensional isolated motion is hard to observe, short of going into space, but let us for the moment assume that our experience in one and two dimensions also holds in three dimensions. We therefore surmise that an object moves uniformly in space provided there are no external influences.

Newton's First Law

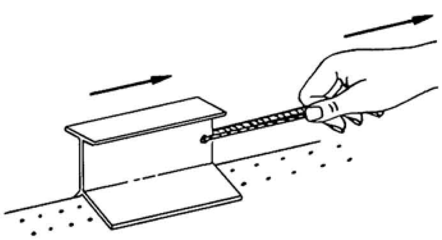
In our discussion of the air track experiments, we glossed over an important point. Motion has meaning only with respect to a particular coordinate system, and in describing motion it is essential to specify the coordinate system we are using. For example, in describing motion along the air track, we implicitly used a coordinate system fixed to the track. However, we are free to choose any coordinate system we please, including systems which are moving with respect to the track. In a coordinate system moving uniformly with respect to the track, the undisturbed rider moves with constant velocity. Such a coordinate system is called an *inertial system*. Not all coordinate systems are inertial; in a coordinate system accelerating with respect to the track, the undisturbed rider does not have constant velocity. However, it is always possible to find a coordinate system with respect to which isolated bodies move uniformly. This is the essence of Newton's first law of motion.

Newton's first law of motion is the assertion that inertial systems exist.

Newton's first law is part definition and part experimental fact. Isolated bodies move uniformly in inertial systems by virtue of the definition of an inertial system. In constrast, that inertial systems exist is a statement about the physical world.

Newton's first law raises a number of questions, such as what we mean by an "isolated body," but we will defer these temporarily and go on.

Newton's Second Law



We now turn to how the rider on the air track behaves when it is no longer isolated. Suppose that we pull the rider with a rubber band. Nothing happens while the rubber band is loose, but as soon as we pull hard enough to stretch the rubber band, the rider starts to move. If we move our hand ahead of the rider so that the rubber band is always stretched to the same standard length, we find that the rider moves in a wonderfully simple way; its velocity increases uniformly with time. The rider moves with constant acceleration.

Now suppose that we try the same experiment with a different rider, perhaps one a good deal larger than the first. Again, the same rubber band stretched to the standard length produces a constant acceleration, but the acceleration is different from that in the first case. Apparently the acceleration depends not only on what we do to the object, since presumably we do the same thing in each case, but also on some property of the object, which we call *mass*.

We can use our rubber band experiment to *define* what we mean by mass. We start by arbitrarily saying that the first body has a mass m_1. (m_1 could be one unit of mass or x units of mass, where x is any number we choose.) We then *define* the mass of the second body to be

$$m_2 = m_1 \frac{a_1}{a_2},$$

where a_1 is the acceleration of the first body in our rubber band experiment and a_2 is the acceleration of the second body.

Continuing this procedure, we can assign masses to other objects by measuring their accelerations with the standard stretched rubber band. Thus

$$m_3 = m_1 \frac{a_1}{a_3}$$

$$m_4 = m_1 \frac{a_1}{a_4} \qquad \text{etc.}$$

Although this procedure is straightforward, there is no obvious reason why the quantity we define this way is particularly important. For instance, why not consider instead some other property, call it property Z, such that $Z_2 = Z_1(a_1/a_2)^2$? The reason is that mass is useful, whereas property Z (or most other quantities you try) is not. By making further experiments with the air track, for instance by using springs or magnets instead of a rubber band, we find that the ratios of accelerations, hence the mass ratios, are the same no matter how we produce the uniform accelerations, provided that we do the same thing to each body. Thus, mass so defined turns out to be independent of the source of acceleration and appears to be an inherent property of a body. Of course, the actual mass value of an individual body depends on our choice of mass unit. The important thing is that two bodies have a unique mass ratio.

Our definition of mass is an example of an *operational* definition. By operational we mean that the definition is dominantly in terms of experiments we perform and not in terms of abstract concepts, such as "mass is a measure of the resistance of bodies to a change in motion." Of course, there can be many abstract concepts hidden in apparently simple operations. For instance, when we measure acceleration, we tacitly assume that we have a clear understanding of distance and time. Although our intuitive ideas are adequate for our purposes here, we shall see when we discuss relativity that the behavior of measuring rods and clocks is itself a matter for experiment.

A second troublesome aspect of operational definitions is that they are limited to situations in which the operations can actually be performed. In practice this is usually not a problem; physics proceeds by constructing a chain of theory and experiment which allows us to employ convenient methods of measurement ultimately based on the operational definitions. For instance, the most practical way to measure the mass of a mountain is to observe its gravitational pull on a test body, such as a hanging plumb bob. According to the operational definition, we should apply a standard force and measure the mountain's acceleration. Nevertheless, the two methods are directly related conceptually.

We defined mass by experiments on laboratory obiects; we cannot say a priori whether the results are consistent on a much larger or smaller scale. In fact, one of the major goals of physics is to find the limitations of such definitions, for the limitations normally reveal new physical laws. Nevertheless, if an operational definition is to be at all useful, it must have very wide applicability. For instance, our definition of mass holds not only for everyday objects on the earth but also, to a very high degree, for planetary motion, motion on an enormously larger scale. It should not surprise us, however, if eventually we find situations in which the operations are no longer useful.

Now that we have defined mass, let us turn our attention to *force*.

We describe the operation of acting on the test mass with a stretched rubber band as "applying" a force. (Note that we have sidestepped the question of what a force is and have limited ourselves to describing how to produce it—namely, by stretching a rubber band by a given amount.) When we apply the force, the test mass accelerates at some rate, a. If we apply two standard stretched rubber bands, side by side, we find that the mass accelerates at the rate $2a$, and if we apply them in opposite directions, the acceleration is zero. The effects of the rubber bands add algebraically for the case of motion in a straight line.

We can establish a force scale by defining the unit force as the force which produces unit acceleration when applied to the unit mass. It follows from our experiments that F units of force accelerate the unit mass by F units of acceleration and, from our definition of mass, it will produce $F \times (1/m)$ units of acceleration in mass m. Hence, the acceleration produced by force F acting on mass m is $a = F/m$ or, in a more familiar order, $F = ma$. In the International System of units (SI), the unit of force is the *newton* (N), the unit of mass is the *kilogram* (kg), and acceleration is in meters per second^2 (m/s^2). Units are discussed further in Sec. 2.3.

So far we have limited our experiments to one dimension. Since acceleration is a vector, and mass, as far as we know, is a scalar, we expect that force is also a vector. It is natural to think of the force as pointing in the direction of the acceleration it produces when acting alone. This assumption appears trivial, but it is not—its justification lies in experiment. We find that forces obey the *principle of superposition:* The acceleration produced by several forces acting on a body is equal to the vector sum of the accelerations produced by each of the forces acting separately. Not only does this confirm the vector nature of force, but it also enables us to analyze problems by considering one force at a time.

Combining all these observations, we conclude that the total force $\mathbf{F}$ on a body of mass m is $\mathbf{F} = \Sigma \mathbf{F}_i$, where $\mathbf{F}_i$ is the ith applied force. If $\mathbf{a}$ is the net acceleration, and $\mathbf{a}_i$ the acceleration due to $\mathbf{F}_i$ alone, then we have

$$\begin{aligned}\mathbf{F} &= \Sigma \mathbf{F}_i \\ &= \Sigma m\mathbf{a}_i \\ &= m\Sigma \mathbf{a}_i \\ &= m\mathbf{a}\end{aligned}$$

or

$$\mathbf{F} = m\mathbf{a}.$$

This is Newton's second law of motion. It will underlie much of our subsequent discussion.

It is important to understand clearly that force is not merely a matter of definition. For instance, if the air track rider starts accelerating, it is not sufficient to claim that there is a force acting defined by $\mathbf{F} = m\mathbf{a}$. Forces always arise from *interactions* between systems, and if we ever found an acceleration without an interaction, we would be in a terrible mess. It is the interaction which is physically significant and which is responsible for the force. For this reason, when we isolate a body sufficiently from its surroundings, we expect the body to move uniformly in an inertial system. Isolation means eliminating interactions. You may question whether it is always possible to isolate a body. Fortunately, as far as we know, the answer is yes. All known interactions decrease with distance. (The forces which extend over the greatest distance are the familiar gravitational and Coulomb forces. They decrease as $1/r^2$, where r is the distance. Most forces decrease much more rapidly. For example, the force between separated atoms decreases as $1/r^7$.) By moving the test body sufficiently far from everything else, the interactions can be reduced as much as desired.

Newton's Third Law

The fact that force is necessarily the result of an interaction between two systems is made explicit by Newton's third law. The third law states that forces always appear in pairs: if body b exerts force $\mathbf{F}_a$ on body a, then there must be a force $\mathbf{F}_b$ acting on body b, due to body a, such that $\mathbf{F}_b = -\mathbf{F}_a$. There is no such thing as a lone force without a partner. As we shall see in the next chapter, the third law leads directly to the powerful law of conservation of momentum.

We have argued that a body can be isolated by removing it sufficiently far from other bodies. However, the following problem arises. Suppose that an isolated body starts to accelerate in defiance of Newton's second law. What prevents us from explaining away the difficulty by attributing the acceleration to carelessness in isolating the system? If this option is open to us, Newton's second law becomes meaningless. We need an independent way of telling whether or not there is a physical interaction on a system. Newton's third law provides such a test. If the acceleration of a body is the result of an outside force, then somewhere in the universe there must be an equal and opposite force acting on another body. If we find such a force, the dilemma is resolved; the body was not completely isolated. The interaction may be new and interesting, but as long as the forces are equal and opposite, Newton's laws are satisfied.

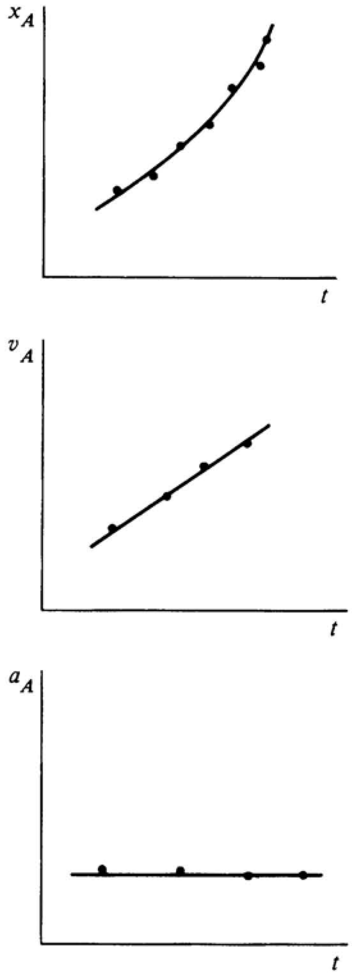
If an isolated body accelerates and we cannot find some external object which suffers an equal and opposite force, then we are in trouble. As far as we know this has never occurred. Thus Newton's third law is not only a vitally important dynamical tool, but it is also an important logical element in making sense of the first two laws.

Newton's second law $\mathbf{F} = m\mathbf{a}$ holds true only in inertial systems. The existence of inertial systems seems almost trivial to us, since the earth provides a reasonably good inertial reference frame for everyday observations. However, there is nothing trivial about the concept of an inertial system, as the following example shows.

Example 2.1 Astronauts in Space—Inertial Systems and Fictitious Forces

Two spaceships are moving in empty space chasing an unidentified flying object, possibly a flying saucer. The captains of the two ships, A and B, must find out if the saucer is flying freely or if it is accelerating. A, B, and the saucer are all moving along a straight line.

The captain of A sets to work and measures the distance to the saucer as a function of time. In principle, he sets up a coordinate system along the line of motion with his ship as origin and notes the position of the saucer, which he calls $x_A(t)$. (In practice he uses his radar set to measure the distance to the saucer.) From $x_A(t)$ he calculates the velocity



$v_A = \dot{x}_A$ and the acceleration $a_A = \ddot{x}_A$. The results are shown in the sketches. The captain of A concludes that the saucer has a positive acceleration $a_A = 1{,}000$ m/s². He therefore assumes that its engines are on and that the force on the saucer is

$$\begin{aligned} F_A &= a_A M \\ &= 1{,}000M \text{ newtons,} \end{aligned}$$

where M is the saucer's mass in kilograms.

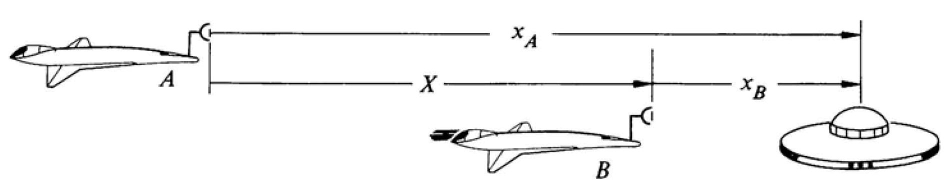
The captain of B goes through the same procedure. He finds that the acceleration is $a_B = 950$ m/s² and concludes that the force on the saucer is

$$\begin{aligned} F_B &= a_B M \\ &= 950M \text{ newtons.} \end{aligned}$$

This presents a serious problem. There is nothing arbitrary about force; if different observers obtain different values for the force, at least one of them must be mistaken. The captains of A and B have confidence in the laws of mechanics, so they set about resolving the discrepancy. In particular, they recall that Newton's laws hold only in inertial systems. How can they decide whether or not their systems are inertial?

A's captain sets out by checking to see if all his engines are off. Since they are, he suspects that he is not accelerating and that his spaceship defines an inertial system. To check that this is the case, he undertakes a simple but sensitive experiment. He observes that a pencil, carefully released at rest, floats without motion. He concludes that the pencil's acceleration is negligible and that he is in an inertial system. The reasoning is as follows: as long as he holds the pencil it must have the same instantaneous velocity and acceleration as the spaceship. However, there are no forces acting on the pencil after it is released, assuming that we can neglect gravitational or electrical interactions with the spaceship, air currents, etc. The pencil, then, can be presumed to represent an isolated body. If the spaceship is itself accelerating, it will catch up with the pencil—the pencil will appear to accelerate relative to the cabin. Otherwise, the spaceship must itself define an inertial system.

The determination of the force on the saucer by the captain of A must be correct because A is in an inertial system. But what can we say about the observations made by the captain of B? To answer this problem, we look at the relation of x_A and x_B. From the sketch,



$$x_A(t) = x_B(t) + X(t),$$

where $X(t)$ is the position of B relative to A. Differentiating twice with respect to time, we have

$$\ddot{x}_A = \ddot{x}_B + \ddot{X}. \qquad 1$$

Since system A is inertial, Newton's second law for the saucer is

$$F_{\text{true}} = M\ddot{x}_A \qquad 2$$

where F_{true} is the true force on the saucer.

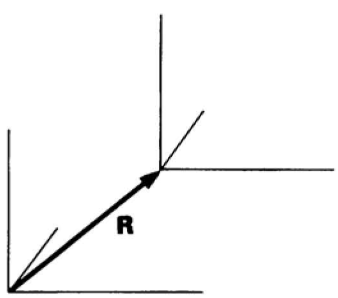
What about the observations made by the captain of B? The apparent force observed by B is

$$F_{B,\text{apparent}} = M\ddot{x}_B. \qquad 3$$

Using the results of (1) and (2), we have

$$\begin{aligned} F_{B,\text{apparent}} &= M\ddot{x}_A - M\ddot{X} \\ &= F_{\text{true}} - M\ddot{X}. \end{aligned} \qquad 4$$

B will not measure the true force unless $\ddot{X} = 0$. However, $\ddot{X} = 0$ only when B moves uniformly with respect to A. As we suspect, this is not the case here. The captain of B has accidently left on a rocket engine, and he is accelerating away from A at 50 m/s^2. After shutting off the engine, he obtains the same value for the force on the saucer as does A.



Although we considered only motion along a line in Example 2.1, it is easy to generalize the result to three dimensions. If $\mathbf{R}$ is the vector from the origin of an inertial system to the origin of another coordinate system, we have

$$\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} - M\ddot{\mathbf{R}}.$$

If $\ddot{\mathbf{R}} = 0$, then $\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}}$, which means that the second coordinate system is also inertial. In fact, we have merely proven what we asserted earlier, namely, that any system moving uniformly with respect to an inertial system is also inertial.

Sometimes we would like to carry out measurements in noninertial systems. What can we do to get the correct equations of motion? The answer lies in the relation $\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} - M\ddot{\mathbf{R}}$. We can think of the last term as an additional force, which we call a *fictitious force*. (The term fictitious indicates that there is no real interaction involved.) We then write

$$\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} + \mathbf{F}_{\text{fictitious}},$$

where $\mathbf{F}_{\text{fictitious}} = -M\ddot{\mathbf{R}}$. Here M is the mass of the particle and $\ddot{\mathbf{R}}$ is the acceleration of the noninertial system with respect to any inertial system.

