

## 3.1 Introduction

In the last chapter we made a gross simplification by treating nature as if it were composed of point particles rather than real, *extended* bodies. Sometimes this simplification is justified—as in the study of planetary motion, where the size of the planets is of little consequence compared with the vast distances which characterize our solar system, or in the case of elementary particles moving through an accelerator, where the size of the particles, about  $10^{-15}$  m, is minute compared with the size of the machine. However, these cases are unusual. Much of the time we deal with large bodies which may have elaborate structure. For instance, consider the landing of a spacecraft on the moon. Even if we could calculate the gravitational field of such an irregular and inhomogeneous body as the moon, the spacecraft itself is certainly not a point particle—it has spiderlike legs, gawky antennas, and a lumpy body.

Furthermore, the methods of the last chapter fail us when we try to analyze systems such as rockets in which there is a flow of mass. Rockets accelerate forward by ejecting mass backward; it is hard to see how to apply  $\mathbf{F} = M\mathbf{a}$  to such a system.

In this chapter we shall generalize the laws of motion to overcome these difficulties. We begin by restating Newton's second law in a slightly modified form. In Chap. 2 we wrote the law in the familiar form

$$\mathbf{F} = M\mathbf{a}.$$
 3.1

This is not quite the way Newton wrote it. He chose to write

$$\mathbf{F} = \frac{d}{dt} M \mathbf{v}.$$
 3.2

For a particle in newtonian mechanics, M is a constant and  $(d/dt)(M\mathbf{v}) = M(d\mathbf{v}/dt) = M\mathbf{a}$ , as before. The quantity  $M\mathbf{v}$ , which plays a prominent role in mechanics, is called *momentum*. Momentum is the product of a vector  $\mathbf{v}$  and a scalar M. Denoting momentum by  $\mathbf{p}$ , Newton's second law becomes

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$
. 3.3

This form is preferable to  $\mathbf{F} = M\mathbf{a}$  because it is readily generalized to complex systems, as we shall soon see, and because momentum

# 3.2 Dynamics of a System of Particles

Consider a system of interacting particles. One example of such a system is the sun and planets, which are so far apart compared with their diameters that they can be treated as simple particles to good approximation. All particles in the solar system interact via gravitational attraction; the chief interaction is with the sun, although the interaction of the planets with each other also influences their motion. In addition, the entire solar system is attracted by far off matter.

At the other extreme, the system could be a billiard ball resting on a table. Here the particles are atoms (disregarding for now the fact that atoms are not point particles but are themselves composed of smaller particles) and the interactions are primarily interatomic electric forces. The external forces on the billiard ball include the gravitational force of the earth and the contact force of the tabletop.

We shall now prove some simple properties of physical systems. We are free to choose the boundaries of the system as we please, but once the choice is made, we must be consistent about which particles are included in the system and which are not. We suppose that the particles in the system interact with particles outside the system as well as with each other. To make the argument general, consider a system of N interacting particles with masses  $m_1, m_2, m_3, \ldots, m_N$ . The position of the *j*th particle is  $\mathbf{r}_j$ , the force on it is  $\mathbf{f}_j$ , and its momentum is  $\mathbf{p}_j = m_j \dot{\mathbf{r}}_j$ . The equation of motion for the *j*th particle is

$$\mathbf{f}_j = \frac{d\mathbf{p}_j}{dt}$$
 3.4

The force on particle j can be split into two terms:

$$\mathbf{f}_j = \mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}}.$$
 3.5

Here  $\mathbf{f}_{j}^{\text{int}}$ , the *internal* force on particle j, is the force due to all other particles in the system, and  $\mathbf{f}_{j}^{\text{ext}}$ , the *external* force on particle j, is the force due to sources outside the system. The equation of motion becomes

$$\mathbf{f}_{j}^{\text{int}} + \mathbf{f}_{j}^{\text{ext}} = \frac{d\mathbf{p}_{j}}{dt}$$
 3.6

Now let us focus on the system as a whole by the following stratagem: add all the equations of motion of all the particles in the system.

$$\mathbf{f}_{1}^{\text{int}} + \mathbf{f}_{1}^{\text{ext}} = \frac{d\mathbf{p}_{1}}{dt}$$

$$\cdots$$

$$\mathbf{f}_{j}^{\text{int}} + \mathbf{f}_{j}^{\text{ext}} = \frac{d\mathbf{p}_{j}}{dt}$$

$$\cdots$$

$$\mathbf{f}_{N}^{\text{int}} + \mathbf{f}_{N}^{\text{ext}} = \frac{d\mathbf{p}_{N}}{dt}$$

$$3.7$$

The result of adding these equations can be written

$$\Sigma \mathbf{f}_{j^{\text{int}}} + \Sigma \mathbf{f}_{j^{\text{ext}}} = \sum \frac{d\mathbf{p}_{j}}{dt}$$
 3.8

The summations extend over all particles,  $j = 1, \ldots, N$ .

The second term,  $\Sigma f_{j}^{ext}$ , is the sum of all external forces acting on all the particles. It is the *total external force* acting on the system,  $\mathbf{F}_{ext}$ .

 $\Sigma \mathbf{f}_{j}^{\mathrm{ext}} \equiv \mathbf{F}_{\mathrm{ext}}.$ 

The first term in Eq. (3.8),  $\Sigma \mathbf{f}_{j}^{\text{int}}$ , is the sum of all internal forces acting on all the particles. According to Newton's third law, the forces between any two particles are equal and opposite so that their sum is zero. It follows that the sum of all the forces between all the particles is also zero; the internal forces cancel in pairs. Hence

 $\Sigma \mathbf{f}_{i^{\mathrm{int}}} = \mathbf{0}.$ 

Equation (3.8) then simplifies to

$$\mathbf{F}_{\text{ext}} = \sum \frac{d\mathbf{p}_j}{dt}$$
 3.9

The right hand side can be written  $\Sigma(d\mathbf{p}_j/dt) = (d/dt)\Sigma\mathbf{p}_j$ , since the derivative of a sum is the sum of the derivatives.  $\Sigma\mathbf{p}_j$  is the *total momentum* of the system, which we designate by **P**.

$$\mathbf{P} \equiv \Sigma \mathbf{p}_j. \tag{3.10}$$

With this substitution, Eq. (3.9) becomes

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt}.$$
 3.11

In words, the total external force applied to a system equals the rate of change of the system's momentum. This is true irrespective of the details of the interaction;  $\mathbf{F}_{\text{ext}}$  could be a single force acting on a single particle, or it could be the resultant of many tiny interactions involving each particle of the system.

## Example 3.1 The Bola

The bola is a weapon used by gauchos for entangling animals. It consists of three balls of stone or iron connected by thongs. The gaucho whirls the bola in the air and hurls it at the animal. What can we say about its motion?



Consider a bola with masses  $m_1$ ,  $m_2$ , and  $m_3$ . The balls are pulled by the binding thong and by gravity. (We neglect air resistance.) Since the constraining forces depend on the instantaneous positions of all three balls, it is a real problem even to write the equation of motion of one ball. However, the total momentum obeys the simple equation

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} = \mathbf{f}_{1}^{\text{ext}} + \mathbf{f}_{2}^{\text{ext}} + \mathbf{f}_{3}^{\text{ext}}$$
$$= m_{1}\mathbf{g} + m_{2}\mathbf{g} + m_{3}\mathbf{g}$$
or

$$\frac{d\mathbf{P}}{dt} = M\mathbf{g}$$

where M is the total mass. This equation represents an important first step in finding the detailed motion. The equation is identical to that of a single particle of mass M with momentum **P**. This is a familiar fact

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to the gaucho who forgets that he has a complicated system when he hurls the bola; he instinctively aims it like a single mass.