Fictitious forces are useful in solving certain problems, but they must be treated with care. They generally cause more confusion then they are worth at this stage of your studies, and for that reason we shall avoid them for the present and agree to use inertial systems only. Later on, in Chap. 8, we shall examine fictitious forces in detail and learn how to deal with them.

Although Newton's laws can be stated in a reasonably clear and consistent fashion, it should be realized that there are fundamental difficulties which cannot be argued away. We shall return to these in later chapters after we have had a chance to become better acquainted with the concepts of newtonian physics. Some points, however, are well to bear in mind now.

1. You have had to take our word that the experiments we used to define mass and to develop the second law of motion really give the results claimed. It should come as no surprise (although it was a considerable shock when it was first discovered) that this is not always so. For instance, the mass scale we have set up is no longer consistent when the particles are moving at high speeds. It turns out that instead of the mass we defined, called the rest mass m_0, a more useful quantity is $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the speed of light and v is the speed of the particle. For the case $v \ll c$, m and m_0 differ negligibly. The reason that our tabletop experiments did not lead us to the more general expression for mass is that even for the largest everyday velocities, say the velocity of a spacecraft going around the earth, $v/c \approx 3 \times 10^{-5}$, and m and m_0 differ by only a few parts in 10^{10}.

2. Newton's laws describe the behavior of point masses. In the case where the size of the body is small compared with the interaction distance, this offers no problem. For instance, the earth and sun are so small compared with the distance between them that for many purposes their motion can be adequately described by considering the motion of point masses located at the center of each. However, the approximation that we are dealing with point masses is fortunately not essential, and if we wish to describe the motion of large bodies, we can readily generalize Newton's laws, as we shall do in the next chapter. It turns out to be not much more difficult to discuss the motion of a rigid body composed of 10^{24} atoms than the motion of a single point mass.

3. Newton's laws deal with particles and are poorly suited for describing a continuous system such as a fluid. We cannot directly apply $\mathbf{F} = m\mathbf{a}$ to a fluid, for both the force and the mass are continuously distributed. However, newtonian mechanics can be extended to deal with fluids and provides the underlying principles of fluid mechanics.

One system which is particularly troublesome for our present formulation of newtonian mechanics is the electromagnetic field. Paradoxes can arise when such a field is present. For instance, two charged bodies which interact electrically actually interact via the electric fields they create. The interaction is not instantaneously transmitted from one particle to the other but propagates at the velocity of light. During the propagation time there is an apparent breakdown of Newton's third law; the forces on the particles are not equal and opposite. Similar problems arise in considering gravitational and other interactions. However, the problem lies not so much with newtonian mechanics as with its misapplication. Simply put, fields possess mechanical properties like momentum and energy which must not be overlooked. From this point of view there is no such thing as a simple two particle system. However, for many systems the fields can be taken into account and the paradoxes can be resolved within the newtonian framework.

2.3 Standards and Units

Length, time, and mass play a fundamental role in every branch of physics. These quantities are defined in terms of certain fundamental physical standards which are agreed to by the scientific community. Since a particular standard generally does not have a convenient size for every application, a number of systems of units have come into use. For example, the centimeter, the angstrom, and the yard are all units of length, but each is defined in terms of the standard meter. There are a number of systems of units in widespread use, the choice being chiefly a matter of custom and convenience. This section presents a brief description of the current standards and summarizes the units which we shall encounter.

The Fundamental Standards

The fundamental standards play two vital roles. In the first place, the precision with which these standards can be defined and reproduced limits the ultimate accuracy of experiments. In some cases the precision is almost unbelievably high—time, for instance, can be measured to a few parts in 10^{12}. In addition, agreeing to a standard for a physical quantity simultaneously provides an operational definition for that quantity. For example, the modern view is that time is what is measured by clocks, and that the properties of time can be understood only by observing the properties of clocks. This is not a trivial point; the rates of all clocks are affected by motion and by gravity (as we shall discuss in Chaps. 8 and 12), and unless we are willing to accept the fact that time itself is altered by motion and gravity, we are led into contradictions.

Once a physical quantity has been defined in terms of a measurement procedure, we must appeal to experiment, not to preconceived notions, to understand its properties. To contrast this viewpoint with a nonoperational approach, consider, for example, Newton's definition of time: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external." This may be intuitively and philosophically appealing, but it is hard to see how such a definition can be applied. The idea is metaphysical and not of much use in physics.

Once we have agreed on the operation underlying a particular physical quantity, the problem is to construct the most precise practical standard. Until recently, physical standards were man-made, in the sense that they consisted of particular objects to which all other measurements had to be referred. Thus, the unit length, the meter, was defined to be the distance between two scratches on a platinum bar. Such man-made standards have a number of disadvantages. Since the standard must be carefully preserved, actual measurements are often done with secondary standards, which causes a loss of accuracy. Furthermore, the precision of a man-made standard is intrinsically limited. In the case of the standard meter, precision was found to be limited by fuzziness in the engraved lines which defined the meter interval. When more accurate optical techniques for locating position were developed in the latter part of the nineteenth century, it was realized that the standard meter bar was no longer adequate.

Length is now defined by a natural, rather than man-made, standard. The meter is defined to be a given multiple of the wavelength of a particular spectral line. The advantage of such a unit is that anyone who has the required optical equipment can reproduce it. Also, as the instrumentation improves, the accuracy of the standard will correspondingly increase. Most of the standards of physics are now natural.

Here is a brief account of the current status of the standards of length, time, and mass.

Length The meter was intended to be one ten-millionth of the distance from the equator to the pole of the earth along the Dunkirk-Barcelona line. This cannot be measured accurately (in fact it changes due to distortions of the earth), and in 1889 it was agreed to define the meter as the separation between two scratches in a platinum-iridium bar which is preserved at the International Bureau of Weights and Measures, Sèvres, France. In 1960 the meter was redefined to be 1,650,763.73 wavelengths of the orange-red line of krypton 86. The accuracy of this standard is a few parts in 10^8.

Recent advances in laser techniques provide methods which should allow the velocity of light to be measured to better than 1 part in 10^8. It is likely that the velocity of light will replace length as a fundamental quantity. In this case the unit of length would be derived from velocity and time.

Time Time has traditionally been measured in terms of rotation of the earth. Until 1956 the basic unit, the second, was defined as 1/86,400 of the mean solar day. Unfortunately, the period of rotation of the earth is not very uniform. Variations of up to one part in 10^7 per day occur due to atmospheric tides and changes in the earth's core. The motion of the earth around the sun is not influenced by these perturbations, and until recently the mean solar year was used to define the second. Here the accuracy was a few parts in 10^9. Fortunately, time can now be measured in terms of a natural atomic frequency. In 1967 the second was defined to be the time required to execute 9,192,631,770 cycles of a hyperfine transition in cesium 133. This transition frequency can be reliably measured to a few parts in 10^{12}, which means that time is by far the most accurately determined fundamental quantity.

Mass Of the three fundamental units, only mass is defined in terms of a man-made standard. Originally, the kilogram was defined to be the mass of 1,000 cubic centimeters of water at a temperature of 4 degrees Centigrade. The definition is difficult to apply, and in 1889 the kilogram was defined to be the mass of a platinum-iridium cylinder which is maintained at the International Bureau of Weights and Measures. Secondary standards can be compared with it to an accuracy of one part in 10^9. Perhaps someday we will learn how to define the kilogram in terms of a natural unit, such as the mass of an atom. However, at present nobody knows how to count reliably the large number of atoms needed to constitute a useful sample. Perhaps you can discover a method.

Systems of Units

Although the standards for mass, length, and time are accepted by the entire scientific community, there are a variety of systems of units which differ in the scaling factors. The most widely used system of units is the International System, abbreviated SI (for Système International d'Unités). It is the legal system in most countries. The SI units are *meter*, *kilogram*, and *second;* SI replaces the former mks system. The related cgs system, based on the centimeter, gram, and second, is also commonly used. A third system, the English system of units, is used for nonscientific measurements in Britain and North America, although Britain is in the process of switching to the metric system. It is essential to know how to work problems in any system of units. We shall work chiefly with SI units, with occasional use of the cgs system and one or two lapses into the English system.

Here is a table listing the names of units in the SI, cgs, and English systems.

	SI	CGS	ENGLISH
Length	1 meter (m)	1 centimeter (cm)	1 foot (ft)
Mass	1 kilogram (kg)	1 gram (g)	1 slug
Time	1 second (s)	1 second (s)	1 second (s)
Acceleration	1 m/s^2	1 cm/s^2	1 ft/s^2
Force	1 newton (N) $= 1\ kg{\cdot}m/s^2$	1 dyne $= 1\ g{\cdot}cm/s^2$	1 pound (lb) $= 1\ slug{\cdot}ft/s^2$

Some useful relations between these units systems are:

1 m = 100 cm	1 in = $\frac{1}{12}$ ft ≈ 2.54 cm
1 kg = 1000 g	1 slug ≈ 14.6 kg
1 N = 10^5 dyne	1 N ≈ 0.224 lb

The word pound sometimes refers to a unit of mass. In this context it stands for the mass which experiences a gravitational force of one pound at the surface of the earth, approximately 0.454 kg. We shall avoid this confusing usage.

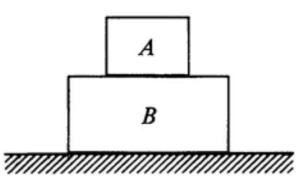
2.4 Some Applications of Newton's Laws

Newton's laws are meaningless equations until we know how to apply them. A number of steps are involved which, once learned, are so natural that the procedure becomes intuitive. Our aim in this section is to outline a method of analyzing physical problems and to illustrate it by examples. A note of reassurance lest you feel that matters are presented too dogmatically: There are many ways of attacking most problems, and the procedure we suggest is certainly not the only one. In fact, no cut-and-dried procedure can ever substitute for intelligent analytical thinking. However, the systematic method suggested here will be helpful in getting started, and we urge you to master it even if you should later resort to shortcuts or a different approach.

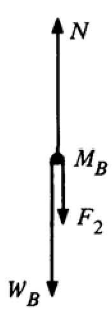
Here are the steps:

1. Mentally divide the system[1] into smaller systems, each of which can be treated as a point mass.
2. Draw a force diagram for each mass as follows:
   a. Represent the body by a point or simple symbol, and label it.
   b. Draw a force vector on the mass for each force acting on it.

Point 2*b* can be tricky. Draw only forces acting *on* the body, not forces exerted *by* the body. The body may be attached to strings, pushed by other bodies, etc. We replace all these physical interactions with other bodies by a system of forces; according to Newton's laws, only forces acting *on* the body influence its motion.



As an example, here are two blocks at rest on a table top. The force diagram for A is shown at left. F_1 is the force exerted on block A by block B, and W_A is the force of gravity on A, called the *weight.*



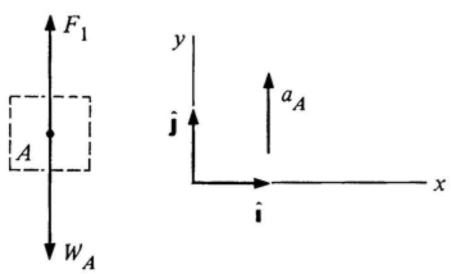
Similarly, we can draw the force diagram for block B. W_B is the force of gravity on B, N is the normal (perpendicular) force exerted by the table top on B, and F_2 is the force exerted by A on B. There are no other physical interactions that would produce a force on B.

It is important not to confuse a force with an acceleration; draw only *real* forces. Since we are using only inertial systems for the present, all the forces are associated with physical interactions. For every force you should be able to answer the question, "What

[1] We use "system" here to mean a collection of physical objects rather than a coordinate system. The meaning should be clear from the context.

exerts this force on the body?" (We shall see how to use so-called fictitious forces in Chap. 8.[1])

3. Introduce a coordinate system. The coordinate system must be inertial—that is, it must be fixed to an inertial frame. With the force diagram as a guide, write separately the component equations of motion for each body. By equation of motion we mean an equation of the form $F_{1x} + F_{2x} + \cdots = Ma_x$, where the x component of each force on the body is represented by a term on the left hand side of the equation. The algebraic sign of each component must be consistent with the force diagram and with the choice of coordinate system.



For instance, returning to the force diagram for block A, Newton's second law gives

$$\mathbf{F}_1 + \mathbf{W}_A = m_A\mathbf{a}_A.$$

Since $\mathbf{F}_1 = F_1\hat{\mathbf{j}}$, $\mathbf{W}_A = -W_A\hat{\mathbf{j}}$, we have

$$0 = m_A(\mathbf{a}_A)_x$$

and

$$F_1 - W_A = m_A(\mathbf{a}_A)_y.$$

The x equation of motion is trivial and normally we omit it, writing simply

$$F_1 - W_A = m_A a_A.$$

The equation of motion for B is

$$N - F_2 - W_B = m_B a_B.$$

4. If two bodies in the same system interact, the forces between them must be equal and opposite by Newton's third law. These relations should be written explicitly.

For example, in the case of the two blocks on the tabletop, $\mathbf{F}_1 = -\mathbf{F}_2$. Hence

$$F_1 = F_2.$$

Note that Newton's third law never relates two forces acting on the *same* body; forces on two different bodies must be involved.

[1] The most notorious fictitious force is the centrifugal force. Long experience has shown that using this force before one has a really solid grasp of Newton's laws invariably causes confusion. Besides, it is only one of several fictitious forces which play a role in rotating systems. For both these reasons, we shall strictly avoid centrifugal forces for the present.

5. In many problems, bodies are constrained to move along certain paths. A pendulum bob, for instance, moves in a circle, and a block sliding on a tabletop is constrained to move in a plane. Each constraint can be described by a kinematical equation known as a *constraint* equation. Write each constraint equation.

Sometimes the constraints are implicit in the statement of the problem. For the two blocks on the tabletop, there is no vertical acceleration, and the constraint equations are

$$(\mathbf{a}_A)_y = 0 \qquad (\mathbf{a}_B)_y = 0.$$

6. Keep track of which variables are known and which are unknown. The force equations and the constraint equations should provide enough relations to allow every unknown to be found. If an equation is overlooked, there will be too few equations for the unknowns.

Completing the problem of the two blocks on the table, we have

$$\left.\begin{aligned} F_1 - W_A &= m_A a_A \\ N - F_2 - W_B &= m_B a_B \end{aligned}\right\} \text{Equations of motion}$$

$$F_1 = F_2 \qquad \text{From Newton's third law}$$

$$\left.\begin{aligned} a_A &= 0 \\ a_B &= 0 \end{aligned}\right\} \text{Constraint equations}$$

All that remains is the mathematical task of solving the equations. We find

$$F_1 = F_2 = W_A$$
$$N = W_A + W_B.$$

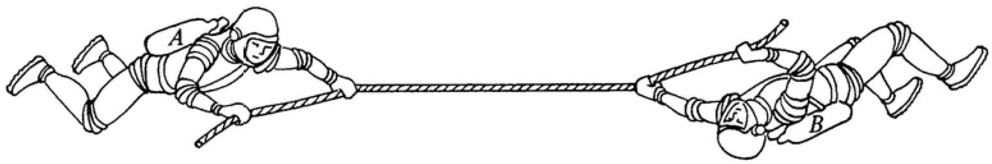
Here are a few examples which illustrate the application of Newton's laws.

The main point of the first example is to help us distinguish between the force we apply to an object and the force it exerts on us. Physiologically, these forces are often confused. If you push a book across a table, the force you feel is not the force that makes the book move; it is the force the book exerts on you. According to Newton's third law, these two forces are always equal and opposite. If one force is limited, so is the other.

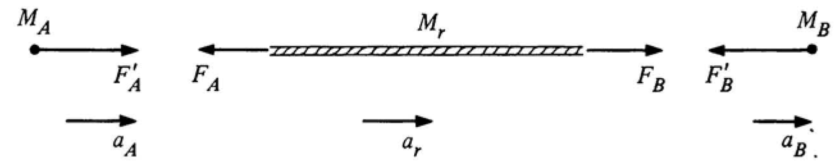
Example 2.2 **The Astronauts' Tug-of-war**

Two astronauts, initially at rest in free space, pull on either end of a rope. Astronaut Alex played football in high school and is stronger than astronaut Bob, whose hobby was chess. The maximum force with which

Alex can pull, F_A, is larger than the maximum force with which Bob can pull, F_B. Their masses are M_A and M_B, and the mass of the rope, M_r, is negligible. Find their motion if each pulls on the rope as hard as he can.



Here are the force diagrams. For clarity, we show the rope as a line.



Note that the forces F_A and F_B exerted by the astronauts act on the *rope*, not on the astronauts. The forces exerted by the rope on the astronauts are $F_A{}'$ and $F_B{}'$. The diagram shows the directions of the forces and the coordinate system we have adopted; acceleration to the right is positive.

By Newton's third law,

$$\begin{aligned} F_A' &= F_A \\ F_B' &= F_B. \end{aligned} \tag{1}$$

The equation of motion for the rope is

$$F_B - F_A = M_r a_r. \tag{2}$$

Only motion along the line of the rope is of interest, and we omit the equations of motion in the remaining two directions. There are no constraints, and we proceed to the solution.

Since the mass of the rope, M_r, is negligible, we take $M_r = 0$ in Eq. (2). This gives $F_B - F_A = 0$ or

$$F_B = F_A.$$

The total force on the rope is F_B to the right and F_A to the left. These forces are equal in magnitude, and the total force on the rope is zero. In general, the total force on any body of negligible mass must be effectively zero; a finite force acting on zero mass would produce an infinite acceleration.

Since $F_B = F_A$, Eq. (1) gives $F_A' = F_A = F_B = F_B'$. Hence

$$F_A' = F_B'.$$

The astronauts each pull with the *same* force. Physically, there is a limit to how hard Bob can grip the rope; if Alex tries to pull too hard, the rope slips through Bob's fingers. The force Alex can exert is limited by the strength of Bob's grip. If the rope were tied to Bob, Alex could exert his maximum pull.

The accelerations of the two astronauts are

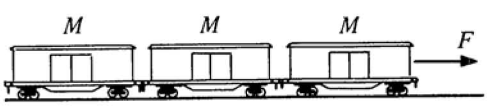
$$a_A = \frac{F'_A}{M_A}$$

$$a_B = \frac{-F'_B}{M_B} = \frac{-F'_A}{M_B}.$$

The negative sign means that a_B is to the left. In many problems the directions of some acceleration or force components are initially unknown. In writing the equations of motion, any choice is valid, provided we are consistent with the convention assumed in the force diagram. If the solution yields a negative sign, the acceleration or force is opposite to the direction assumed.

The next example shows that in order for a compound system to accelerate, there must be a net force on each part of the system.

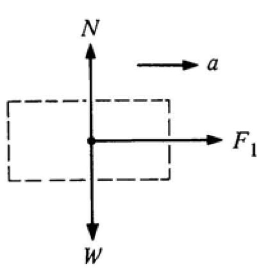
Example 2.3 **Freight Train**



Three freight cars of mass M are pulled with force F by a locomotive. Friction is negligible. Find the forces on each car.

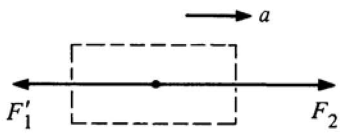
Before drawing the force diagram, it is worth thinking about the system as a whole. Since the cars are joined, they are constrained to have the same acceleration. Since the total mass is $3M$, the acceleration is

$$a = \frac{F}{3M}.$$



A force diagram for the last car is shown at the left. W is the weight and N is the upward force exerted by the track. The vertical acceleration is zero, so that $N = W$. F_1 is the force exerted by the next car. We have

$$F_1 = Ma = M\left(\frac{F}{3M}\right) = \frac{F}{3}.$$

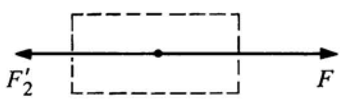


Now let us consider the middle car. The vertical forces are as before, and we omit them. F_1' is the force exerted by the last car, and F_2 is the force exerted by the first car. The equation of motion is

$$F_2 - F_1' = Ma.$$

By Newton's third law, $F_1' = F_1 = F/3$. Since $a = F/3M$, we have

$$F_2 = M\left(\frac{F}{3M}\right) + \frac{F}{3}$$
$$= \frac{2F}{3}.$$

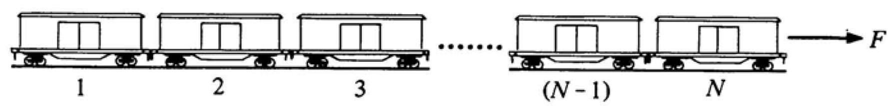


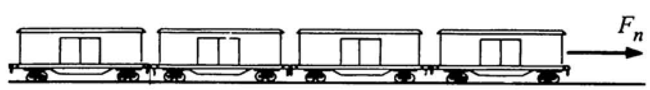
The horizontal forces on the first car are F, to the right, and

$$F_2' = F_2 = \frac{2F}{3},$$

to the left. Each car experiences a total force $F/3$ to the right.

Here is a slightly more general way to look at the problem. Consider a string of N cars, each of mass M, pulled by a force F. The accelera-





tion is $a = F/(NM)$. To find the force F_n pulling the last n cars, note that F_n must give the mass nM an acceleration $F/(NM)$. Hence

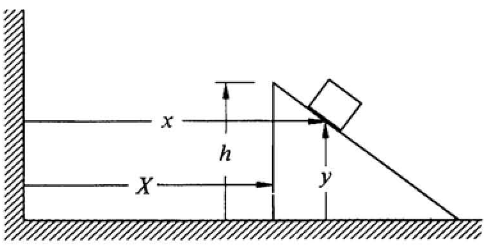
$$F_n = nM\frac{F}{NM}$$
$$= \frac{n}{N}F.$$

The force is proportional to the number of cars pulled.

In systems composed of several bodies, the accelerations are often related by constraints. The equations of constraint can sometimes be found by simple inspection, but the most general approach is to start with the coordinate geometry, as shown in the next example.

Example 2.4 **Constraints**

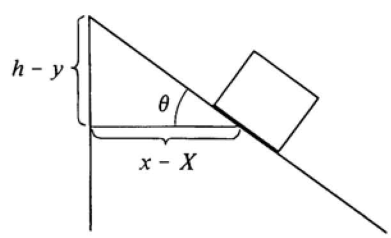
a. WEDGE AND BLOCK



A block moves on a wedge which in turns moves on a horizontal table, as shown in the sketch. The wedge angle is θ. How are the accelerations of the block and the wedge related?

As long as the wedge is in contact with the table, we have the trivial constraint that the vertical acceleration of the wedge is zero. To find the less obvious constraint, let X be the horizontal coordinate of the end of the wedge and let x and y be the horizontal and vertical coordinates of the block, as shown. Let h be the height of the wedge.

From the geometry, we see that



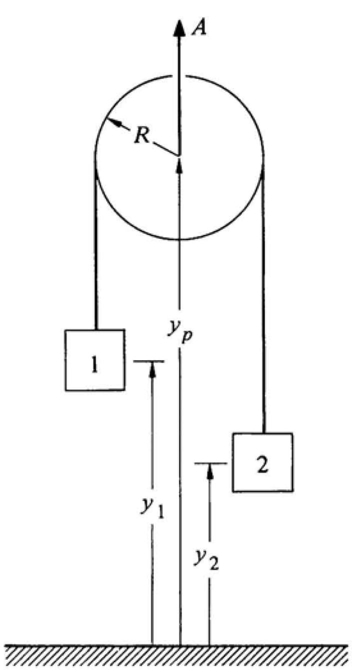
$$(x - X) = (h - y)\cot\theta.$$

Differentiating twice with respect to time, we obtain the equation of constraint

$$\ddot{x} - \ddot{X} = -\ddot{y}\cot\theta. \tag{1}$$

A few comments are in order. Note that the coordinates are inertial. We would have trouble using Newton's second law if we measured the position of the block with respect to the wedge; the wedge is accelerating and cannot specify an inertial system. Second, unimportant parameters, like the height of the wedge, disappear when we take time derivatives, but they can be useful in setting up the geometry. Finally, constraint equations are independent of applied forces. For example, even if friction between the block and wedge affects their accelerations, Eq. (1) is valid as long as the bodies are in contact.

b. MASSES AND PULLEY



Two masses are connected by a string which passes over a pulley accelerating upward at rate A, as shown. Find how the accelerations of the bodies are related. Assume that there is no horizontal motion.

We shall use the coordinates shown in the drawing. The length of the string, l, is constant. Hence, if y_p is measured to the center of the pulley of radius R,

$$l = \pi R + (y_p - y_1) + (y_p - y_2). \tag{2}$$

Differentiating twice with respect to time, we find the constraint condition

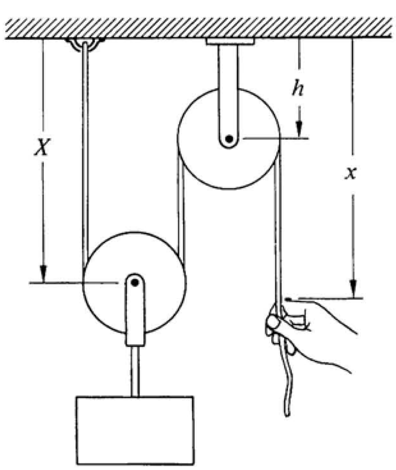
$$0 = 2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2.$$

Using $A = \ddot{y}_p$, we have

$$A = \tfrac{1}{2}(\ddot{y}_1 + \ddot{y}_2).$$

c. PULLEY SYSTEM

The pulley system shown on the opposite page is used to hoist the block. How does the acceleration of the end of the rope compare with the



acceleration of the block? Using the coordinates indicated, the length of the rope is given by

$$l = X + \pi R + (X - h) + \pi R + (x - h),$$

where R is the radius of the pulleys. Hence

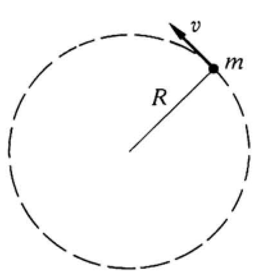
$$\ddot{X} = -\tfrac{1}{2}\ddot{x}.$$

The block accelerates half as fast as the hand, and in the opposite direction.

Our examples so far have involved linear motion only. Let us look at the dynamics of rotational motion.

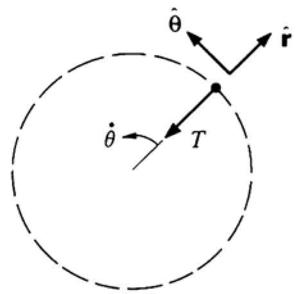
A particle undergoing circular motion must have a radial acceleration. This sometimes causes confusion, since our intuitive idea of acceleration usually relates to change in speed rather than to change in direction of motion. For this reason, we start with as simple an example as possible.

Example 2.5 **Block on String 1**



Mass m whirls with constant speed v at the end of a string of length R. Find the force on m in the absence of gravity or friction.

The only force on m is the string force T, which acts toward the center, as shown in the diagram. It is natural to use polar coordinates. Note that according to the derivation in Sec. 1.9, the radial acceleration is $a_r = \ddot{r} - r\dot{\theta}^2$, where $\dot{\theta}$ is the angular velocity. a_r is positive outward. Since $\mathbf{T}$ is directed toward the origin, $\mathbf{T} = -T\hat{\mathbf{r}}$ and the radial equation of motion is



$$\begin{aligned} -T &= ma_r \\ &= m(\ddot{r} - r\dot{\theta}^2). \end{aligned}$$

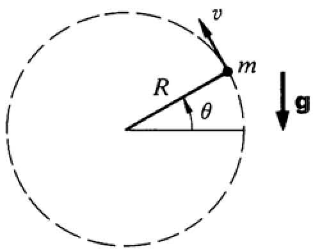
$\ddot{r} = \ddot{R} = 0$ and $\dot{\theta} = v/R$. Hence $a_r = -R(v/R)^2 = -v^2/R$ and

$$T = \frac{mv^2}{R}.$$

Note that T is directed toward the origin; there is no outward force on m. If you whirl a pebble at the end of a string, *you* feel an outward force. However, the force you feel does not act on the pebble, it acts on you. This force is equal in magnitude and opposite in direction to the force with which you pull the pebble, assuming the string's mass to be negligible.

In the following example both radial and tangential acceleration play a role in circular motion.

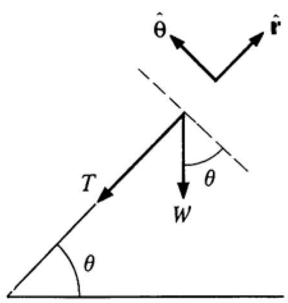
Example 2.6 **Block on String 2**



Mass m is whirled on the end of a string length R. The motion is in a vertical plane in the gravitational field of the earth. The forces on m are the weight W down, and the string force T toward the center. The instantaneous speed is v, and the string makes angle θ with the horizontal. Find T and the tangential acceleration at this instant.

The lower diagram shows the forces and unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. The radial force is $-T - W \sin\theta$, so the radial equation of motion is

$$-(T + W\sin\theta) = ma_r$$
$$= m(\ddot{r} - r\dot{\theta}^2). \qquad 1$$



The tangential force is $-W\cos\theta$. Hence

$$-W\cos\theta = ma_\theta$$
$$= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \qquad 2$$

Since $r = R =$ constant, $a_r = -R(\dot{\theta}^2) = -v^2/R$, and Eq. (1) gives

$$T = \frac{mv^2}{R} - W\sin\theta.$$

The string can pull but not push, so that T cannot be negative. This requires that $mv^2/R \geq W\sin\theta$. The maximum value of $W\sin\theta$ occurs when the mass is vertically up; in this case $mv^2/R > W$. If this condition is not satisfied, the mass does not follow a circular path but starts to fall; $\ddot{r}$ is no longer zero.

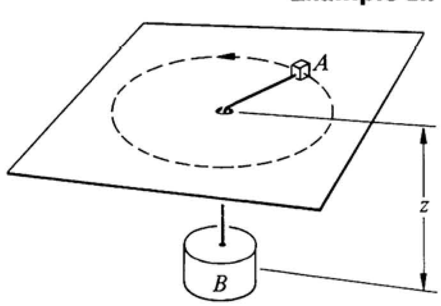
The tangential acceleration is given by Eq. (2). Since $\dot{r} = 0$ we have

$$a_\theta = R\ddot{\theta}$$
$$= -\frac{W\cos\theta}{m}.$$

The mass does not move with constant speed; it accelerates tangentially. On the downswing the tangential speed increases, on the upswing it decreases.

The next example involves rotational motion, translational motion, and constraints.

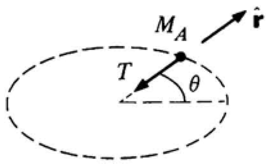
Example 2.7 **The Whirling Block**



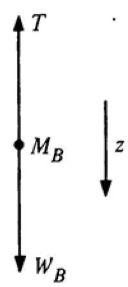
A horizontal frictionless table has a small hole in its center. Block A on the table is connected to block B hanging beneath by a string of negligible mass which passes through the hole.

Initially, B is held stationary and A rotates at constant radius r_0 with steady angular velocity ω_0. If B is released at $t = 0$, what is its acceleration immediately afterward?

The force diagrams for A and B after the moment of release are shown in the sketches.



The vertical forces acting on A are in balance and we need not consider them. The only horizontal force acting on A is the string force T. The forces on B are the string force T and the weight W_B.



It is natural to use polar coordinates r, θ for A, and a single linear coordinate z for B, as shown in the force diagrams. As usual, the unit vector $\hat{\mathbf{r}}$ is radially outward. The equations of motion are

$$-T = M_A(\ddot{r} - r\dot{\theta}^2) \qquad \text{Radial} \tag{1}$$

$$0 = M_A(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad \text{Tangential} \tag{2}$$

$$W_B - T = M_B\ddot{z} \qquad \text{Vertical.} \tag{3}$$

Since the length of the string, l, is constant, we have

$$r + z = l. \tag{4}$$

Differentiating Eq. (4) twice with respect to time gives us the constraint equation

$$\ddot{r} = -\ddot{z}. \tag{5}$$

The negative sign means that if A moves inward, B falls. Combining Eqs. (1), (3), and (5), we find

$$\ddot{z} = \frac{W_B - M_A r\dot{\theta}^2}{M_A + M_B}.$$

It is important to realize that although acceleration can change instantaneously, velocity and position cannot. Thus immediately after B is released, $r = r_0$ and $\dot{\theta} = \omega_0$. Hence

$$\ddot{z}(0) = \frac{W_B - M_A r_0 \omega_0{}^2}{M_A + M_B}. \tag{6}$$

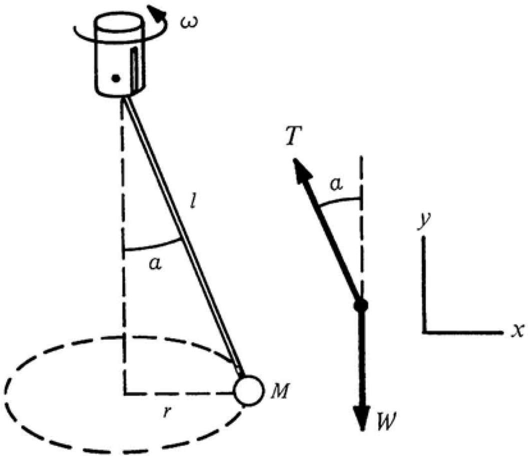
$z(0)$ can be positive, negative, or zero depending on the value of the numerator in Eq. (6); if ω_0 is large enough, block B will begin to rise after release.