## **Center of Mass**

According to Eq. (3.11),

$$\mathbf{F} = \frac{d\mathbf{P}}{dt},$$
 3.12

where we have dropped the subscript ext with the understanding that **F** stands for the external force. This result is identical to the equation of motion of a single particle, although in fact it refers to a system of particles. It is tempting to push the analogy between Eq. (3.12) and single particle motion even further by writing

$$\mathbf{F} = M\mathbf{\ddot{R}}, \qquad 3.13$$

where M is the total mass of the system and **R** is a vector yet to be defined. Since **P** =  $\Sigma m_j \dot{\mathbf{r}}_j$ , Eq. (3.12) and (3.13) give

$$M\ddot{\mathbf{R}} = \frac{d\mathbf{P}}{dt} = \Sigma m_j \ddot{\mathbf{r}}_j,$$

which is true if

$$\mathbf{R} = \frac{1}{M} \Sigma m_j \mathbf{r}_j. \tag{3.14}$$

**R** is a vector from the origin to the point called the *center of* mass. The system behaves as if all the mass is concentrated at the center of mass and all the external forces act at that point.

We are often interested in the motion of comparatively rigid bodies like baseballs or automobiles. Such a body is merely a system of particles which are fixed relative to each other by strong internal forces; Eq. (3.13) shows that with respect to external forces, the body behaves as if it were a point particle. In Chap. 2, we casually treated every body as if it were a particle; we see now that this is justified provided that we focus attention on the center of mass.

You may wonder whether this description of center of mass motion isn't a gross oversimplification—experience tells us that an extended body like a plank behaves differently from a compact body like a rock, even if the masses are the same and we apply the same force. We are indeed oversimplifying. The relation  $\mathbf{F} = M\mathbf{\ddot{R}}$  describes only the translation of the body (the motion of its center of mass); it does not describe the body's orientation in space. In Chaps. 6 and 7 we shall investigate the rotation of extended bodies, and it will turn out that the rotational motion of a body depends both on its shape and the point where the forces are applied. Nevertheless, as far as translation of the center of mass is concerned,  $\mathbf{F} = M\mathbf{\ddot{R}}$  tells the whole story. This result is true for any system of particles, not just for those fixed in rigid objects, as long as the forces between the particles obey Newton's third law. It is immaterial whether or not the particles move relative to each other and whether or not there happens to be any matter at the center of mass.

## Example 3.2 Drum Major's Baton

A drum major's baton consists of two masses  $m_1$  and  $m_2$  separated by a thin rod of length l. The baton is thrown into the air. The problem is to find the baton's center of mass and the equation of motion for the center of mass.

Let the position vectors of  $m_1$  and  $m_2$  be  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . The position vector of the center of mass, measured from the same origin, is

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2},$$
 1

where we have neglected the mass of the thin rod. The center of mass lies on the line joining  $m_1$  and  $m_2$ . To show this, suppose first that the tip of **R** does not lie on the line, and consider the vectors  $\mathbf{r}'_1$ ,  $\mathbf{r}'_2$  from the tip of **R** to  $m_1$  and  $m_2$ . From the sketch we see that

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R}$$
$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R}$$

Using Eq. (1) gives

$$\mathbf{r}_{1}' = \mathbf{r}_{1} - \frac{m_{1}\mathbf{r}_{1}}{m_{1} + m_{2}} - \frac{m_{2}\mathbf{r}_{2}}{m_{1} + m_{2}}$$
$$= \frac{m_{2}}{m_{1} + m_{2}} (\mathbf{r}_{1} - \mathbf{r}_{2})$$
$$\mathbf{r}_{2}' = \mathbf{r}_{2} - \frac{m_{1}\mathbf{r}_{1}}{m_{1} + m_{2}} - \frac{m_{2}\mathbf{r}_{2}}{m_{1} + m_{2}}$$
$$= -\left(\frac{m_{1}}{m_{1} + m_{2}}\right) (\mathbf{r}_{1} - \mathbf{r}_{2}).$$





 $\mathbf{r}'_1$  and  $\mathbf{r}'_2$  are proportional to  $\mathbf{r}_1 - \mathbf{r}_2$ , the vector from  $m_1$  to  $m_2$ . Hence  $\mathbf{r}'_1$  and  $\mathbf{r}'_2$  lie along the line joining  $m_1$  and  $m_2$ , as shown. Furthermore,

$$r'_{1} = \frac{m_{2}}{m_{1} + m_{2}} |\mathbf{r}_{1} - \mathbf{r}_{2}|$$
$$= \frac{m_{2}}{m_{1} + m_{2}} l$$

and

$$r'_{2} = \frac{m_{1}}{m_{1} + m_{2}} |\mathbf{r}_{1} - \mathbf{r}_{2}|$$
$$= \frac{m_{1}}{m_{1} + m_{2}} l.$$

Assuming that friction is negligible, the external force on the baton is

$$\mathbf{F} = m_1 \mathbf{g} + m_2 \mathbf{g}.$$

The equation of motion of the center of mass is

$$(m_1 + m_2)\mathbf{R} = (m_1 + m_2)\mathbf{g}$$

The center of mass follows the parabolic trajectory of a single mass in a uniform gravitational field. With the methods developed in Chap. 6, we shall be able to find the motion of  $m_1$  and  $m_2$  about the center of mass, completing the solution to the problem.

Although it is a simple matter to find the center of mass of a system of particles, the procedure for locating the center of mass of an extended body is not so apparent. However, it is a straightforward task with the help of calculus. We proceed by dividing the body into N mass elements. If  $\mathbf{r}_j$  is the position of the jth element, and  $m_j$  is its mass, then

$$\mathbf{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \mathbf{r}_j.$$

The result is not rigorous, since the mass elements are not true particles. However, in the limit where N approaches infinity, the size of each element approaches zero and the approximation becomes exact.

$$\mathbf{R} = \lim_{N \to \infty} \frac{1}{M} \sum_{j=1}^{N} m_j \mathbf{r}_j$$

This limiting process defines an integral. Formally

$$\lim_{N\to\infty}\sum_{j=1}^{\infty}m_j\mathbf{r}_j=\int \mathbf{r} \ dm,$$



where dm is a differential mass element. Then

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm. \tag{3.15}$$

To visualize this integral, think of dm as the mass in an element of volume dV located at position **r**. If the mass density at the element is  $\rho$ , then  $dm = \rho \, dV$  and

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \rho \, dV.$$

This integral is called a volume integral. Although it is important to know how to find the center of mass of rigid bodies, we shall only be concerned with a few simple cases here, as illustrated by the following two examples. Further examples are given in Note 3.1 at the end of the chapter.

## Example 3.3 Center of Mass of a Nonuniform Rod

A rod of length L has a nonuniform density.  $\lambda$ , the mass per unit length of the rod, varies as  $\lambda = \lambda_0(s/L)$ , where  $\lambda_0$  is a constant and s is the distance from the end marked 0. Find the center of mass.

It is apparent that **R** lies on the rod. Let the origin of the coordinate system coincide with the end of the rod, 0, and let the x axis lie along the rod so that s = x. The mass in an element of length dx is  $dm = \lambda dx = \lambda_0 x dx/L$ . The rod extends from x = 0 to x = L and the total mass is



 $M = \int dm$ =  $\int_0^L \lambda \, dx$ =  $\int_0^L \frac{\lambda_0 x \, dx}{L}$ =  $\frac{1}{2} \lambda_0 L.$ 

The center of mass is at

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \lambda \, dM$$
$$= \frac{2}{\lambda_0 L} \int_0^L \left( x \hat{\mathbf{i}} + 0 \hat{\mathbf{j}} + 0 \hat{\mathbf{k}} \right) \frac{\lambda_0 x \, dx}{L}$$
$$= \frac{2}{L^2} \frac{\hat{\mathbf{i}}}{3} \, x^3 \Big|_0^L$$
$$= \frac{2}{3} L \hat{\mathbf{i}}.$$



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Example 3.4

## e 3.4 Center of Mass of a Triangular Sheet



Consider the two dimensional case of a uniform right triangular sheet of mass M, base b, height h, and small thickness t. If we divide the sheet into small rectangular areas of side  $\Delta x$  and  $\Delta y$ , as shown, then the volume of each element is  $\Delta V = t \Delta x \Delta y$ , and

$$\mathbf{R} \approx \frac{\sum m_i \mathbf{r}_i}{M}$$
$$= \frac{\sum \rho_i t \ \Delta x \ \Delta y \mathbf{r}_i}{M},$$

where j is the label of one of the volume elements and  $\rho_j$  is the density. Because the sheet is uniform,

$$\rho_i = \text{constant} = \frac{M}{V} = \frac{M}{At},$$

where A is the area of the sheet.