The apparently simple problem in the next example has some unexpected subtleties.

Example 2.8 The Conical Pendulum

Mass M hangs by a massless rod of length l which rotates at constant angular frequency ω, as shown in the drawing on the next page. The mass moves with steady speed in a circular path of constant radius. Find α, the angle the string makes with the vertical.

We start with the force diagram. T is the string force and W is the weight of the bob. (Note that there are no other forces on the bob. If this is not clear, you are most likely confusing an acceleration with a



force—a serious error.) The vertical equation of motion is

$$T \cos \alpha - W = 0$$

because y is constant and $\ddot{y}$ is therefore zero.

To find the horizontal equation of motion note that the bob is accelerating in the $\hat{\mathbf{r}}$ direction at rate $a_r = -\omega^2 r$. Then

$$-T \sin \alpha = -Mr\omega^2. \tag{2}$$

Since $r = l \sin \alpha$ we have

$$T \sin \alpha = Ml\omega^2 \sin \alpha \tag{3}$$

or

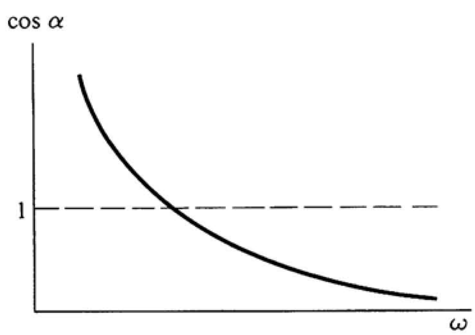
$$T = Ml\omega^2. \tag{4}$$

Combining Eqs. (1) and (3) gives

$$Ml\omega^2 \cos \alpha = W.$$

As we shall discuss in Sec. 2.5, $W = Mg$, where M is the mass and g is known as the acceleration due to gravity. We obtain

$$\cos \alpha = \frac{g}{l\omega^2}.$$

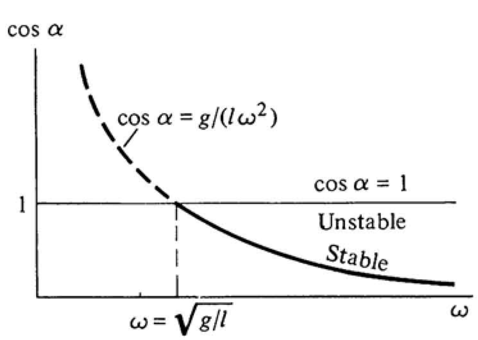


This appears to be the desired solution. For $\omega \to \infty$, $\cos \alpha \to 0$ and $\alpha \to \pi/2$. At high speeds the bob flies out until it is almost horizontal. However, at low speeds the solution does not make sense. As $\omega \to 0$, our solution predicts $\cos \alpha \to \infty$, which is nonsense since $\cos \alpha \leq 1$. Something has gone wrong. Here is the trouble.

Our solution predicts $\cos \alpha > 1$ for $\omega < \sqrt{g/l}$. When $\omega = \sqrt{g/l}$, $\cos \alpha = 1$ and $\sin \alpha = 0$; the bob simply hangs vertically. In going from Eq. (2) to Eq. (3) we divided both sides of Eq. (2) by $\sin \alpha$ and, in this case we divided by 0, which is not permissible. However, we see that we have overlooked a second possible solution, namely, $\sin \alpha = 0$, $T = W$, which is true for all values of ω. The solution corresponds to the pendulum hanging straight down. Here is a plot of the complete solution.

Physically, for $\omega \leq \sqrt{g/l}$ the only acceptable solution is $\alpha = 0$, $\cos \alpha = 1$. For $\omega > \sqrt{g/l}$ there are two acceptable solutions:

1. $\cos \alpha = 1$
2. $\cos \alpha = \dfrac{g}{l\omega^2}.$



Solution 1 corresponds to the bob rotating rapidly but hanging vertically. Solution 2 corresponds to the bob flying around at an angle with the vertical. For $\omega > \sqrt{g/l}$, solution 1 is unstable—if the system is in that state and is slightly perturbed, it will jump outward. Can you see why this is so?

The moral of this example is that you have to be sure that the mathematics makes good physical sense.

2.5 The Everyday Forces of Physics

When a physicist sets out to design an accelerator, he uses the laws of mechanics and his knowledge of electric and magnetic forces to determine the paths that the particles will follow. Predicting motion from known forces is an important part of physics and underlies most of its applications. Equally important, however, is the converse process of deducing the physical interaction by observing the motion; this is how new laws are discovered. A classic example is Newton's deduction of the law of gravitation from Kepler's laws of planetary motion. The current attempt to understand the interactions between elementary particles from high energy scattering experiments provides a more contemporary illustration.

Unscrambling experimental observations to find the force can be difficult. In a facetious mood, Eddington once said that force is the mathematical expression we put into the left hand side of Newton's second law to obtain results that agree with observed motions. Fortunately, force has a more concrete physical reality.

Much of our effort in the following chapters will be to learn how systems behave under applied forces. If every pair of particles in the universe had its own special interaction, the task would be impossible. Fortunately, nature is kinder than this. As far as we know, there are only four fundamentally different types of interactions in the universe: gravity, electromagnetic interactions, the so-called weak interaction, and the strong interaction.

Gravity and the electromagnetic interactions can act over a long range because they decrease only as the inverse square of the distance. However, the gravitational force always attracts, whereas electrical forces can either attract or repel. In large systems, electrical attraction and repulsion cancel to a high degree, and gravity alone is left. For this reason, gravitational forces dominate the cosmic scale of our universe. In contrast, the world immediately around us is dominated by the electrical forces, since they are far stronger than gravity on the atomic scale. Electrical forces are responsible for the structure of atoms, molecules, and more complex forms of matter, as well as the existence of light.

The weak and strong interactions have such short ranges that they are important only at nuclear distances, typically 10^{-15} m.

They are negligible even at atomic distances, 10^{-10} m. As its name implies, the strong interaction is very strong, much stronger than the electromagnetic force at nuclear distances. It is the "glue" that binds the atomic nucleus, but aside from this it has little effect in the everyday world. The weak interaction plays a less dramatic role; it mediates in the creation and destruction of neutrinos—particles of no mass and no charge which are essential to our understanding of matter but which can be detected only by the most arduous experiments.

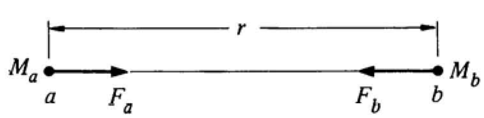
Our object in the remainder of the chapter is to become familiar with the forces which are important in everyday mechanics. Two of these, the forces of gravity and electricity, are fundamental and cannot be explained in simpler terms. The other forces we shall discuss, friction, the contact force, and the viscous force, can be understood as the macroscopic manifestation of interatomic forces.

Gravity, Weight, and the Gravitational Field

Gravity is the most familiar of the fundamental forces. It has close historical ties to the development of mechanics; Newton discovered the law of universal gravitation in 1666, the same year that he formulated his laws of motion. By calculating the motion of two gravitating particles, he was able to derive Kepler's empirical laws of planetary motion. (By accomplishing all this by age 26, Newton established a tradition which still maintains—that great advances are often made by young physicists.)

According to Newton's law of gravitation, two particles attract each other with a force directed along their line of centers. The magnitude of the force is proportional to the product of the masses and decreases as the inverse square of the distance between the particles.

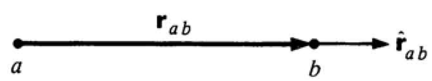
In verbal form the law is bulky and hard to use. However, we can reduce it to a simple mathematical expression.



Consider two particles, a and b, with masses M_a and M_b, respectively, separated by distance r. Let $\mathbf{F}_b$ be the force exerted on particle b by particle a. Our verbal description of the magnitude of the force is summarized by

$$|\mathbf{F}_b| = \frac{GM_aM_b}{r^2}.$$

G is a constant of proportionality called the *gravitational constant*. Its value is found by measuring the force between masses in a known geometry. The first measurements were performed by Henry Cavendish in 1771 using a torsion balance. The modern value of G is 6.67×10^{-11} N·m²/kg². (G is the least accurately known of the fundamental constants. Perhaps you can devise a new way to measure it more precisely.) Experimentally, G is the same for all materials—aluminum, lead, neutrons, or what have you. For this reason, the law is called the universal law of gravitation.



The gravitational force between two particles is *central* (along the line of centers) and attractive. The simplest way to describe these properties is to use vectors. By convention, we introduce a vector $\mathbf{r}_{ab}$ *from* the particle exerting the force, particle a in this case, *to* the particle experiencing the force, particle b. Note that $|\mathbf{r}_{ab}| = r$. Using the unit vector $\hat{\mathbf{r}}_{ab} = \mathbf{r}_{ab}/r$, we have

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}.$$

The negative sign indicates that the force is attractive. The force on a due to b is

$$\mathbf{F}_a = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ba} = +\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab} = -\mathbf{F}_b,$$

since $\hat{\mathbf{r}}_{ba} = -\hat{\mathbf{r}}_{ab}$. The forces are equal and opposite, and Newton's third law is automatically satisfied.

The gravitational force has a unique and mysterious property. Consider the equation of motion of particle b under the gravitational attraction of particle a.

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}$$

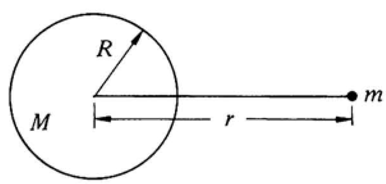
$$= M_b\mathbf{a}_b$$

or

$$\mathbf{a}_b = -\frac{GM_a}{r^2}\hat{\mathbf{r}}_{ab}.$$

The acceleration of a particle under gravity is independent of its mass! There is a subtle point connected with our cancelation of M_b, however. The "mass" (*gravitational* mass) in the law of gravitation, which measures the strength of gravitational interaction, is operationally distinct from the "mass" (*inertial* mass) which characterizes inertia in Newton's second law. Why gravitational mass is proportional to inertial mass for all matter is one of the great mysteries of physics. However, the proportionality has been experimentally verified to very high accuracy, approximately 1 part in 10^{11}; we shall have more to say about this in Chap. 8.

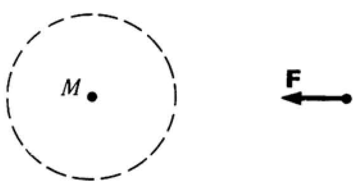
The Gravitational Force of a Sphere The law of gravitation applies only to particles. How can we find the gravitational force on a particle due to an extended body like the earth? Fortunately, the gravitational force obeys the *law of superposition:* the force due to a collection of particles is the vector sum of the forces exerted by the particles individually. This allows us to mentally divide the body into a collection of small elements which can be treated as particles. Using integral calculus, we can sum the forces from all the particles. This method is applied in Note 2.1 to calculate the force between a particle of mass m and a uniform thin spherical shell of mass M and radius R. The result is



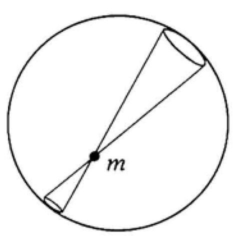
$$\mathbf{F} = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \qquad r > R$$

$$\mathbf{F} = 0 \qquad r < R,$$

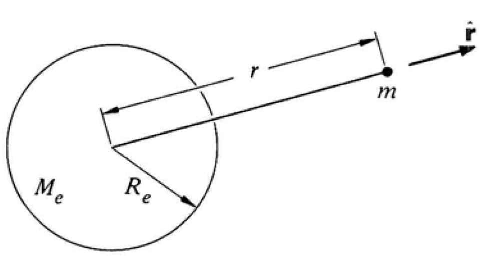
where r is the distance from the center of the shell to the particle. If the particle lies outside the shell, the force is the same as if all the mass of the shell were concentrated at its center.



The reason the gravitational force vanishes inside the spherical shell can be seen by a simple argument due to Newton. Consider the two small mass elements marked out by a conical surface with its apex at m. The amount of mass in each element is proportional to its surface area. The area increases as (distance)^2. However, the strength of the force varies as $1/(\text{distance})^2$. Thus the forces of the two mass elements are equal and opposite, and cancel. The total force on m is zero, because we can pair up all the elements of the shell this way.



A uniform solid sphere can be regarded as a succession of thin spherical shells, and it follows that for particles outside it, a sphere behaves gravitationally as if its mass were concentrated at its center. This result also holds if the density of the sphere varies with radius, provided the mass distribution is spherically symmetric. For example, although the earth has a dense core, the mass distribution is nearly spherically symmetric, so that to good approximation the gravitational force of the earth on a mass m at distance r is



$$\mathbf{F} = -\frac{GM_em}{r^2}\hat{\mathbf{r}} \qquad r \geq R_e,$$

where M_e is the mass of the earth and R_e is its radius.

At the surface of the earth, the gravitational force is

$$\mathbf{F} = -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}},$$

and the acceleration due to gravity is

$$\mathbf{a} = \frac{\mathbf{F}}{m}$$
$$= -\frac{GM_e}{R_e^2}\hat{\mathbf{r}}.$$

As we expect, the acceleration is independent of m. GM_e/R_e^2 is usually called g. Sometimes g is written as a vector directed down. toward the center of the earth.

$$\mathbf{g} = -\frac{GM_e}{R_e^2}\hat{\mathbf{r}}$$

Numerically, $|g|$ is approximately 9.8 m/s^2 = 980 cm/s^2 $\approx$ 32 ft/s^2.

By convention, g usually stands for the downward acceleration of an object measured with respect to the earth's surface. This differs slightly from the true gravitational acceleration because of the rotation of the earth, a point we shall return to in Chap. 8. g increases by about five parts per thousand from the equator to the poles. About half this variation is due to the slight flattening of the earth about the poles, and the remainder arises from the earth's rotation. Local mass concentrations also affect g; a variation in g of ten parts per million is typical.

The acceleration due to gravity decreases with altitude. We can estimate this effect by taking differentials of the expression

$$g(r) = \frac{GM_e}{r^2}.$$

We have

$$\Delta g(r) = \frac{dg}{dr}\Delta r = -\frac{2GM_e}{r^3}\Delta r$$
$$= -\frac{2g}{r}\Delta r.$$

The fractional change in g with altitude is

$$\frac{\Delta g}{g} = -\frac{2\,\Delta r}{r}.$$

At the earth's surface, $r = 6 \times 10^6$ m, and g decreases by one part per million for an increase in altitude of 3 m.

Weight We define the weight of a body near the earth to be the gravitational force exerted on it by the earth. At the surface of the earth the weight of a mass m is

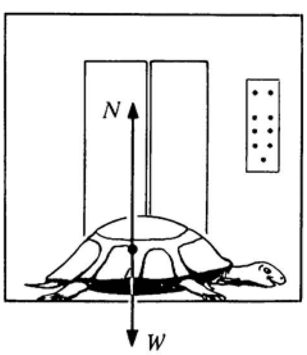
$$\begin{aligned}\mathbf{W} &= -G\frac{M_e m}{R_e^2}\hat{\mathbf{r}} \\ &= m\mathbf{g}.\end{aligned}$$

The unit of weight is the newton (SI), dyne (cgs), or pound (English). Since $g = 9.8\ \text{m/s}^2$, the weight of 1 kg mass is 9.8 N. An automobile which weighs 3,200 lb has mass

$$m = \frac{W}{g} = \frac{3{,}200\ \text{lb}}{32\ \text{ft/s}^2} = 100\ \text{slugs}.$$

Our definition of weight is unambiguous. According to our definition, the weight of a body is not affected by its motion. However, weight is often used in another sense. In this sense, the magnitude of the weight is the magnitude of the force which must be exerted on a body by its surroundings to keep it at rest; its direction is the direction of gravitational attraction. The next example illustrates the difference between these two definitions.

Example 2.9 Turtle in an Elevator



An amiable turtle of mass M stands in an elevator accelerating at rate a Find N, the force exerted on him by the floor of the elevator.

The forces acting on the turtle are N and the weight, the true gravitational force $W = Mg$. Taking up to be the positive direction, we have

$$\begin{aligned}N - W &= Ma \\ N &= Mg + Ma \\ &= M(g + a).\end{aligned}$$

This result illustrates the two senses in which weight is used. In the sense that weight is the gravitational force, the weight of the turtle, Mg, is independent of the motion of the elevator. In contrast, the weight of the turtle has magnitude $N = M(g + a)$, if the magnitude of the weight is taken to be the magnitude of the force exerted by the elevator on the turtle. If the turtle were standing on a scale, the scale would indicate a weight N. With this definition, the turtle's weight increases when the elevator accelerates up.

If the elevator accelerates down, a is negative and N is less than Mg. If the downward acceleration equals g, N becomes zero, and the turtle "floats" in the elevator. The turtle is then said to be in a state of weightlessness.

Although the two definitions of weight are both commonly used and are both acceptable, we shall generally consider weight to mean the true gravitational force. This is consistent with our resolve to refer all motion to inertial systems and helps us to keep the real forces on a body distinct. If the acceleration due to gravity is g, the real gravitational force on a body of mass m is $W = mg$.

Our definition of weight has one minor drawback. As we saw in the last example, a scale does not read mg in an accelerating system. As we have already pointed out, systems at rest on the earth's surface have a small acceleration due to the earth's rotation, so that the reading of a scale is not the true gravitational force on a mass. However, the effect is small, and we shall treat the surface of the earth as an inertial system for the present.

The Gravitational Field The gravitational force on particle b due to particle a is

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab},$$

where $\hat{\mathbf{r}}_{ab}$ is a unit vector which points from a toward b. The ratio $\mathbf{F}_b/M_b$, which is independent of M_b, is called the *gravitational field* due to M_a. Denoting the field by $\boldsymbol{G}_a$, we have

$$\begin{aligned}\boldsymbol{G}_a &= \frac{\mathbf{F}_b}{M_b}\\ &= -G\frac{M_a}{r^2}\hat{\mathbf{r}}_{ab}.\end{aligned}$$

In general, if the gravitational field at a point in space is $\boldsymbol{G}$, the gravitational force on mass M at that point is

$$\mathbf{F} = M\boldsymbol{G}.$$

The dimension of gravitation field is force/mass = acceleration. The acceleration of mass M by gravitational field $\boldsymbol{G}$ is given by

$$\begin{aligned}\mathbf{F} &= M\mathbf{a}\\ &= M\boldsymbol{G}\end{aligned}$$

or

$$\mathbf{a} = \boldsymbol{G}.$$

We see that the gravitational field at a point is numerically equal to the gravitational acceleration experienced by a body located there. For example, the gravitational field of the earth is **g**.

For the present we can regard the gravitational field as a mathematical convenience that allows us to focus on the source of the gravitational attraction. However, the concept of field has a broader significance in physics. Fields have important physical properties, such as the ability to store or transmit energy and momentum. Until recently, the dynamical properties of the gravitational field were chiefly of theoretical interest, since their effects were too small to be observed. However, there is now lively experimental activity in searching for such dynamical features as gravitational waves and "black holes."

The Electrostatic Force

We mention the electrostatic force only in passing since its full implications are better left to a more detailed study of electricity and magnetism. The salient feature of the electrostatic force between two particles is that the force, like gravity, is an inverse square central force. The force depends upon a fundamental property of the particle called its *electric charge* q. There are two different kinds of electric charge: like charges repel, unlike charges attract.

For the sake of convenience, we distinguish the two different kinds of charges by associating an algebraic sign with q, and for this reason we talk about negative and positive charges. The electrostatic force F_b on charge q_b due to charge q_a is given by Coulomb's law:

$$\mathbf{F}_b = k\,\frac{q_a q_b}{r^2}\,\hat{\mathbf{r}}_{ab}.$$

k is a constant of proportionality and $\hat{\mathbf{r}}_{ab}$ is a unit vector which points from a to b. If q_a and q_b are both negative or both positive, the force is repulsive, but if the charges are of different sign, $\mathbf{F}_b$ is attractive.

In the SI system, the unit of charge is the *coulomb*, abbreviated C. (The coulomb is defined in terms of electric currents and magnetic forces.) In this system, k is found by experiment to be

$$k = 8.99 \times 10^9\ \mathrm{N{\cdot}m^2/C^2}.$$

In analogy with the gravitational field, we can define the electric field **E** as the electric force on a body divided by its charge. The electric field at **r** due to a charge q at the origin is

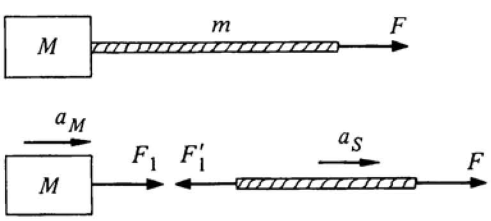
$$\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}}.$$

Contact Forces

By contact forces we mean the forces which are transmitted between bodies by short-range atomic or molecular interactions. Examples include the pull of a string, the surface force of sliding friction, and the force of viscosity between a moving body and a fluid. One of the achievements of twentieth century physics is that these forces can now be explained in terms of the fundamental properties of matter. However, our approach will emphasize the empirical properties of these forces and the techniques for dealing with them in physical problems, with only brief mention of their microscopic origins.

Tension—The Force of a String We have been taking the "string" force for granted, having some primitive idea of this kind of force. The following example is intended to help put these ideas into sharper focus.

Example 2.10 **Block and String 3**



Consider a block of mass M in free space pulled by a string of mass m. A force F is applied to the string, as shown. What is the force that the string "transmits" to the block?

The sketch shows the force diagrams. F_1 is the force of the string on the block, F_1' is the force of the block on the string, a_M is the acceleration of the block, and a_S is the acceleration of the string. The equations of motion are

$$F_1 = Ma_M$$
$$F - F_1' = ma_S.$$

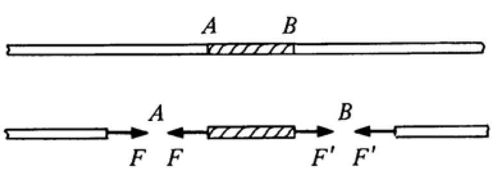
Assuming that the string is inextensible, it accelerates at the same rate as the block, giving the constraint equation $a_S = a_M$. Furthermore, $F_1 = F_1'$ by Newton's third law. Solving for the acceleration, we find that

$$a = \frac{F}{M+m},$$

as we expect, and

$$F_1 = F_1'$$
$$= \frac{M}{M+m}F.$$

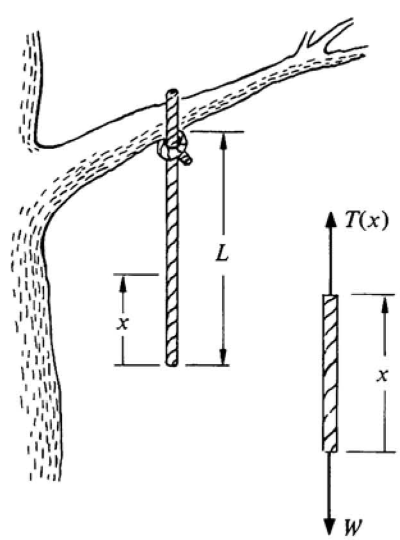
The force on the block is less than F; the string does not transmit the full applied force. However, if the mass of the string is negligible compared with the block, $F_1 = F$ to good approximation.



We can think of a string as composed of short sections interacting by contact forces. Each section pulls the sections to either side of it, and by Newton's third law, it is pulled by the adjacent sections. The magnitude of the force acting between adjacent sections is called *tension*. There is no direction associated with tension. In the sketch, the tension at A is F and the tension at B is F'.

Although a string may be under considerable tension (for example a string on a guitar), if the tension is uniform, the net string force on each small section is zero and the section remains at rest unless external forces act on it. If there are external forces on the section, or if the string is accelerating, the tension generally varies along the string, as Examples 2.11 and 2.12 show.

Example 2.11 **Dangling Rope**



A uniform rope of mass M and length L hangs from the limb of a tree. Find the tension a distance x from the bottom.

The force diagram for the lower section of the rope is shown in the sketch. The section is pulled up by a force of magnitude $T(x)$, where $T(x)$ is the tension at x. The downward force on the rope is its weight $W = Mg(x/L)$. The total force on the section is zero since it is at rest. Hence

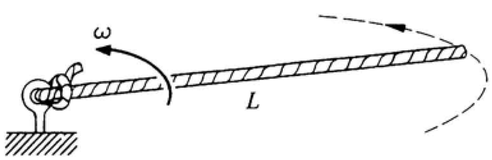
$$T(x) = \frac{Mg}{L}x.$$

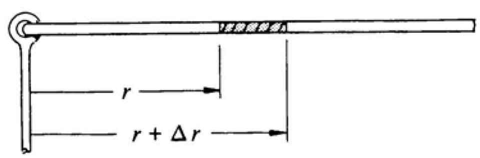
At the bottom of the rope the tension is zero, while at the top the tension equals the total weight of the rope Mg.

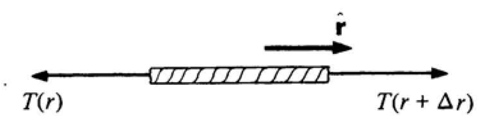
The next example cannot be solved by direct application of Newton's second law. However, by treating each small section of the system as a particle, and taking the limit using calculus, we can obtain a differential equation which leads to the solution.

The technique is so useful that it is employed time and again in physics.

Example 2.12 **Whirling Rope**







A uniform rope of mass M and length L is pivoted at one end and whirls with uniform angular velocity ω. What is the tension in the rope at distance r from the pivot? Neglect gravity.

Consider the small section of rope between r and $r + \Delta r$. The length of the section is Δr and its mass is $\Delta m = M\,\Delta r/L$. Because of its circular motion, the section has a radial acceleration. Therefore, the forces pulling either end of the section cannot be equal, and we conclude that the tension must vary with r.

The inward force on the section is $T(r)$, the tension at r, and the outward force is $T(r + \Delta r)$. Treating the section as a particle, its inward radial acceleration is $r\omega^2$. [This point can be confusing; it is just as reasonable to take the acceleration to be $(r + \Delta r)\omega^2$. However, we shall shortly take the limit $\Delta r \to 0$, and in this limit the two expressions give the same result.]

The equation of motion for the section is

$$\begin{aligned} T(r + \Delta r) - T(r) &= -(\Delta m) r\omega^2 \\ &= -\frac{Mr\omega^2\,\Delta r}{L}. \end{aligned}$$

The problem is to find $T(r)$, but we are not yet ready to do this. However, by dividing the last equation by Δr and taking the limit $\Delta r \to 0$, we can find an exact expression for dT/dr.

$$\begin{aligned} \frac{dT}{dr} &= \lim_{\Delta r \to 0} \frac{T(r + \Delta r) - T(r)}{\Delta r} \\ &= -\frac{Mr\omega^2}{L} \end{aligned}$$

To find the tension, we integrate.

$$dT = -\frac{M\omega^2}{L}\, r\, dr$$

$$\int_{T_0}^{T(r)} dT = -\int_0^r \frac{M\omega^2}{L}\, r\, dr,$$

where T_0 is the tension at $r = 0$.

$$T(r) - T_0 = -\frac{M\omega^2}{L}\frac{r^2}{2}$$

or

$$T(r) = T_0 - \frac{M\omega^2}{2L} r^2.$$

To evaluate T_0 we need one additional piece of information. Since the end of the rope at $r = L$ is free, the tension there must be zero. We have

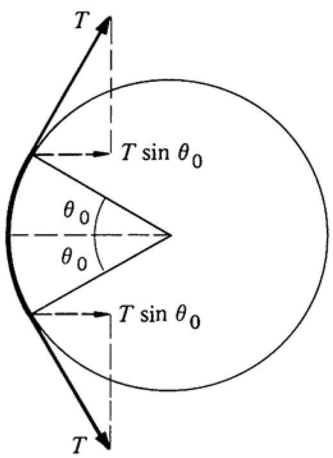
$$T(L) = 0 = T_0 - \tfrac{1}{2}M\omega^2 L.$$

Hence, $T_0 = \frac{1}{2}M\omega^2 L$, and the final result can be written

$$T(r) = \frac{M\omega^2}{2L}(L^2 - r^2).$$

When a pulley is used to change the direction of a rope under tension, there is a reaction force on the pulley. As every sailor knows, the force on the pulley depends on the tension and the angle through which the rope is deflected. Working out this problem in detail provides another illustration of how calculus can be applied to a physical problem.

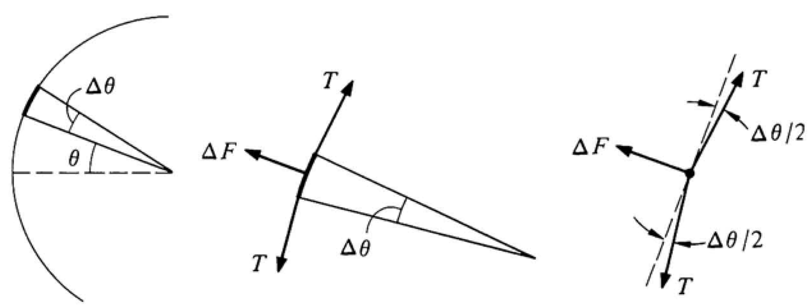
Example 2.13 Pulleys



A string with constant tension T is deflected through angle $2\theta_0$ by a smooth fixed pulley. What is the force on the pulley?

Intuitively, the magnitude of the force is $2T \sin \theta_0$. To prove this result, we shall find the force due to each element of the string and then add them vectorially.

Consider the section of string between θ and $\theta + \Delta\theta$. The force diagram is drawn below, center. ΔF is the outward force due to the pulley

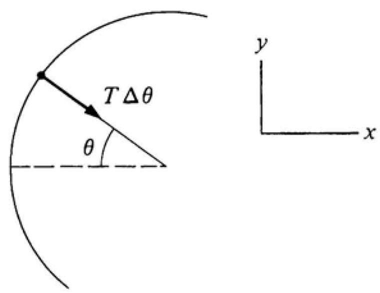


The tension in the string is constant, but the forces T at either end of the element are not parallel. Since we shall shortly take the limit $\Delta\theta \to 0$, we can treat the element like a particle. For equilibrium, the total force is zero. We have

$$\Delta F - 2T \sin \frac{\Delta\theta}{2} = 0.$$

For small $\Delta\theta$, $\sin(\Delta\theta/2) \approx \Delta\theta/2$ and

$$\Delta F = 2T \frac{\Delta\theta}{2} = T\Delta\theta.$$

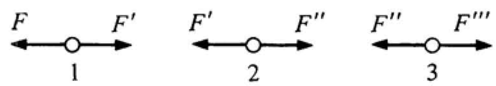


Thus the element exerts an inward radial force of magnitude $T\,\Delta\theta$ on the pulley.

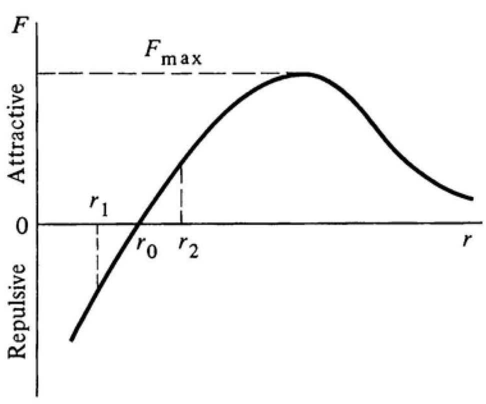
The element at angle θ exerts a force in the x direction of $(T\,\Delta\theta)\cos\theta$. The total force in the x direction is $\Sigma T\cos\theta\,\Delta\theta$, where the sum is over all elements of the string which are touching the pulley. In the limit $\Delta\theta \rightarrow 0$, the sum becomes an integral. The total force in the x direction is therefore

$$\int_{-\theta_0}^{\theta_0} T\cos\theta\,d\theta = 2T\sin\theta_0.$$

Tension and Atomic Forces The force on each element of a string in equilibrium is zero. Nevertheless, the string will break if the tension is too large. We can understand this qualitatively by looking at strings from the atomic viewpoint. An idealized model of a string is a single long chain of molecules. Suppose that force F is applied to molecule 1 at the end of the string. The force diagrams for molecules 1 and 2 are shown in the sketch below. In



equilibrium, $F = F'$ and $F' = F''$, so that $F'' = F$. We see that the string "transmits" the force F. To understand how this comes about, we need to look at the nature of intermolecular forces.



Qualitatively, the force between two molecules depends on the distance r between them, as shown in the drawing. The intermolecular force is repulsive at small distances, is zero at some separation r_0, and is attractive for $r > r_0$. For large values of r the force falls to zero. There are no scales on our sketch, but r_0 is typically a few angstroms (1 Å $= 10^{-10}$ m).