We can carry out the sum by summing first over the  $\Delta x$ 's and then over the  $\Delta y$ 's, instead of over the single index j. This gives a double sum which can be converted to a double integral by taking the limit, as follows:

$$\mathbf{R} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{M}{At}\right) \left(\frac{t}{M}\right) \Sigma \Sigma \mathbf{r}_i \,\Delta x \,\Delta y$$
$$= \frac{1}{A} \iint \mathbf{r} \, dx \, dy.$$

Let  $\mathbf{r} = x\mathbf{\hat{i}} + y\mathbf{\hat{j}}$  be the position vector of an element dx dy. Then, writing  $\mathbf{R} = X\mathbf{\hat{i}} + Y\mathbf{\hat{j}}$ , we have

$$\mathbf{R} = X\mathbf{\hat{i}} + Y\mathbf{\hat{j}}$$
  
=  $\frac{1}{A} \iint (x\mathbf{\hat{i}} + y\mathbf{\hat{j}}) dx dy$   
=  $\frac{1}{A} \left( \iint x dx dy \right) \mathbf{\hat{i}} + \frac{1}{A} \left( \iint y dx dy \right) \mathbf{\hat{j}}.$ 

Hence the coordinates of the center of mass are given by

$$X = \frac{1}{A} \iint x \, dx \, dy$$
$$Y = \frac{1}{A} \iint y \, dx \, dy.$$

The double integrals may look strange, but they are easily evaluated. Consider first the double integral

$$X = \frac{1}{A} \iint x \, dx \, dy.$$

This integral instructs us to take each element, multiply its area by its x coordinate, and sum the results. We can do this in stages by first considering the elements in a strip parallel to the y axis. The strip runs from y = 0 to y = xh/b. Each element in the strip has the same x coordinate, and the contribution of the strip to the double integral is

$$\frac{1}{A} x \, dx \int_0^{xh/b} dy = \frac{h}{bA} x^2 \, dx.$$

Finally, we sum the contributions of all such strips x = 0 to x = b to find

$$X = \frac{h}{bA} \int_0^b x^2 \, dx = \frac{h}{bA} \frac{b^3}{3}$$
$$= \frac{hb^2}{3A}$$



Since 
$$A = \frac{1}{2}bh$$

$$X = \frac{2}{3}b.$$

Similarly,

$$Y = \frac{1}{A} \int_0^b \left( \int_0^{xh/b} y \, dy \right) dx$$
$$= \frac{h^2}{2Ab^2} \int_0^b x^2 \, dx = \frac{h^2b}{6A}$$
$$= \frac{1}{3}h.$$

Hence

$$\mathbf{R} = \frac{2}{3}b\mathbf{\hat{i}} + \frac{1}{3}h\mathbf{\hat{j}}$$

Although the coordinates of  $\mathbf{R}$  depend on the particular coordinate system we choose, the position of the center of mass with respect to the triangular plate is, of course, independent of the coordinate system.



Often physical arguments are more useful than mathematical analysis. For instance, to find the center of mass of an irregular plane object, let it hang from a pivot and draw a plumb line from the pivot. The center of mass will hang directly below the pivot (this may be intuitively be obvious, and it can easily be proved



with the methods of Chap. 6), and it is somewhere on the plumb line. Repeat the procedure with a different pivot point. The two lines intersect at the center of mass.

## Example 3.5 Center of Mass Motion

A rectangular box is held with one corner resting on a frictionless table and is gently released. It falls in a complex tumbling motion, which we are not yet prepared to solve because it involves rotation. However, there is no difficulty in finding the trajectory of the center of mass.



The external forces acting on the box are gravity and the normal force of the table. Neither of these has a horizontal component, and so the center of mass must accelerate vertically. For a uniform box, the center of mass is at the geometrical center. If the box is released from rest, then its center falls straight down.

## 3.3 Conservation of Momentum

In the last section we found that the total external force  $\mathbf{F}$  acting on a system is related to the total momentum  $\mathbf{P}$  of the system by

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}$$

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case  $\mathbf{F} = 0$ , and  $d\mathbf{P}/dt = 0$ . The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system is constant. This is the law of conservation of momentum. As we shall show, this apparently simple law can provide powerful insights into complicated systems.

## Example 3.6 Spring Gun Recoil



A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation  $\theta$ . The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is  $v_0$ . What is the final motion of the gun?

Take the physical system to be the gun and marble. Gravity and the normal force of the table act on the system. Both these forces are vertical. Since there are no horizontal external forces, the x component of the vector equation  $\mathbf{F} = d\mathbf{P}/dt$  is

$$0 = \frac{dP_x}{dt}$$

According to Eq. (1),  $P_x$  is conserved:

$$P_{x,\text{initial}} = P_{x,\text{final}}.$$



Let the initial time be prior to firing the gun. Then  $P_{x,\text{initial}} = 0$ , since the system is initially at rest. After the marble has left the muzzle, the gun recoils with some speed  $V_f$ , and its final horizontal momentum is  $MV_f$ , to the left. Finding the final velocity of the marble involves a subtle point, however. Physically, the marble's acceleration is due to the force of the gun, and the gun's recoil is due to the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so that at the instant the marble and the gun part company, the gun has its final speed  $V_f$ . At that same instant the speed of the marble relative to the gun is  $v_0$ . Hence, the final horizontal speed of the marble relative to the table is  $v_0 \cos \theta - V_f$ . By conservation of horizontal momentum, we therefore have

$$0 = m(v_0 \cos \theta - V_f) - M V_f$$

or

$$V_f = \frac{mv_0\cos\theta}{M+m}$$

By using conservation of momentum we found the final motion of the system in a few steps. To show the advantage of this method, let us repeat the problem using Newton's laws directly.

Let  $\mathbf{v}(t)$  be the velocity of marble at time t and let  $\mathbf{V}(t)$  be the velocity of the gun. While the marble is being fired, it is acted on by the spring, by gravity, and by friction forces with the muzzle wall. Let the net force on the marble be  $\mathbf{f}(t)$ . The x equation of motion for the marble is

$$m\frac{dv_x}{dt} = f_x(t).$$

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Formal integration of Eq. (3) gives

$$mv_x(t) = mv_x(0) + \int_0^t f_x dt.$$
 4

The external forces are all vertical, and therefore the horizontal force  $f_x$  on the marble is due entirely to the gun. By Newton's third law, there is a reaction force  $-f_x$  on the gun due to the marble. No other horizontal forces act on the gun, and the horizontal equation of motion for the gun is therefore

$$M \, \frac{dV_x}{dt} = -f_x(t),$$

which can be integrated to give

$$MV_x(t) = MV_x(0) - \int_0^t f_x dt.$$
 5

We can eliminate the integral by combining Eqs. (4) and (5):

$$MV_x(t) + mv_x(t) = MV_x(0) + mv_x(0).$$
 6

We have rediscovered that the horizontal component of momentum is conserved.

What about the motion of the center of mass? Its horizontal velocity is

$$\dot{R}_x(t) = \frac{MV_x(t) + mv_x(t)}{M + m}$$

Using Eq. (6), the numerator can be rewritten to give

$$\dot{R}_x(t) = \frac{MV_x(0) + mv_x(0)}{M + m} = 0,$$

since the system is initially at rest.  $R_x$  is constant, as we expect.

We did not include the small force of air friction. Would the center of mass remain at rest if we had included it?