When there is no applied force, the molecules must be a distance r_0 apart; otherwise the intermolecular forces would make the string contract or expand. As we pull on the string, the molecules move apart slightly, say to $r = r_2$, where the intermolecular attractive force just balances the applied force so that the total force on each molecule is zero. If the string were stiff like a metal rod, we could push as well as pull. A push makes the molecules move slightly together, say to $r = r_1$, where the intermolecular repulsive force balances the applied force. The change in the length depends on the slope of the interatomic force curve at r_0. The steeper the curve, the less the stretch for a given pull.

The attractive intermolecular force has a maximum value $F_{\max}$, as shown in the sketch. If the applied pull is greater than $F_{\max}$, the intermolecular force is too weak to restore balance—the molecules continue to separate and the string breaks.

For a real string or rod, the intermolecular forces act in a three dimensional lattice work of atoms. The breaking strength of most materials is considerably less than the limit set by F_{max}. Breaks occur at points of weakness, or "defects," in the lattice, where the molecular arrangement departs from regularity. Microscopic metal "whiskers" seem to be nearly free from defects, and they exhibit breaking strengths close to the theoretical maximum.

The Normal Force The force exerted by a surface on a body in contact with it can be resolved into two components, one perpendicular to the surface and one tangential to the surface. The perpendicular component is called the *normal* force and the tangential component is called *friction.*

The origin of the normal force is similar to the origin of tension in a string. When we put a book on a table, the molecules of the book exert downward forces on the molecules of the table. The molecules composing the upper layers of the tabletop move downward until the repulsion of the molecules below balances the force applied by the book. From the atomic point of view, no surface is perfectly rigid. Although compression always occurs, it is often too slight to notice, and we shall neglect it and treat surfaces as rigid.

The normal force on a body, generally denoted by N, has the following simple property: for a body resting on a surface, N is equal and opposite to the resultant of all other forces which act on the body in a direction perpendicular to the surface. For instance, when you stand still, the normal force exerted by the ground is equal to your weight. However, when you walk, the normal force fluctuates as you accelerate up and down.

Friction Friction cannot be described by a simple formula, but macroscopic mechanics is hard to understand without some idea of the properties of friction.

Friction arises when the surface of one body moves, or tries to move, along the surface of a second body. The magnitude of the force of friction varies in a complicated way with the nature of the surfaces and their relative velocity. In fact, the only thing we can always say about friction is that it opposes the motion which would occur in its absence. For instance, suppose that we try to push a book across a table. If we push gently, the book remains at rest; the force of friction assumes a value equal and opposite to the tangential force we apply. In this case, the force of friction assumes whatever value is needed to keep the book at rest. However, the friction force cannot increase indefinitely. If we push hard enough, the book starts to slide. For many surfaces the maximum value of the friction is found to be equal to μN, where N is the normal force and μ is the *coefficient of friction.*

When a body slides across a surface, the friction force is directed opposite to the instantaneous velocity and has magnitude μN. Experimentally, the force of sliding friction decreases slightly when bodies begin to slide, but for the most part we shall neglect this effect. For two given surfaces the force of sliding friction is essentially independent of the area of contact.

It may seem strange that friction is independent of the area of contact. The reason is that the actual area of contact on an atomic scale is a minute fraction of the total surface area. Friction occurs because of the interatomic forces at these minute regions of atomic contact. The fraction of the geometric area in atomic contact is proportional to the normal force divided by the geometric area. If the normal force is doubled, the area of atomic contact is doubled and the friction force is twice as large. However, if the geometric area is doubled while the normal force remains the same, the fraction of area in atomic contact is halved and the actual area in atomic contact—hence the friction force—remains constant. (Nonrigid bodies, like automobile tires, are more complicated. A wide tire is generally better than a narrow one for good acceleration and braking.)

In summary, we take the force of friction f to behave as follows:

1. For bodies not in relative motion,

$$0 \le f \le \mu N.$$

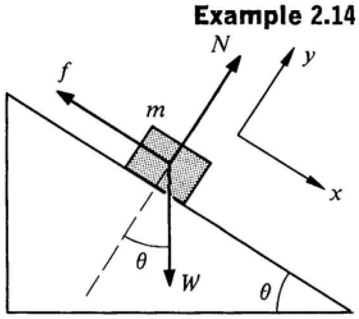
f opposes the motion that would occur in its absence.

2. For bodies in relative motion,

$$f = \mu N.$$

f is directed opposite to the relative velocity.

Example 2.14 **Block and Wedge with Friction**



A block of mass m rests on a fixed wedge of angle θ. The coefficient of friction is μ. (For wooden blocks, μ is of the order of 0.2 to 0.5.) Find the value of θ at which the block starts to slide.

In the absence of friction, the block would slide down the plane; hence the friction force f points up the plane. With the coordinates shown, we have

$$m\ddot{x} = W \sin\theta - f$$

and

$$m\ddot{y} = N - W \cos \theta$$
$$= 0.$$

When sliding starts, f has its maximum value μN, and $\ddot{x} = 0$. The equations then give

$$W \sin \theta_{\max} = \mu N$$
$$W \cos \theta_{\max} = N.$$

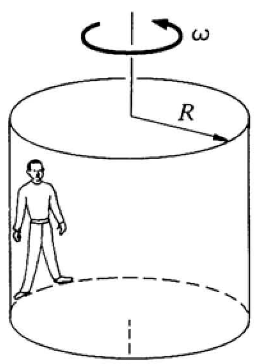
Hence,

$$\tan \theta_{\max} = \mu.$$

Notice that as the wedge angle is gradually increased from zero, the friction force grows in magnitude from zero toward its maximum value μN, since before the block begins to slide we have

$$f = W \sin \theta \qquad \theta \leq \theta_{\max}.$$

Example 2.15 **The Spinning Terror**

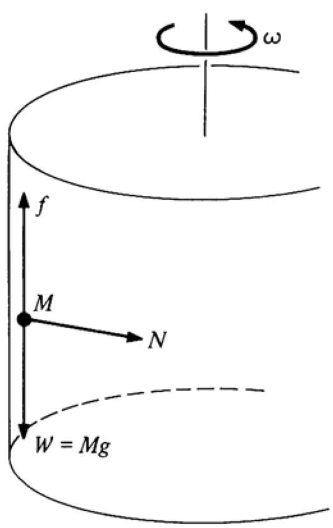


The Spinning Terror is an amusement park ride—a large vertical drum which spins so fast that everyone inside stays pinned against the wall when the floor drops away. What is the minimum steady angular velocity ω which allows the floor to be dropped away safely?

Suppose that the radius of the drum is R and the mass of the body is M. Let μ be the coefficient of friction between the drum and M. The forces on M are the weight W, the friction force f, and the normal force exerted by the wall, N, as shown below.

The radial acceleration is $R\omega^2$ toward the axis, and the radial equation of motion is

$$N = MR\omega^2.$$



By the law of static friction,

$$f \leq \mu N = \mu MR\omega^2.$$

Since we require M to be in vertical equilibrium,

$$f = Mg,$$

and we have

$$Mg \leq \mu MR\omega^2$$

or

$$\omega^2 \geq \frac{g}{\mu R}.$$

The smallest value of ω that will work is

$$\omega_{\min} = \sqrt{\frac{g}{\mu R}}.$$

For cloth on wood μ is at least 0.3, and if the drum has radius 6 ft, then $\omega_{\min} = [32/(0.3 \times 6)]^{\frac{1}{2}} = 4$ rad/s. The drum must make at least $\omega/2\pi =$ 0.6 turns per second.

Viscosity

A body moving through a liquid or gas is retarded by the force of viscosity exerted on it by the fluid. Unlike the friction force between dry surfaces, the viscous force has a simple velocity dependence; it is proportional to the velocity. At high speeds other forces due to turbulence occur and the total drag force can have a complicated velocity dependence. (Sports car designers use a force proportional to the square of the speed to account for the drag forces.) However, in many practical cases viscosity is the only important drag force.

Viscosity arises because a body moving through a medium exerts forces which set the nearby fluid into motion. By Newton's third law the fluid exerts a reaction force on the body.

We can write the viscous retarding force in the form

$$\mathbf{F}_v = -C\mathbf{v},$$

where C is a constant which depends on the fluid and the geometry of the body. $\mathbf{F}_v$ is always along the line of motion, because it is proportional to $\mathbf{v}$. The negative sign assures that $\mathbf{F}_v$ opposes the motion. For objects of simple shape moving through a gas at low pressure, C can be calculated from first principles. We shall treat it as an empirical constant.

When the only force on a body is the viscous retarding force, the equation of motion is

$$-C\mathbf{v} = m\frac{d\mathbf{v}}{dt}.$$

What we have here is a differential equation for $\mathbf{v}$. Since the force is along the line of motion, only the magnitude of $\mathbf{v}$ changes[1]

[1] Formally, this is proved as follows. Since $\mathbf{v} = v\hat{\mathbf{v}}$, $d\mathbf{v}/dt = dv/dt\,\hat{\mathbf{v}} + v\,d\hat{\mathbf{v}}/dt$. The equation of motion is $-Cv\hat{\mathbf{v}} = m\,dv/dt\,\hat{\mathbf{v}} + mv\,d\hat{\mathbf{v}}/dt$. Because $\hat{\mathbf{v}}$ is a unit vector, $d\hat{\mathbf{v}}/dt$ is perpendicular to $\hat{\mathbf{v}}$. The other terms of the equation lie in the $\hat{\mathbf{v}}$ direction, so that $d\hat{\mathbf{v}}/dt$ must be zero. The same conclusion follows more directly from the simple physical argument that a force directed along the line of motion can change the speed but cannot change the direction of motion.

and the vector equation reduces to the scalar equation

$$-Cv = m\frac{dv}{dt}$$

or

$$m\frac{dv}{dt} + Cv = 0.$$

The task of solving such a differential equation occurs often in physics. A few differential equations are so simple and occur so frequently that it is helpful to be thoroughly familiar with them and their solutions. The equation of the form $m\,dv/dt + Cv = 0$ is one of the most common, and the following example should make you feel at home with it.

Example 2.16 **Free Motion in a Viscous Medium**

A body of mass m released with velocity v_0 in a viscous fluid is retarded by a force Cv. Find the motion, supposing that no other forces act.

The equation of motion is

$$m\frac{dv}{dt} + Cv = 0,$$

which we can rewrite in the standard form

$$\frac{dv}{dt} + \frac{C}{m}v = 0. \qquad 1$$

If you are familiar with the properties of the exponential function e^{ax}, then you know that $(d/dx)e^{ax} = ae^{ax}$, or $(d/dx)e^{ax} - ae^{ax} = 0$. This suggests that we use a trial solution $v = e^{at}$, where a is a constant to be determined. Then $dv/dt = ae^{at}$, and substituting this in Eq. (1) gives us

$$ae^{at} + \frac{C}{m}e^{at} = 0.$$

This holds true at all times if $a = -C/m$. Hence, a solution is

$$v = e^{-Ct/m}.$$

However, this cannot be the correct solution; v has the dimension of velocity whereas the exponential function is dimensionless. Let us try

$$v = Ae^{-Ct/m},$$

where A is a constant. Substituting this in Eq. (1) gives

$$-\frac{C}{m}Ae^{-Ct/m} + \frac{C}{m}Ae^{-Ct/m} = 0,$$

so that the solution is acceptable. But A can be any constant, whereas our solution must be quite specific. To evaluate A we make use of the given initial condition. An *initial condition* is a specific piece of information about the motion at some particular time. We were given that $v = v_0$ at $t = 0$. Hence

$$v(t = 0) = Ae^0 = v_0.$$

Since $e^0 = 1$, it follows that $A = v_0$, and the full solution is

$$v = v_0 e^{-Ct/m}.$$

We solved Eq. (1) by what might be called a common sense approach—we simply guessed the answer. This particular equation can also be solved by formal integration after appropriate "separation of the variables."

$$\frac{dv}{dt} + \frac{C}{m}v = 0$$

$$\frac{dv}{v} = -\frac{C}{m}dt$$

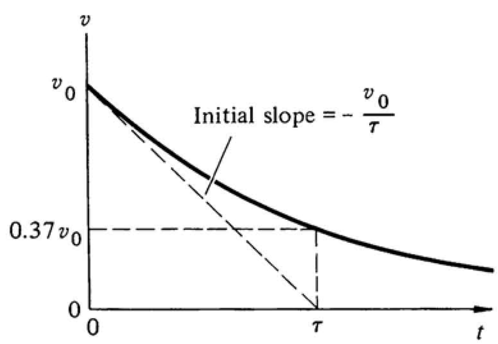
$$\int_{v_0}^{v} \frac{dv}{v} = -\int_0^t \frac{C}{m}dt$$

Note the correspondence between the limits: v is the velocity at time t and v_0 is the velocity at time 0.

$$\ln\frac{v}{v_0} = -\frac{C}{m}t$$

$$\frac{v}{v_0} = e^{(-C/m)t}$$

$$v = v_0 e^{-Ct/m}.$$



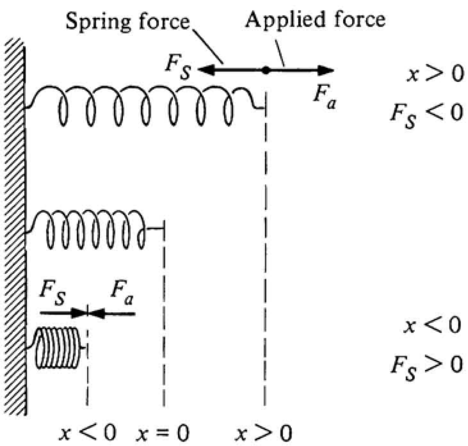
Before leaving this problem, let us look at the solution in a little more detail. The velocity decreases exponentially in time. If we let $\tau = m/c$, then we have $v = v_0 e^{-t/\tau}$. τ is a *characteristic time* for the system; it is the time for the velocity to drop to $e^{-1} \approx 0.37$ of its original velocity.

The Linear Restoring Force: Hooke's Law, the Spring, and Simple Harmonic Motion

In the mid-seventeenth century Robert Hooke discovered that the extension of a spring is proportional to the applied force, both for positive and negative displacements. The force F_S exerted by a stretched spring is given by Hooke's law

$$F_S = -kx,$$

where k is a constant called the *spring constant* and x is the displacement of the end of the spring from its equilibrium position. The magnitude of F_S increases linearly with displacement. The



negative sign indicates that F_S is a restoring force; the spring force is always in the direction that tends to restore the spring to its equilibrium length. A force obeying Hooke's law is called a *linear restoring force.*

If the spring is stretched by an applied force F_a, then $x > 0$ and F_S is negative, directed toward the origin.

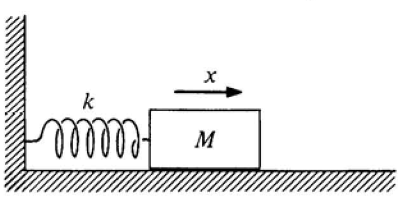
If the spring is compressed by F_a, then $x < 0$ and F_S is positive.

Hooke's law is essentially empirical and breaks down for large displacements. Taking a jaundiced view of affairs, we could rephrase Hooke's law as "extension is proportional to force, as long as it is." However, this misses the important point. For sufficiently small displacements Hooke's law is remarkably accurate, not only for springs but also for practically every system near equilibrium. Consequently, the motion of a system under a linear restoring force occurs persistently throughout physics.

By looking at the intermolecular force curve on page 91, we can see why the linear restoring force is so common. If the force curve is linear in the neighborhood of the equilibrium point, then the force is proportional to the displacement from equilibrium. This is almost always the case; a sufficiently short segment of a curve is generally linear to good approximation. Only in pathological cases does the force curve have no linear component. It is also apparent that the linear approximation necessarily breaks down for large displacements. We shall return to these considerations in Chap. 4.

In the following example we investigate simple harmonic motion—the motion of a mass under a linear restoring force. We shall again encounter a differential equation. Like the equation for viscous drag, the differential equation for simple harmonic motion occurs frequently and is well worth learning to recognize early in the game. Fortunately, the solution has a simple form.

Example 2.17 Spring and Block—The Equation for Simple Harmonic Motion



A block of mass M is attached to one end of a horizontal spring, the other end of which is fixed. The block rests on a horizontal frictionless surface. What motion is possible for the block?