The essential step in our derivation of the law of conservation of momentum was to use Newton's third law. Thus, conservation of momentum appears to be a natural consequence of newtonian mechanics. It has been found, however, that conservation of momentum holds true even in areas where newtonian mechanics proves inadequate, including the realms of quantum mechanics and relativity. In addition, conservation of momentum can be generalized to apply to systems like the electromagnetic field, which possess momentum but not mass. For these reasons, conservation of momentum is generally regarded as being more fundamental than newtonian mechanics. From this point of view, Newton's third law is a simple consequence of conservation of momentum for interacting particles. For our present purposes it is purely a matter of taste whether we wish to regard Newton's third law or conservation of momentum as more fundamental.

## Example 3.7 Earth, Moon, and Sun—a Three Body System

Newton was the first to calculate the motion of two gravitating bodies. As we shall discuss in Chap. 9, two bodies of mass  $M_1$  and  $M_2$  bound by gravity move so that  $\mathbf{r}_{12}$  traces out an ellipse. The sketch shows the motion in a frame in which the center of mass is at rest. (Note that the center of mass of two particles lies on the line joining them.)





There is no general analytical solution for the motion of three gravitating bodies, however. In spite of this, we can explain many of the important features of the motion with the help of the concept of center of mass.

At first glance, the motion of the earth-moon-sun system appears to be quite complex. In the absence of the sun, the earth and moon would execute elliptical motion about their center of mass. As we shall now show, that center of mass orbits the sun like a single planet, to good approximation. The total motion is the simple result of two simultaneous elliptical orbits.





The center of mass of the earth-moon-sun system lies at

$$\mathbf{R} = \frac{M_{e}\mathbf{R}_{e} + M_{m}\mathbf{R}_{m} + M_{s}\mathbf{R}_{s}}{M_{e} + M_{m} + M_{s}},$$

where  $M_{e}$ ,  $M_{m}$ , and  $M_{s}$  are the masses of the earth, moon, and sun, respectively. The sun's mass is so large compared with the mass of the earth or the moon that  $\mathbf{R}_{0} \approx \mathbf{R}_{s}$ , and to good approximation the center of mass of the three body system lies at the center of the sun. Since external forces are negligible, the sun is effectively at rest in an inertial frame and it is natural to use a coordinate system with its origin at the center of the sun so that  $\mathbf{R} = 0$ .

Let  $r_e$  and  $r_m$  be the positions of the earth and moon with respect to the sun, and let us focus for the moment on the system composed of the earth and moon. Their center of mass lies at

$$\mathbf{R}_{em} = \frac{M_{e}\mathbf{r}_{e} + M_{m}\mathbf{r}_{m}}{M_{e} + M_{m}}$$

The external force on the earth-moon system is the gravitational pull of the sun:

$$\mathbf{F} = -GM_s \left( \frac{M_e}{r_e^2} \, \hat{\mathbf{r}}_e + \frac{M_m}{r_m^2} \, \hat{\mathbf{r}}_m \right) \cdot$$

The equation of motion of the center of mass is

$$(M_e + M_m)\ddot{\mathbf{R}}_{em} = \mathbf{F}.$$

The earth and moon are so close compared with their distance from the sun that we shall not make a large error if we assume  $r_e \approx r_m \approx R_{em}$ . With this approximation,

$$(M_e + M_m)\ddot{\mathbf{R}}_{em} \approx \frac{-GM_s}{R^2} (M_e \hat{\mathbf{r}}_e + M_m \hat{\mathbf{r}}_m)$$
$$= \frac{-GM_s (M_e + M_m)\hat{\mathbf{R}}_{em}}{R^2}.$$

The center of mass of the earth and moon moves like a planet of mass  $M_e + M_m$  about the sun. The total motion is the combination of this elliptical motion and the elliptical motion of the earth and moon about their center of mass, as illustrated on the opposite page. (The drawing is not to scale: the center of mass of the earth-moon system lies within the earth, and the moon's orbit is always concave toward the sun. Also, the plane of the moon's orbit is inclined by 5° with respect to the earth's orbit around the sun.)







## **Center of Mass Coordinates**

Often a problem can be simplified by the right choice of coordinates. The center of mass coordinate system, in which the origin lies at the center of mass, is particularly useful. The drawing illustrates the case of a two particle system with masses  $m_1$  and  $m_2$ . In the initial coordinate system, x, y, z, the particles are located at  $\mathbf{r}_1$  and  $\mathbf{r}_2$  and their center of mass is at

$$=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}$$

We now set up the center of mass coordinate system, x', y', z', with its origin at the center of mass. The origins of the old and new system are displaced by **R**. The center of mass coordinates of the two particles are

$$\begin{aligned} \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} \\ \mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R}. \end{aligned}$$

R

Center of mass coordinates are the natural coordinates for an isolated two body system. For such a system the motion of the center of mass is trivial—it moves uniformly. Furthermore,  $m_1\mathbf{r}'_1 + m_2\mathbf{r}'_2 = 0$  by the definition of center of mass, so that if the motion of one particle is known, the motion of the other particle follows directly. Here is an example.

### Example 3.8 The Push Me-Pull You

Two identical blocks a and b both of mass m slide without friction on a straight track. They are attached by a spring of length l and spring constant k. Initially they are at rest. At t = 0, block a is hit sharply, giving it an instantaneous velocity  $v_0$  to the right. Find the velocities for subsequent times. (Try this yourself if there is a linear air track available—the motion is quite unexpected.)

Since the system slides freely after the collision, the center of mass moves uniformly and therefore defines an inertial frame.

Let us transform to center of mass coordinates. The center of mass lies at

$$R = \frac{mr_a + mr_b}{m + m}$$
$$= \frac{1}{2} (r_a + r_b).$$

As expected, R is always halfway between a and b. The center of mass coordinates of a and b are

$$r'_{a} = r_{a} - R$$
  
=  $\frac{1}{2}(r_{a} - r_{b})$   
 $r'_{b} = r_{b} - R$   
=  $-\frac{1}{2}(r_{a} - r_{b})$   
=  $-r'_{a}$ .

The sketch below shows these coordinates.





The instantaneous length of the spring is  $r_a - r_b - l = r'_a - r'_b - l$ , where *l* is the unstretched length of the spring. The magnitude of the spring force is  $k(r'_a - r'_b - l)$ . The equations of motion in the center of mass system are

$$\begin{split} m\ddot{r}'_{a} &= -k(r'_{a} - r'_{b} - l) \\ m\ddot{r}'_{b} &= +k(r'_{a} - r'_{b} - l), \end{split}$$

where l is the unstretched length of the spring. The form of these equations suggests that we subtract them, obtaining

$$m(\ddot{r}'_a - \ddot{r}'_b) = -2k(r'_a - r'_b - l).$$

It is natural to introduce the departure of the spring from its equilibrium length as a variable. Letting  $u = r'_a - r'_b - l$ , we have

$$m\ddot{u}+2ku=0.$$

This is the equation for simple harmonic motion which we discussed in Example 2.14. The solution is

$$u = A \sin \omega t + B \cos \omega t,$$

where  $\omega = \sqrt{2k/m}$ . Since the spring is unstretched at t = 0, u(0) = 0 which requires B = 0. Furthermore, since  $u = r'_a - r'_b - l = r_a - r_b - l$ , we have at t = 0

$$\dot{u}(0) = v_a(0) - v_b(0)$$
  
=  $A \omega \cos(0)$   
=  $v_0$ ,

so that

 $A = v_0/\omega$ 

and

 $u = (v_0/\omega) \sin \omega t.$ 

Since  $v_a' - v_b' = \dot{u}$ , and  $v_a' = -v_b'$ , we have

$$v_a' = -v_b' = \frac{1}{2}v_0 \cos \omega t.$$

The laboratory velocities are

$$v_a = \dot{R} + v'_a$$
$$v_b = \dot{R} + v'_b.$$

Since  $\dot{R}$  is constant, it is always equal to its initial value

$$\dot{R} = \frac{1}{2} [v_a(0) + v_b(0)] \\ = \frac{1}{2} v_0.$$

Putting these together gives

$$v_a = \frac{v_0}{2} (1 + \cos \omega t)$$
$$v_b = \frac{v_0}{2} (1 - \cos \omega t).$$

The masses move to the right on the average, but they alternately come to rest in a push me-pull you fashion.