Since the spring force is the only horizontal force acting on the block, the equation of motion is

$$M\ddot{x} = -kx$$

or

$$\ddot{x} + \frac{k}{M}x = 0,$$

where x is measured from the equilibrium position. It is convenient to write

$$\omega = \sqrt{\frac{k}{M}}.$$

The equation takes the standard form

$$\ddot{x} + \omega^2 x = 0.$$

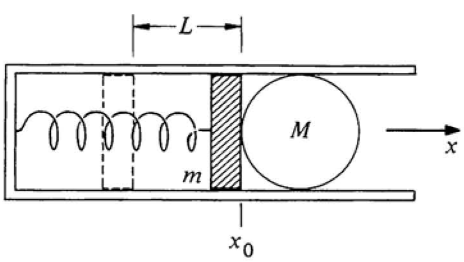
You should learn to recognize the mathematical form of this equation, since it arises in many different physical contexts. It is called the equation of *simple harmonic motion* (SHM). Without going into the theory of differential equations, we simply write down the solution

$$x = A \sin \omega t + B \cos \omega t.$$

ω is known as the *angular frequency* of the motion. By substitution it is easy to show that this solution satisfies the original equation for arbitrary values of A and B. The theory of differential equations tells us that there are no further nontrivial solutions. The main point here, however, is to become familiar with the mathematical form of the SHM differential equation and the form of its solution. We shall derive the solution in Example 4.2, but this purely mathematical process does not concern us now.

As we show in the following example, the constants A and B are to be determined from the initial conditions. We shall show that A and B can be found by knowing the position and velocity at some particular time.

Example 2.18 **The Spring Gun—An Example Illustrating Initial Conditions**



The piston of a spring gun has mass m and is attached to one end of a spring with spring constant k. The projectile is a marble of mass M. The piston and marble are pulled back a distance L from the equilibrium position and suddenly released. What is the speed of the marble as it loses contact with the piston? Neglect friction.

Let the x axis be along the direction of motion with the origin at the unstretched position. The position of the piston is given by

$$x(t) = A \sin \omega t + B \cos \omega t, \qquad 1$$

where $\omega = \sqrt{k/(m + M)}$. This equation holds up to the time the marble and piston lose contact. The velocity is

$$\begin{aligned} v(t) &= \dot{x}(t) \\ &= \omega A \cos \omega t - \omega B \sin \omega t. \qquad 2 \end{aligned}$$

There are two arbitrary constants in the solution, A and B, and to evaluate them we need two pieces of information. We know that at $t = 0$, when the spring is released, the position and velocity are given by

$$x(0) = -L$$
$$v(0) = 0.$$

Using these values in Eqs. (1) and (2), we find

$$\begin{aligned} -L &= x(0) \\ &= A \sin(0) + B \cos(0) \\ &= B, \end{aligned}$$

and

$$\begin{aligned} 0 &= v(0) \\ &= \omega A \cos(0) - \omega B \sin(0) \\ &= \omega A. \end{aligned}$$

Hence

$$B = -L$$
$$A = 0.$$

Then, from the time of release until the time when the marble leaves the piston, the motion is described by the equations

$$x(t) = -L \cos \omega t \qquad 3$$
$$v(t) = \omega L \sin \omega t. \qquad 4$$

When do the marble and piston lose contact? The piston can only push, not pull, on the marble, and when the piston begins to slow down, contact is lost and the marble moves on at a constant velocity. From Eq. (4), we see that the time t_m at which the velocity reaches a maximum is given by

$$\omega t_m = \frac{\pi}{2}.$$

Substituting this in Eq. (3), we find

$$\begin{aligned} x(t_m) &= -L \cos \frac{\pi}{2} \\ &= 0. \end{aligned}$$

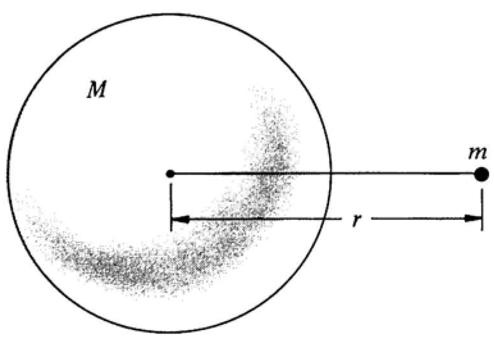
The marble loses contact as the spring passes its equilibrium point, as we expect, since the spring force retards the piston for $x > 0$.

From Eq. (4), the final speed of the marble is

$$v_{\max} = v(t_m) = \omega L \sin\frac{\pi}{2} = \sqrt{\frac{k}{m+M}}\,L.$$

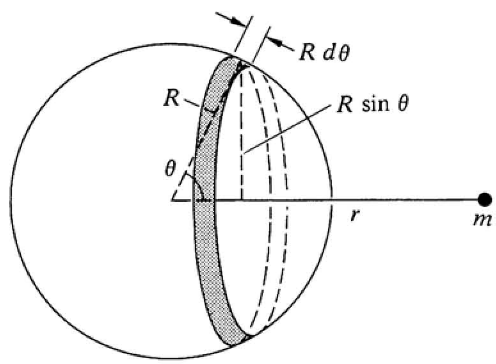
For the highest speeds, k and L should be large and $m + M$ should be small.

Note 2.1 The Gravitational Attraction of a Spherical Shell



In this note we calculate the gravitational force between a uniform thin spherical shell of mass M and a particle of mass m located a distance r from its center. We shall show that the magnitude of the force is GMm/r^2 if the particle is outside the shell and zero if the particle is inside.

To attack the problem, we divide the shell into narrow rings and add their forces by using integral calculus. Let R be the radius of the shell and t its thickness, $t \ll R$. The ring at angle θ, which subtends angle $d\theta$, has circumference $2\pi R \sin\theta$, width $R\,d\theta$, and thickness t. Its volume is

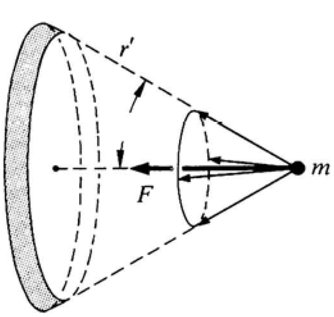


$$dV = 2\pi R^2 t \sin\theta\, d\theta$$

and its mass is

$$\rho\, dV = 2\pi R^2 t\rho \sin\theta\, d\theta = \frac{M}{2}\sin\theta\, d\theta,$$

where $\rho = M/(4\pi R^2 t)$ is the density of the shell.



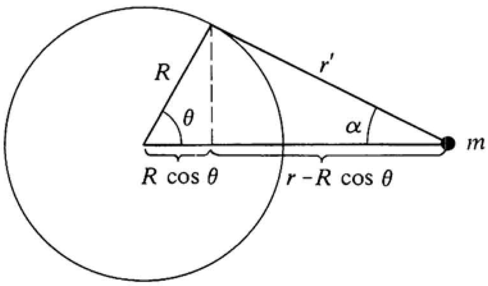
Each part of the ring is the same distance r' from m. The force on m due to a small section of the ring points toward that section. By symmetry, the transverse force components for the whole ring add vectorially to zero. Since the angle α between the force vector and the line of centers is the same for all sections of the ring, the force components along the line of centers add to give

$$dF = \frac{Gm\rho\, dV}{r'^2}\cos\alpha$$

for the whole ring.

The force due to the entire shell is

$$F = \int dF$$
$$= \int \frac{Gm\rho\, dV}{r'^2} \cos \alpha.$$



The problem now is to express all the quantities in the integrand in terms of one variable, say the polar angle θ. From the sketch, $\cos \alpha = (r - R \cos \theta)/r'$, and $r' = \sqrt{r^2 + R^2 - 2rR \cos \theta}$. Since

$\rho\, dV = M \sin \theta\, d\theta/2$,

we have

$$F = \left(\frac{GMm}{2}\right) \int_0^\pi \frac{(r - R \cos \theta) \sin \theta\, d\theta}{(r^2 + R_2 - 2rR \cos \theta)^{\frac{3}{2}}}.$$

A convenient substitution for evaluating this integral is $u = r - R \cos \theta$, $du = R \sin \theta\, d\theta$. Hence

$$F = \left(\frac{GMm}{2R}\right) \int_{r-R}^{r+R} \frac{u\, du}{(R^2 - r^2 + 2ru)^{\frac{3}{2}}}. \qquad 1$$

This integral is listed in standard tables. The result is

$$F = \frac{GMm}{2R} \frac{1}{2r^2} \left(\sqrt{R^2 - r^2 + 2ru} - \frac{r^2 - R^2}{\sqrt{R^2 - r^2 + 2ru}} \right) \Bigg|_{r-R}^{r+R}$$
$$= \frac{GMm}{4Rr^2} \left[(r + R) - (r - R) - (r^2 - R^2) \left(\frac{1}{r + R} - \frac{1}{r - R} \right) \right]$$
$$= \frac{GMm}{r^2} \qquad r > R.$$

For $r > R$, the shell acts gravitationally as though all its mass were concentrated at its center.

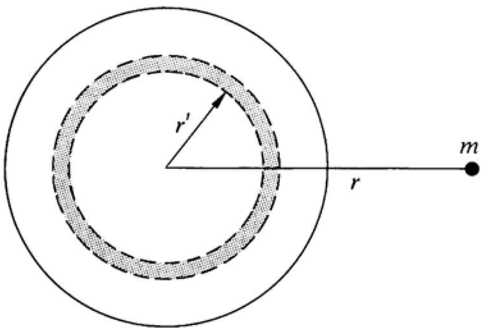
There is one subtlety in our evaluation of the integral. The term $\sqrt{r^2 + R^2 - 2rR}$ is inherently positive, and we must take

$$\sqrt{r^2 + R^2 - 2rR} = r - R,$$

since $r > R$. If the particle is inside the shell, the magnitude of the force is still given by Eq. (1). However, in this case $r < R$, and we must take $\sqrt{r^2 + R^2 - 2rR} = R - r$ in the evaluation. We find

$$F = \frac{GMm}{4Rr^2} \left[(R + r) - (R - r) - (r^2 - R^2) \left(\frac{1}{R + r} - \frac{1}{R - r} \right) \right]$$
$$= 0 \qquad r < R.$$

A solid sphere can be thought of as a succession of spherical shells. It is not hard to extend our results to this case when the density of the sphere $\rho(r')$ is a function only of radial distance r' from the center of



the sphere. The mass of a spherical shell of radius r' and thickness dr' is $\rho(r')4\pi r'^2\,dr'$. The force it exerts on m is

$$dF = \frac{GM}{r^2}\,\rho(r')4\pi r'^2\,dr'.$$

Since the force exerted by every shell is directed toward the center of the sphere, the total force is

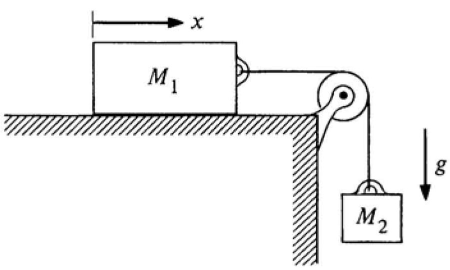
$$F = \frac{Gm}{r^2}\int_0^R \rho(r')4\pi r'^2\,dr'.$$

However, the integral is simply the total mass of the sphere, and we find that for $r > R$, the force between m and the sphere is identical to the force between two particles separated a distance r.

Problems

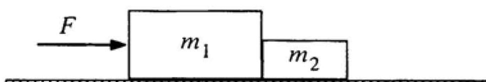
2.1 A 5-kg mass moves under the influence of a force $\mathbf{F} = (4t^2\hat{\mathbf{i}} - 3t\hat{\mathbf{j}})$ N, where t is the time in seconds (1 N = 1 newton). It starts from the origin at $t = 0$. Find: (*a*) its velocity; (*b*) its position; and (*c*) $\mathbf{r} \times \mathbf{v}$, for any later time.

Ans. clue. (c) If $t = 1$ s, $\mathbf{r} \times \mathbf{v} = 6.7 \times 10^{-3}\hat{\mathbf{k}}$ m^2/s



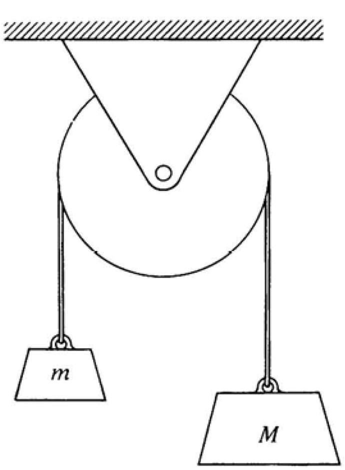
2.2 The two blocks shown in the sketch are connected by a string of negligible mass. If the system is released from rest, find how far block M_1 slides in time t. Neglect friction.

Ans. clue. If $M_1 = M_2$, $x = gt^2/4$



2.3 Two blocks are in contact on a horizontal table. A horizontal force is applied to one of the blocks, as shown in the drawing. If $m_1 = 2$ kg, $m_2 = 1$ kg, and $F = 3$ N, find the force of contact between the two blocks.

2.4 Two particles of mass m and M undergo uniform circular motion about each other at a separation R under the influence of an attractive force F. The angular velocity is ω radians per second. Show that $R = (F/\omega^2)(1/m + 1/M)$.



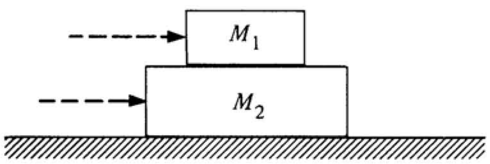
2.5 The Atwood's machine shown in the drawing has a pulley of negligible mass. Find the tension in the rope and the acceleration of M.

Ans. clue. If $M = 2m$, $T = \frac{2}{3}Mg$, $A = \frac{1}{3}g$

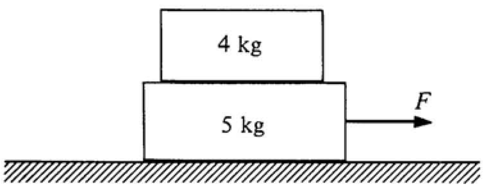
2.6 In a concrete mixer, cement, gravel, and water are mixed by tumbling action in a slowly rotating drum. If the drum spins too fast the ingredients stick to the drum wall instead of mixing.

Assume that the drum of a mixer has radius R and that it is mounted with its axle horizontal. What is the fastest the drum can rotate without the ingredients sticking to the wall all the time? Assume $g = 32$ ft/s^2.

Ans. clue. If $R = 2$ ft, $\omega_{\max} = 4$ rad/s ≈ 38 rotations per minute

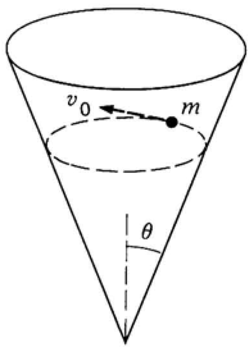


2.7 A block of mass M_1 rests on a block of mass M_2 which lies on a frictionless table. The coefficient of friction between the blocks is μ. What is the maximum horizontal force which can be applied to the blocks for them to accelerate without slipping on one another if the force is applied to (*a*) block 1 and (*b*) block 2?



2.8 A 4-kg block rests on top of a 5-kg block, which rests on a frictionless table. The coefficient of friction between the two blocks is such that the blocks start to slip when the horizontal force F applied to the lower block is 27 N. Suppose that a horizontal force is now applied only to the upper block. What is its maximum value for the blocks to slide without slipping relative to each other?

Ans. $F = 21.6$ N



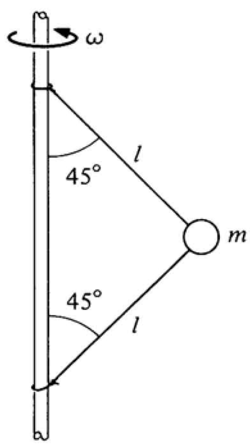
2.9 A particle of mass m slides without friction on the inside of a cone. The axis of the cone is vertical, and gravity is directed downward. The apex half-angle of the cone is θ, as shown.

The path of the particle happens to be a circle in a horizontal plane. The speed of the particle is v_0.

Draw a force diagram and find the radius of the circular path in terms of v_0, g, and θ.

2.10 Find the radius of the orbit of a synchronous satellite which circles the earth. (A synchronous satellite goes around the earth once every 24 h, so that its position appears stationary with respect to a ground station.) The simplest way to find the answer and give your results is by expressing all distances in terms of the earth's radius.

Ans. $6.6R_e$



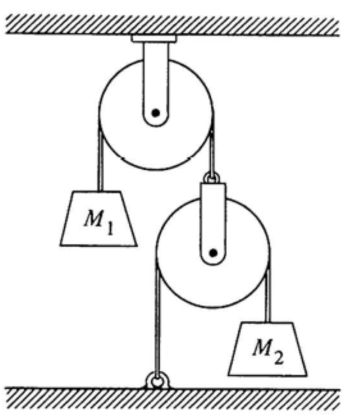
2.11 A mass m is connected to a vertical revolving axle by two strings of length l, each making an angle of 45° with the axle, as shown. Both the axle and mass are revolving with angular velocity ω. Gravity is directed downward.

a. Draw a clear force diagram for m.

b. Find the tension in the upper string, T_{up}, and lower string, T_{low}.