# 3.4 Impulse and a Restatement of the Momentum Relation

The relation between force and momentum is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}$$
. 3.16

As a general rv, any law of physics which can be expressed in terms of derivatives can also be written in an integral form. The integral form of the force-momentum relationship is

$$\int_0^t \mathbf{F} \, dt = \mathbf{P}(t) - \mathbf{P}(0). \tag{3.17}$$

The change in momentum of a system is given by the integral of force with respect to time. This form contains essentially the same physical information as Eq. (3.16), but it gives a new way of looking at the effect of a force: the change in momentum is the time integral of the force. To produce a given change in the momentum in time interval t requires only that  $\int_0^t \mathbf{F} dt$  have the appropriate value; we can use a small force acting for much of the time or a large force acting for only part of the interval. The integral  $\int_0^t \mathbf{F} dt$  is called the *impulse*. The word impulse calls to mind a short, sharp shock, as in Example 3.8, where we talked of giving a blow to a mass at rest so that its final velocity was  $v_0$ . However, the physical definition of impulse can just as well be applied to a weak force acting for a long time. Change of momentum depends only on  $\int \mathbf{F} dt$ , independent of the detailed time dependence of the force.

Here are two examples involving impulse.

## Example 3.9 Rubber Ball Rebound

A rubber ball of mass 0.2 kg falls to the floor. The ball hits with a speed of 8 m/s and rebounds with approximately the same speed. High speed photographs show that the ball is in contact with the floor for  $10^{-3}$  s. What can we say about the force exerted on the ball by the floor?

The momentum of the ball just before it hits the floor is  $\mathbf{P}_a = -1.6\hat{\mathbf{k}}$ kg·m/s and its momentum  $10^{-3}$  s later is  $\mathbf{P}_b = +1.6\hat{\mathbf{k}}$  kg·m/s. Since  $\int_{t_a}^{t_b} \mathbf{F} dt = \mathbf{P}_b - \mathbf{P}_a$ ,  $\int_{t_a}^{t_b} \mathbf{F} dt = 1.6\hat{\mathbf{k}} - (-1.6\hat{\mathbf{k}}) = 3.2\hat{\mathbf{k}}$  kg·m/s. Although the exact variation of  $\mathbf{F}$  with time is not known, it is easy to find the average force exerted by the floor on the ball. If the collision time is  $\Delta t = t_b - t_a$ , the average force  $\mathbf{F}_{av}$  acting during the collision is

$$\mathbf{F}_{\mathrm{av}}\,\Delta t\,=\,\int_{t_a}^{t_a\,+\,\Delta t}\,\mathbf{F}\,dt.$$

Since  $\Delta t = 10^{-3}$  s,

$$\mathbf{F}_{av} = \frac{3.2\hat{\mathbf{k}} \text{ kg·m/s}}{10^{-3} \text{ s}} = 3,200\hat{\mathbf{k}} \text{ N}.$$

The average force is directed upward, as we expect. In more familiar units, 3,200 N  $\approx$  720 lb—a sizable force. The instantaneous force on the ball is even larger at the peak, as the sketch shows. If the ball hits a resilient surface, the collision time is longer and the peak force is less.

Actually, there is a weakness in our treatment of the rubber ball rebound. In calculating the impulse  $\int \mathbf{F} dt$ ,  $\mathbf{F}$  is the total force. This includes the gravitational force, which we have neglected. Proceeding more carefully, we write

$$F = F_{floor} + F_{grav}$$
$$= F_{floor} - Mg\hat{k}.$$

The impulse equation then becomes

$$\int_0^{10^{-3}} \mathbf{F}_{\text{floor}} \, dt \, - \, \int_0^{10^{-3}} Mg \hat{\mathbf{k}} \, dt \, = \, 3.2 \hat{\mathbf{k}} \, \text{kg·m/s.}$$

The impulse due to the gravitational force is

$$-\int_0^{10^{-3}} Mg\mathbf{\hat{k}} dt = -Mg\mathbf{\hat{k}} \int_0^{10^{-3}} dt = -(0.2)(9.8)(10^{-3})\mathbf{\hat{k}}$$
  
= -1.96 × 10^{-3}\mathbf{\hat{k}} kg·m/s.

This is less than one-thousandth of the total impulse, and we can neglect it with little error. Over a long period of time, gravity can produce a large change in the ball's momentum (the ball gains speed as it falls, for example). In the short time of contact, however, gravity contributes little momentum change compared with the tremendous force exerted by the floor. Contact forces during a short collision are generally so



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huge that we can neglect the impulse due to other forces of moderate strength, such as gravity or friction.

The last example reveals why a quick collision is more violent than a slow collision, even when the initial and final velocities are identical. This is the reason that a hammer can produce a force far greater than the carpenter could produce on his own; the hard hammerhead rebounds in a very short time compared with the time of the hammer swing, and the force driving the hammer is correspondingly amplified. Many devices to prevent bodily injury in accidents are based on the same considerations, but applied in reverse—they essentially prolong the time of the collision. This is the rationale for the hockey player's helmet, as well as the automobile seat belt. The following example shows what can happen in even a relatively mild collision, as when you jump to the ground.

## Example 3.10 How to Avoid Broken Ankles

Animals, including humans, instinctively reduce the force of impact with the ground by flexing while running or jumping. Consider what happens to someone who hits the ground with his legs rigid.

Suppose a man of mass M jumps to the ground from height h, and that his center of mass moves downward a distance s during the time of collision with the ground. The average force during the collision is

$$F = \frac{Mv_0}{t},$$
 1

where *t* is the time of the collision and  $v_0$  is the velocity with which he hits the ground. As a reasonable approximation, we can take his acceleration due to the force of impact to be constant, so that the man comes uniformly to rest. In this case the collision time is given by  $v_0 = 2s/t$ , or

$$t = \frac{2s}{v_0}$$

Inserting this in Eq. (1) gives

$$F = \frac{M v_0^2}{2s}$$

For a body in free fall for distance h,

$$v_0{}^2 = 2gh.$$

Inserting this in Eq. (2) gives

$$F = Mg \frac{h}{s}$$

If the man hits the ground rigidly in a vertical position, his center of mass will not move far during the collision. Suppose that his center of mass moves 1 cm, which roughly means that his height momentarily decreases by approximately 2 cm. If he jumps from a height of 2 m, the force is 200 times his weight!

Consider the force on a 90-kg ( $\approx$  200-lb) man jumping from a height of 2 m. The force is

F = 90 kg imes 9.8 m/s<sup>2</sup> imes 200

= 1.8  $\times$  10  $^{\rm 5}$  N.

Where is a bone fracture most likely to occur? The force is a maximum at the feet, since the mass above a horizontal plane through the man decreases with height. Thus his ankles will break, not his neck. If the area of contact of bone at each ankle is  $5 \text{ cm}^2$ , then the force per unit area is

$$\frac{F}{A} = \frac{1.8 \times 10^5 \text{ N}}{10 \text{ cm}^2}$$
$$= 1.8 \times 10^4 \text{ N/cm}^2.$$

This is approximately the compressive strength of human bone, and so there is a good probability that his ankles will snap.

Of course, no one would be so rash as to jump rigidly. We instinctively cushion the impact when jumping by flexing as we hit the ground, in the extreme case collapsing to the ground. If the man's center of mass drops 50 cm, instead of 1 cm, during the collision, the force is only one-fiftieth as much as we calculated, and there is no danger of compressive fracture.

## 3.5 Momentum and the Flow of Mass

Analyzing the forces on a system in which there is a flow of mass becomes terribly confusing if we try to apply Newton's laws blindly. A rocket provides the most dramatic example of such a system, although there are many other everyday problems where the same considerations apply—for instance, the problem of calculating the reaction force on a fire hose, or of calculating the acceleration of a snowball which grows larger as it rolls downhill.

There is no fundamental difficulty in handling any of these problems provided that we keep clearly in mind exactly what is included in the system. Recall that  $\mathbf{F} = d\mathbf{P}/dt$  [Eq. (3.12)] was established for a system composed of a certain set of particles. When we apply this equation in the integral form,

$$\int_{t_a}^{t_b} \mathbf{F} \, dt = \mathbf{P}(t_b) - \mathbf{P}(t_a),$$

it is essential to deal with the same set of particles throughout the time interval  $t_{\alpha}$  to  $t_{b}$ ; we must keep track of all the particles that were originally in the system. Consequently, the mass of the system cannot change during the time of interest.

### Example 3.11 Mass Flow and Momentum



A spacecraft moves through space with constant velocity **v**. The spacecraft encounters a stream of dust particles which embed themselves in it at rate dm/dt. The dust has velocity **u** just before it hits. At time *t* the total mass of the spacecraft is M(t). The problem is to find the external force **F** necessary to keep the spacecraft moving uniformly. (In practice, **F** would most likely come from the spacecraft's own rocket engines. For simplicity, we can visualize the source **F** to be completely external—an invisible hand, so to speak.)

Let us focus on the short time interval between t and  $t + \Delta t$ . The drawings below show the system at the beginning and end of the interval.



mass added to the satellite during  $\Delta t$ 

Let  $\Delta m$  denote the mass added to the satellite during  $\Delta t$ . The system consists of M(t) and  $\Delta m$ . The initial momentum is

 $\mathbf{P}(t) = M(t)\mathbf{v} + (\Delta m)\mathbf{u}.$ 

The final momentum is

$$\mathbf{P}(t + \Delta t) = M(t)\mathbf{v} + (\Delta m)\mathbf{v}.$$

The change in momentum is

$$\Delta \mathbf{P} = \mathbf{P}(t + \Delta t) - \mathbf{P}(t)$$
$$= (\mathbf{v} - \mathbf{u}) \Delta m.$$

The rate of change of momentum is approximately

$$\frac{\Delta \mathbf{P}}{\Delta t} = (\mathbf{v} - \mathbf{u}) \frac{\Delta m}{\Delta t}$$

In the limit  $\Delta t \rightarrow 0$ , we have the exact result

$$\frac{d\mathbf{P}}{dt} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt}$$

Since  $\mathbf{F} = d\mathbf{P}/dt$ , the required external force is

$$\mathbf{F} = (\mathbf{v} - \mathbf{u}) \frac{dm}{dt}$$

Note that **F** can be either positive or negative, depending on the direction of the stream of mass. If  $\mathbf{u} = \mathbf{v}$ , the momentum of the system is constant, and  $\mathbf{F} = 0$ .

The procedure of isolating the system, focusing on differentials, and taking the limit may appear a trifle formal. However, the procedure is helpful in avoiding errors in a subject where it is easy to become confused. For instance, a frequent error is to argue that  $\mathbf{F} = (d/dt)(m\mathbf{v}) = m(d\mathbf{v}/dt) + \mathbf{v}(dm/dt)$ . In the last example  $\mathbf{v}$  is constant, and the result would be  $\mathbf{F} = \mathbf{v}(dm/dt)$ rather than  $(\mathbf{v} - \mathbf{u})(dm/dt)$ . The difficulty arises from the fact that there are several contributions to the momentum, so that the expression for the momentum of a single particle,  $\mathbf{p} = m\mathbf{v}$ , is not appropriate. The limiting procedure illustrated in the last example avoids such ambiguities.

## Example 3.12 Freight Car and Hopper

Sand falls from a stationary hopper onto a freight car which is moving with uniform velocity v. The sand falls at the rate dm/dt. How much force is needed to keep the freight car moving at the speed v?

In this case, the initial speed of the sand is 0, and

$$\frac{dP}{dt} = (v - u) \left(\frac{dm}{dt}\right) = v \frac{dm}{dt}.$$

The required force is F = v dm/dt. We can understand why this force is needed by considering in detail just what happens to a sand grain as it lands on the surface of the freight car. What would happen if the surface of the freight car were slippery?



### Example 3.13 Leaky Freight Car

Now consider a related case. The same freight car is leaking sand at the rate dm/dt; what force is needed to keep the freight car moving uniformly with speed **v**?

Here the mass is decreasing. However, the velocity of the sand after leaving the freight car is identical to its initial velocity, and its momentum does not change. Since dP/dt = 0, no force is required. (The sand does change its momentum when it hits the ground, and there is a resulting force on the ground, but that does not affect the motion of the freight car.)

The concept of momentum is invaluable in understanding the motion of a rocket. A rocket accelerates by expelling gas at a high velocity; the reaction force of the gas on the rocket accelerates the rocket in the opposite direction. The mechanism is illustrated by the drawings of the cubical chamber containing gas at high pressure.

The gas presses outward on each wall with the force  $\mathbf{F}_{a}$ . (We show only four walls for clarity.) The vector sum of the  $\mathbf{F}_{a}$ 's is zero, giving zero net force on the chamber. Similarly each wall of the chamber exerts a force on the gas  $\mathbf{F}_{b} = -\mathbf{F}_{a}$ ; the net force on the gas is also zero. In the right hand drawings below, one wall



has been removed. The net force on the chamber is  $F_a$ , to the right. The net force on the gas is  $F_b$ , to the left. Hence the gas accelerates to the left, and the chamber accelerates to the right.

To analyze the motion of the rocket in detail, we must equate the external force on the system, **F**, with the rate of change of momentum,  $d\mathbf{P}/dt$ . Consider the rocket at time t. Between t and  $t + \Delta t$  a mass of fuel  $\Delta m$  is burned and expelled as gas with velocity **u** relative to the rocket. The exhaust velocity **u** is determined by the nature of the propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket.

The sketches below show the system at time t and at time



 $t + \Delta t$ . The system consists of  $\Delta m$  plus the remaining mass of the rocket M. Hence the total mass is  $M + \Delta m$ .

The velocity of the rocket at time t is  $\mathbf{v}(t)$ , and at  $t + \Delta t$ , it is  $\mathbf{v} + \Delta \mathbf{v}$ . The initial momentum is

 $\mathbf{P}(t) = (M + \Delta m)\mathbf{v}$ 

and the final momentum is

$$\mathbf{P}(t + \Delta t) = M(\mathbf{v} + \Delta \mathbf{v}) + \Delta m(\mathbf{v} + \Delta \mathbf{v} + \mathbf{u}).$$

The change in momentum is

$$\Delta \mathbf{P} = \mathbf{P}(t + \Delta t) - \mathbf{P}(t)$$
$$= M \Delta \mathbf{v} + (\Delta m)\mathbf{u}.$$

Therefore,

$$\frac{d\mathbf{P}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{P}}{\Delta t}$$
$$= M \frac{d\mathbf{v}}{dt} + \mathbf{u} \frac{dm}{dt}.$$
3.18

Note that we have defined **u** to be positive in the direction of **v**. In most rocket applications, **u** is negative, opposite to **v**. It is inconvenient to have both m and M in the equation. dm/dt is

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the rate of increase of the exhaust mass. Since this mass comes from the rocket,

$$\frac{dm}{dt} = -\frac{dM}{dt}.$$

Using this in Eq. (3.18), and equating the external force to  $d\mathbf{P}/dt$ , we obtain the fundamental rocket equation

$$\mathbf{F} = M \, \frac{d\mathbf{v}}{dt} - \mathbf{u} \, \frac{dM}{dt}$$
 3.19

It may be useful to point out two minor subtleties in our development. The first is that the velocities have been expressed with respect to an inertial frame, not a frame attached to the rocket. The second is that we took the final velocity of the element of exhaust gas to be  $\mathbf{v} + \Delta \mathbf{v} + \mathbf{u}$  rather than  $\mathbf{v} + \mathbf{u}$ . This is correct (consult Example 3.6 on spring gun recoil if you need help in seeing the reason), but actually it makes no difference here, since either expression yields the same final result when the limit is taken. Here are two examples on rockets.