Ans. clue. If $l\omega^2 = \sqrt{2}\,g$, $T_{\text{up}} = \sqrt{2}\,mg$

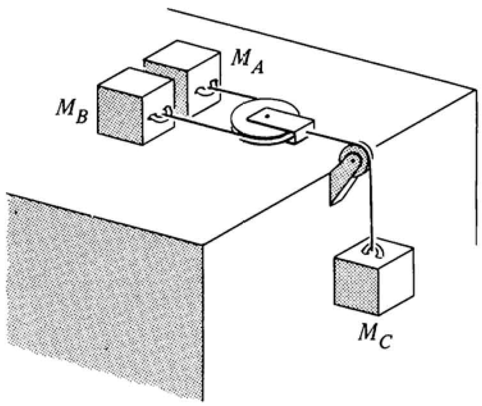
2.12 If you have courage and a tight grip, you can yank a tablecloth out from under the dishes on a table. What is the longest time in which the cloth can be pulled out so that a glass 6 in from the edge comes to rest before falling off the table? Assume that the coefficient of friction of the glass sliding on the tablecloth or sliding on the tabletop is 0.5. (For the trick to be effective the cloth should be pulled out so rapidly that the glass does not move appreciably.)

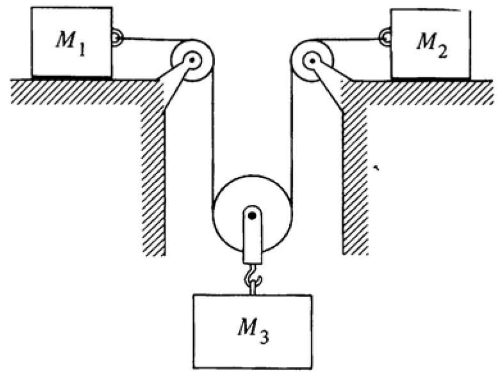


2.13 Masses M_1 and M_2 are connected to a system of strings and pulleys as shown. The strings are massless and inextensible, and the pulleys are massless and frictionless. Find the acceleration of M_1.

Ans. clue. If $M_1 = M_2$, $\ddot{x}_1 = g/5$

2.14 Two masses, A and B, lie on a frictionless table (see below left). They are attached to either end of a light rope of length l which passes around a pulley of negligible mass. The pulley is attached to a rope connected to a hanging mass, C. Find the acceleration of each mass. (You can check whether or not your answer is reasonable by considering special cases—for instance, the cases $M_A = 0$, or $M_A = M_B = M_C$.)





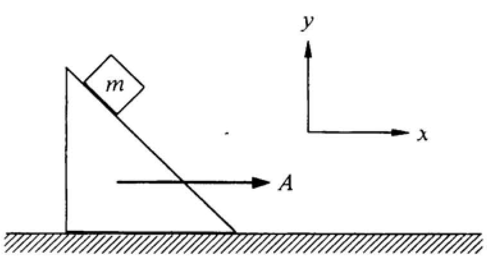
2.15 The system on the right above uses massless pulleys and rope. The coefficient of friction between the masses and horizontal surfaces is μ. Assume that M_1 and M_2 are sliding. Gravity is directed downward

a. Draw force diagrams, and show all relevant coordinates.

b. How are the accelerations related?

c. Find the tension in the rope, T.

Ans. $T = (\mu + 1)g/[2/M_3 + 1/(2M_1) + 1/(2M_2)]$



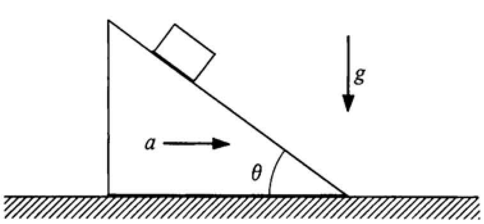
2.16 A 45° wedge is pushed along a table with constant acceleration A. A block of mass m slides without friction on the wedge. Find its acceleration. (Gravity is directed down.)

Ans. clue. If $A = 3g$, $\ddot{y} = g$

2.17 A block rests on a wedge inclined at angle θ. The coefficient of friction between the block and plane is μ.

a. Find the maximum value of θ for the block to remain motionless on the wedge when the wedge is fixed in position.

Ans. $\tan\theta = \mu$



b. The wedge is given horizontal acceleration a, as shown. Assuming that $\tan\theta < \mu$, find the minimum acceleration for the block to remain on the wedge without sliding.

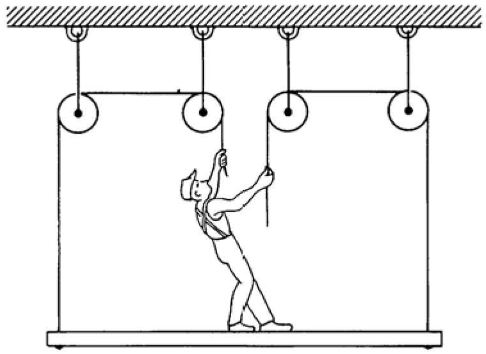
Ans. clue. If $\theta = \pi/4$, $a_{\min} = g(1 - \mu)/(1 + \mu)$

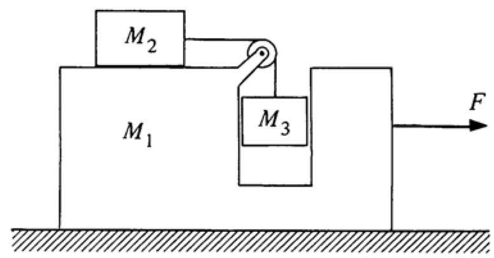
c. Repeat part b, but find the maximum value of the acceleration.

Ans. clue. If $\theta = \pi/4$, $a_{\max} = g(1 + \mu)/(1 - \mu)$

2.18 A painter of mass M stands on a platform of mass m and pulls himself up by two ropes which hang over pulleys, as shown. He pulls each rope with force F and accelerates upward with a uniform acceleration a. Find a—neglecting the fact that no one could do this for long.

Ans. clue. If $M = m$ and $F = Mg$, $a = g$



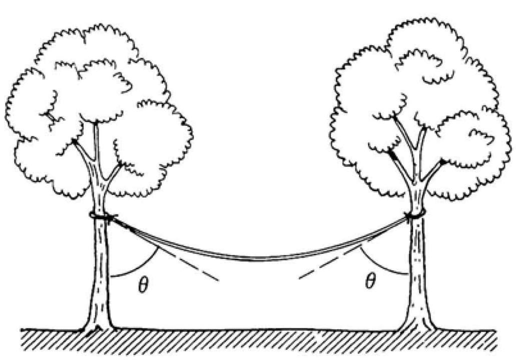


2.19 A "Pedagogical Machine" is illustrated in the sketch above. All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

Ans. clue. For equal masses, $F = 3Mg$

2.20 Consider the "Pedagogical Machine" of the last problem in the case where F is zero. Find the acceleration of M_1.

Ans. $a_1 = -M_2M_3g/(M_1M_2 + M_1M_3 + 2M_2M_3 + M_3^2)$



2.21 A uniform rope of mass m and length l is attached to a block of mass M. The rope is pulled with force F. Find the tension at distance x from the end of the rope. Neglect gravity.

2.22 A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees. Find

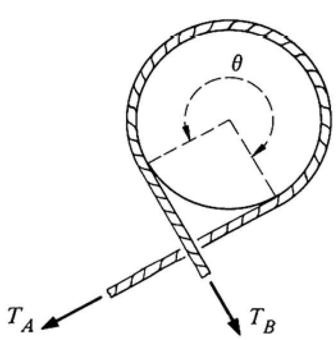
a. The tension at either end of the rope

b. The tension in the middle of the rope

Ans. clue. If $\theta = 45°$, $T_{\text{end}} = W/\sqrt{2}$, $T_{\text{middle}} = W/2$

2.23 A piece of string of length l and mass M is fastened into a circular loop and set spinning about the center of a circle with uniform angular velocity ω. Find the tension in the string. Suggestion: Draw a force diagram for a small piece of the loop subtending a small angle, $\Delta\theta$.

Ans. $T = M\omega^2 l/(2\pi)^2$



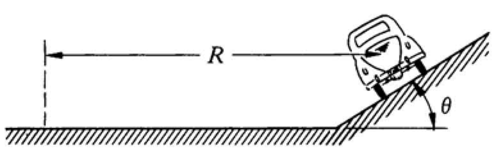
2.24 A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing shows about three-fourths turn). The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B. Can you show that $T_B = T_A e^{-\mu\theta}$, where μ is the coefficient of friction and θ is the total angle subtended by the rope on the drum?

2.25 Find the shortest possible period of revolution of two identical gravitating solid spheres which are in circular orbit in free space about a point midway between them. (You can imagine the spheres fabricated from any material obtainable by man.)

2.26 The gravitational force on a body located at distance R from the center of a uniform spherical mass is due solely to the mass lying at distance $r \leq R$, measured from the center of the sphere. This mass exerts a force as if it were a point mass at the origin.

Use the above result to show that if you drill a hole through the earth and then fall in, you will execute simple harmonic motion about the earth's center. Find the time it takes you to return to your point of departure and show that this is the time needed for a satellite to circle the earth in a low orbit with $r \approx R_e$. In deriving this result, you need to treat the earth as a uniformly dense sphere, and you must neglect all friction and any effects due to the earth's rotation.

2.27 As a variation of the last problem, show that you will also execute simple harmonic motion with the same period even if the straight hole passes far from the earth's center.



2.28 An automobile enters a turn whose radius is R. The road is banked at angle θ, and the coefficient of friction between wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

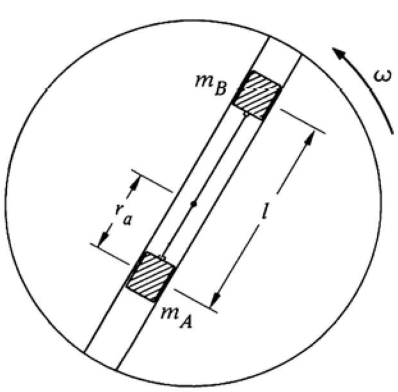
Ans. clue. If $\mu = 1$ and $\theta = \pi/4$, all speeds are possible

2.29 A car is driven on a large revolving platform which rotates with constant angular speed ω. At $t = 0$ a driver leaves the origin and follows a line painted radially outward on the platform with constant speed v_0. The total weight of the car is W, and the coefficient of friction between the car and stage is μ.

a. Find the acceleration of the car as a function of time using polar coordinates. Draw a clear vector diagram showing the components of *acceleration* at some time $t > 0$.

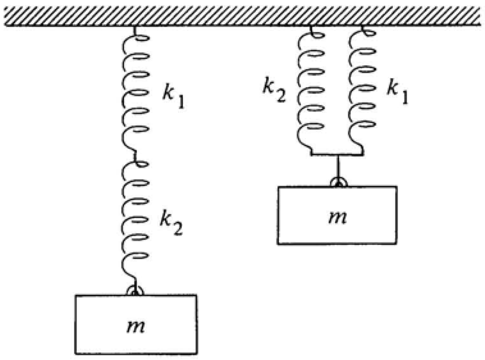
b. Find the time at which the car just starts to skid.

c. Find the direction of the friction force with respect to the instantaneous position vector **r** just before the car starts to skid. Show your result on a clear diagram.



2.30 A disk rotates with constant angular velocity ω, as shown. Two masses, m_A and m_B, slide without friction in a groove passing through the center of the disk. They are connected by a light string of length l, and are initially held in position by a catch, with mass m_A at distance r_A from the center. Neglect gravity. At $t = 0$ the catch is removed and the masses are free to slide.

Find $\ddot{r}_A$ immediately after the catch is removed in terms of m_A, m_B, l, r_A, and ω.



(a) (b)

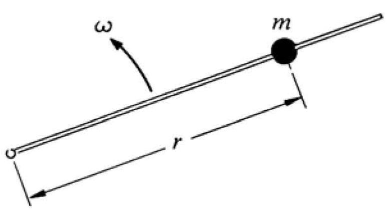
2.31 Find the frequency of oscillation of mass m suspended by two springs having constants k_1 and k_2, in each of the configurations shown.

Ans. clue. If $k_1 = k_2 = k$, $\omega_a = \sqrt{k/2m}$, $\omega_b = \sqrt{2k/m}$

2.32 A wheel of radius R rolls along the ground with velocity V. A pebble is carefully released on top of the wheel so that it is instantaneously at rest on the wheel.

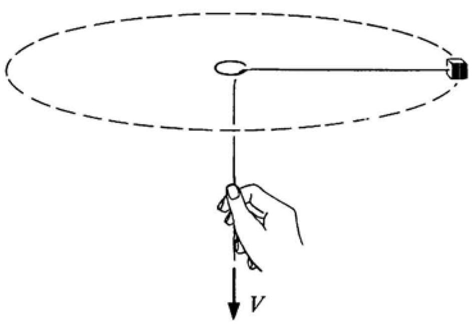
a. Show that the pebble will immediately fly off the wheel if $V > \sqrt{Rg}$.

b. Show that in the case where $V < \sqrt{Rg}$, and the coefficient of friction is $\mu = 1$, the pebble starts to slide when it has rotated through an angle given by $\theta = \arccos[(1/\sqrt{2})(V^2/Rg)] - \pi/4$.



2.33 A particle of mass m is free to slide on a thin rod. The rod rotates in a plane about one end at constant angular velocity ω. Show that the motion is given by $r = Ae^{-\gamma t} + Be^{+\gamma t}$, where γ is a constant which you must find and A and B are arbitrary constants. Neglect gravity.

Show that for a particular choice of initial conditions [that is, $r(t = 0)$ and $v(t = 0)$], it is possible to obtain a solution such that r decreases continually in time, but that for any other choice r will eventually increase. (Exclude cases where the bead hits the origin.)

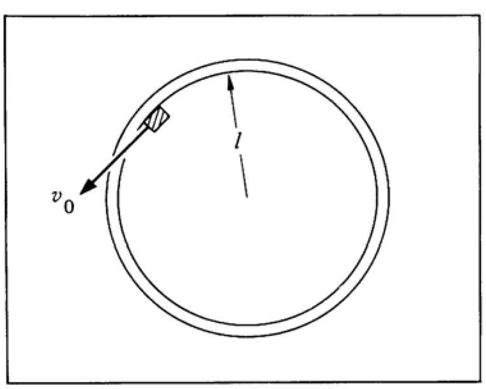


2.34. A mass m whirls around on a string which passes through a ring, as shown. Neglect gravity. Initially the mass is distance r_0 from the center and is revolving at angular velocity ω_0. The string is pulled with constant velocity V starting at $t = 0$ so that the radial distance to the mass decreases. Draw a force diagram and obtain a differential equation for ω. This equation is quite simple and can be solved either by inspection or by formal integration. Find

a. $\omega(t)$.

Ans. clue. For $Vt = r_0/2$, $\omega = 4\omega_0$

b. The force needed to pull the string.



2.35 This problem involves solving a simple differential equation.

A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius l which is fixed to the table. At $t = 0$, the block is moving along the inside of the ring (i.e., in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ.

a. Find the velocity of the block at later times.

Ans. $v_0/[1 + (\mu v_0 t/l)]$

b. Find the position of the block at later times.

2.36 This problem involves a simple differential equation. You should be able to integrate it after a little "playing around."

A particle of mass m moving along a straight line is acted on by a retarding force (one always directed against the motion) $F = be^{\alpha v}$, where

b and α are constants and v is the velocity. At $t = 0$ it is moving with velocity v_0. Find the velocity at later times.

Ans. $v(t) = (1/\alpha) \ln [1/(\alpha b t/m + e^{-\alpha v_0})]$

2.37 The Eureka Hovercraft Corporation wanted to hold hovercraft races as an advertising stunt. The hovercraft supports itself by blowing air downward, and has a big fixed propeller on the top deck for forward propulsion. Unfortunately, it has no steering equipment, so that the pilots found that making high speed turns was very difficult. The company decided to overcome this problem by designing a bowl shaped track in which the hovercraft, once up to speed, would coast along in a circular path with no need to steer. They hired an engineer to design and build the track, and when he finished, he hastily left the country. When the company held their first race, they found to their dismay that the craft took exactly the same time T to circle the track, no matter what its speed. Find the equation for the cross section of the bowl in terms of T.