## Example 3.14 Rocket in Free Space

If there is no external force on a rocket,  $\mathbf{F} = 0$  and its motion is given by

$$M \frac{d\mathbf{v}}{dt} = \mathbf{u} \frac{dM}{dt}$$
  
or  
$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M} \frac{dM}{dt}$$

Generally the exhaust velocity  $\mathbf{u}$  is constant, in which case it is easy to integrate the equation of motion.

$$\int_{t_0}^{t_f} \frac{d\mathbf{v}}{dt} dt = \mathbf{u} \int_{t_0}^{t_f} \frac{1}{M} \frac{dM}{dt} dt$$
$$= \mathbf{u} \int_{M_0}^{M_f} \frac{dM}{M}$$

or

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u} \ln \frac{M_f}{M_0}$$
$$= -\mathbf{u} \ln \frac{M_0}{M_f}$$

If 
$$\mathbf{v}_0 = 0$$
, then  
 $\mathbf{v}_f = -\mathbf{u} \ln \frac{M_0}{M_f}$ .

The final velocity is independent of how the mass is released—the fuel can be expended rapidly or slowly without affecting  $v_f$ . The only important quantities are the exhaust velocity and the ratio of initial to final mass.

The situation is quite different if a gravitational field is present, as shown by the next example.

## Example 3.15 Rocket in a Gravitational Field

If a rocket takes off in a constant gravitational field, Eq. (3.19) becomes

$$M\mathbf{g} = M \frac{d\mathbf{v}}{dt} - \mathbf{u} \frac{dM}{dt},$$

where **u** and **g** are directed down and are assumed to be constant.

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt} + \mathbf{g}.$$

Integrating with respect to time, we obtain

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u} \ln \left( rac{M_f}{M_0} 
ight) + \mathbf{g}(t_f - t_0).$$

Let  $\mathbf{v}_0 = 0$ ,  $t_0 = 0$ , and take velocity positive upward.

$$v_f = u \ln \left(\frac{M_0}{M_f}\right) - gt_f.$$

Now there is a premium attached to burning the fuel rapidly. The shorter the burn time, the greater the velocity. This is why the takeoff of a large rocket is so spectacular—it is essential to burn the fuel as quickly as possible.

## 3.6 Momentum Transport

Nearly everyone has at one time or another been on the receiving end of a stream of water from a hose. You feel a push. If the stream is intense, as in the case of a fire hose, the push can be dramatic—a jet of high pressure water can be used to break through the wall of a burning building.

The push of a water stream arises from the momentum it transfers to you. Unless another external force gives you equal momentum in the opposite direction, off you go. How can a column of water flying through the air exert a force which is every bit as real as a force transmitted by a rigid steel rod? The reason is easy to see if we picture the stream of water as a series of small uniform droplets of mass m, traveling with velocity  $v_0$ . Let the droplets be distance l apart and suppose that the stream is directed against your hand. Assume that the drops collide without rebound and simply run down your arm. Consider the force exerted by your hand on the stream. As each drop hits there is a large force for a short time. Although we do not know the instantaneous force, we can find the impulse  $I_{droplet}$  on each drop due to your hand.

$$I_{\text{droplet}} = \int_{1 \text{ collision}} F dt$$
  
=  $\Delta p$   
=  $m(v_f - v_0)$   
=  $-mv_0$ .

The impulse on your hand is equal and opposite.

$$I_{\text{hand}} = mv_0.$$

The positive sign means that the impulse on the hand is in the same direction as the velocity of the drop. The impulse equals the area under one of the peaks shown in the drawing. If there are many collisions per second, you do not feel the shock of each drop. Rather, you feel the average force  $F_{\rm av}$  indicated by the dashed line in the drawing. The area under  $F_{\rm av}$  during one collision period T (the time between collisions) is identical to the impulse due to one drop.

$$F_{\rm av}T = \int_{1 \text{ collision}} F dt$$

Since  $T = l/v_0$  and  $\int F dt = mv_0$ , the average force is

$$F_{av} = \frac{mv_0}{T}$$
$$= \frac{m}{l} v_0^2$$





Here is another way to find the average force. Consider length L of the stream just about to hit the surface. The number of drops in L is L/l, and since each drop has momentum  $mv_0$ , the total momentum is

$$\Delta p = \frac{L}{l} m v_0.$$

All these drops will strike the wall in time

$$\Delta t = \frac{L}{v_0}.$$

The average force is

$$F_{av} = \frac{\Delta p}{\Delta t}$$
$$= \frac{m}{l} v_0^2.$$

To apply this model to a fluid, consider a stream moving with speed v. If the mass per unit length is  $m/l \equiv \lambda$ , the momentum per unit length is  $\lambda v$  and the rate at which the stream transports momentum to the surface is

$$\frac{dp}{dt} = \lambda v^2. \tag{3.20}$$

If the stream comes to rest at the surface, the force on the surface is

$$F = \lambda v^2. \qquad 3.21$$

## Example 3.16 Momentum Transport to a Surface



A stream of particles of mass m and separation l hits a perpendicular surface with velocity v. The stream rebounds along the original line of motion with velocity v'. The mass per unit length of the incident stream is  $\lambda = m/l$ . What is the force on the surface?

The incident stream transfers momentum to the surface at the rate  $\lambda v^2$ . However, the reflected stream does not carry it away at the rate  $\lambda v'^2$ , since the density of the stream must change at the surface. The number of particles incident on the surface in time  $\Delta t$  is  $v \Delta t/l$  and their total mass is  $\Delta m = mv \Delta t/l$ . Hence, the rate at which mass arrives at the surface is

$$\frac{dm}{dt} = \frac{m}{l}v = \lambda v.$$

The rate at which mass is carried away from the surface is  $\lambda' v'$ . Since mass does not accumulate on the surface, these rates must be equal. Hence  $\lambda' v' = \lambda v$ , and the force on the surface is

$$F = \frac{dp'}{dt} + \frac{dp}{dt} = \lambda' v'^2 + \lambda v^2$$
$$= \lambda v (v' + v).$$

If the stream collides without rebound, then v' = 0 and  $F = \lambda v^2$ , in agreement with our previous result. If the particles undergo perfect reflection, then v' = v, and  $F = 2\lambda v^2$ . The actual force lies somewhere between these extremes.

We can generalize the idea of momentum transport to three dimensions. Consider a stream of fluid which strikes an object and rebounds in some arbitrary direction. For simplicity we assume that the incident stream is uniform and that in time  $\Delta t$ it transports momentum  $\Delta \mathbf{P}_i$ . The direction of  $\Delta \mathbf{P}_i$  is parallel to the initial velocity  $\mathbf{v}_i$  and  $\Delta P_i = \lambda_i v_i^2 \Delta t$ . During the same interval  $\Delta t$  the rebounding stream carries away momentum  $\Delta \mathbf{P}_f$ , where  $\Delta P_f = \lambda_f v_f^2 \Delta t$ ; the direction of  $\Delta \mathbf{P}_f$  is parallel to the final velocity  $\mathbf{v}_f$ . The vectors are shown in the sketch.

The net momentum change of the fluid in  $\Delta t$  is

$$\Delta \mathbf{P}_{\text{fluid}} = \Delta \mathbf{P}_f - \Delta \mathbf{P}_i.$$

The rate of change of the fluid's momentum is

$$\left(\frac{d\mathbf{P}}{dt}\right)_{\text{fluid}} = \left(\frac{d\mathbf{P}}{dt}\right)_f - \left(\frac{d\mathbf{P}}{dt}\right)_i$$

By Newton's second law,  $(d\mathbf{P}/dt)_{\rm fluid}$  equals the force on the fluid due to the object. By Newton's third law, the force on the object due to the fluid is

$$\mathbf{F} = -\left(\frac{d\mathbf{P}}{dt}\right)_{\text{fluid}}$$
$$= \left(\frac{d\mathbf{P}}{dt}\right)_{i} - \left(\frac{d\mathbf{P}}{dt}\right)_{f}$$
$$= \dot{\mathbf{P}}_{i} - \dot{\mathbf{P}}_{f}.$$
 3.22

The sketches illustrate this result.

Unless there is some opposing force, the object will begin to accelerate. If  $\dot{\mathbf{P}}_f = \dot{\mathbf{P}}_i$ , the stream transfers no momentum and  $\mathbf{F} = \mathbf{0}$ .





The force on a moving airplane or boat can be found by considering the effect of a multitude of streams hitting the surface, each with its own velocity. Although the mathematical formalism for analyzing this would lead us too far afield, the physical principle is the same: momentum transport.

### Example 3.17 A Dike at the Bend of a River

The problem is to build a dike at the bend of a river to prevent flooding when the river rises. Obviously the dike has to be strong enough to withstand the static pressure of the river  $\rho gh$ , where  $\rho$  is the density of the water and h is the height from the base of the dike to the surface of the water. However, because of the bend there is an additional pressure, the dynamic pressure due to the rush of water. How does this compare with the static pressure?

We approximate the bend by a circular curve with radius R, and focus our attention on a short length of the curve subtending angle  $\Delta\theta$ . We need only concern ourselves with that section of the river above the base of the dike, and we consider the volume of the river bounded by the bank a, the dike b, and two imaginary surfaces c and d. Momentum is transferred into the volume through surface c and out through surface d at rate  $\dot{P} = \lambda v^2 = \rho A v^2$ . Here A is the cross sectional area of the river lying above the base of the dike, A = hw. (Note that  $\rho A = \lambda = mass$ per unit length of the river.)

However, surfaces c and d are not parallel. The rate of change of the stream's momentum is

$$\dot{\mathbf{P}} = \dot{\mathbf{P}}_d - \dot{\mathbf{P}}_c$$

As we can see from the vector drawing below,  $\dot{\textbf{P}}$  is radially inward and has magnitude

$$|\dot{\mathbf{P}}| = \dot{P} \Delta \theta.$$

The dynamic force on the dike is radially outward, and has the same magnitude,  $\dot{P} \Delta \theta$ . The force is exerted over the area  $(R \Delta \theta)h$ , and the dynamic pressure is therefore







The ratio of dynamic to static pressure is

$$\frac{\text{dynamic pressure}}{\text{static pressure}} = \frac{\rho w v^2}{R} \frac{1}{\rho g h} = \frac{w}{h} \frac{v^2}{Rg}$$
$$= \frac{\text{width}}{\text{depth}} \times \frac{\text{centripetal acceleration}}{g}.$$

For a river in flood with a speed of 10 mi/h (approximately 14 ft/s), a radius of 2,000 ft, a flood height of 3 ft, and a width of 200 ft, the ratio is 0.22, so that the dynamic pressure is by no means negligible. The ratio is even larger near the surface of the river where the static pressure is small.

## Example 3.18 Pressure of a Gas



As a further application of the idea of momentum transport, let us find the pressure exerted by a gas. Although our argument will be somewhat simpleminded, it exhibits the essential ideas and gives the same result as more refined arguments.

Assume that there are n atoms per unit volume of the gas, each having mass m, and that they move randomly. Let us find the force exerted on an area A in the yz plane due to motion of the atoms in the x direction. We make the plausible assumption that it is permissible to neglect motion in the y and z direction, and treat only motion parallel to the x axis. Suppose that all atoms have the same speed,  $v_x$ . The rate at which they hit the surface is  $\frac{1}{2}nAv_x$ , where the factor of  $\frac{1}{2}$  is introduced because the atoms can move in either direction with equal probability. The momentum carried by each atom is  $mv_x$ . It is unlikely that the atoms come to rest after the collision; this would correspond to the freezing of the gas on the walls. On the average, they must leave at the same rate as they arrive, which means that the average change in momentum is  $2mv_x$ . Hence, the rate at which momentum changes due to collisions with area A is

$$\frac{dp}{dt} = \left(\frac{1}{2} nAv_x\right) (2mv_x)$$
$$= mnAv_x^2.$$

The force is

$$F = \frac{dp}{dt}$$
$$= mnAv_x^2$$

and the pressure  $P_x$  on the x surface is

$$P_x = \frac{F}{A}$$
$$= mnv_x^2.$$

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The assumption that  $v_x$  has a fixed value is actually unnecessary. If the atoms have many different instantaneous speeds, then it can be shown that  $v_x^2$  should be replaced by its average  $\overline{v_x^2}$ , and  $P_x = nm\overline{v_x^2}$ . By an identical argument we have  $P_y = mn\overline{v_y^2}$  and  $P_z = nm\overline{v_z^2}$ . However, since the pressure of a gas should not depend on direction, we have  $P_x = P_y = P_z$ , which implies that  $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$ . The mean squared velocity is  $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$ , so that  $\overline{v_x^2} = \frac{1}{3}v^2$  and the pressure is

$$P = \frac{1}{3}nm\overline{v^2}.$$

This is a famous result of the kinetic theory of gas, and it is a crucial point in the argument connecting heat and kinetic energy.

## Note 3.1 Center of Mass

In this Note we shall find the center of mass of some nonsymmetrical objects. These examples are trivial if you have had experience evaluating two or three dimensional integrals. Otherwise, read on.



$$\mathbf{R} = \frac{1}{M} \iint (x\mathbf{\hat{i}} + y\mathbf{\hat{j}})\sigma \ dx \ dy$$

We find M, the mass of the plate, as follows:

$$M = \int_0^b \int_0^a \sigma \, dx \, dy$$
$$= \int_0^b \int_0^a \sigma_0 \frac{x}{a} \frac{y}{b} \, dx \, dy$$

We first integrate over x, treating y as a constant.

$$M = \int_0^b \left( \int_0^a \sigma_0 \frac{x}{a} \frac{y}{b} dx \right) dy$$
$$= \int_0^b \left( \sigma_0 \frac{y}{b} \frac{x^2}{2a} \Big|_{x=0}^{x=a} \right) dy$$
$$= \int_0^b \sigma_0 \frac{y}{b} \frac{a}{2} dy$$
$$= \frac{\sigma_0 a}{2} \frac{y^2}{2b} \Big|_{y=0}^{y=b} = \frac{1}{4} \sigma_0 ab.$$



The x component of **R** is

$$X = \frac{1}{M} \iint x\sigma \, dx \, dy$$
  
=  $\frac{1}{M} \int_0^b \left( \int_0^a x\sigma_0 \frac{xy}{ab} \, dx \right) dy$   
=  $\frac{1}{M} \int_0^b \left( \frac{\sigma_0 y}{ab} \frac{x^3}{3} \Big|_0^a \right) dy$   
=  $\frac{1}{M} \frac{\sigma_0}{ab} \int_0^b \frac{ya^3}{3} \, dy$   
=  $\frac{1}{M} \frac{\sigma_0}{ab} \frac{a^3}{3} \frac{b^2}{2}$   
=  $\frac{4}{\sigma_0 ab} \frac{\sigma_0 a^2 b}{6}$   
=  $\frac{2}{3} a.$ 



z dz



Similarly,  $Y = \frac{2}{3}b$ .

r

dz

2. Find the center of mass of a uniform solid hemisphere of radius Rand mass M.

From symmetry it is apparent that the center of mass lies on the zaxis, as illustrated. Its height above the equatorial plane is

$$Z = \frac{1}{M} \int z \, dM.$$

The integral is over three dimensions, but the symmetry of the situation lets us treat it as a one dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider the circular disk of radius r and thickness dz. Its volume is  $dV = \pi r^2 dz$ , and its mass is  $dM = \rho \ dV = (M/V)(dV)$ , where  $V = \frac{2}{3}\pi R^3$ . Hence,

$$Z = \frac{1}{M} \int \frac{M}{V} z \, dV$$
$$= \frac{1}{V} \int_{z=0}^{R} \pi r^2 z \, dz.$$

To evaluate the integral we need to find r in terms of z. Since  $r^2 = R^2 - z^2$ , we have







$$= \frac{\pi}{V} \left( \frac{1}{2} R^4 - \frac{1}{4} R^4 \right)$$
$$= \frac{\frac{1}{4} \pi R^4}{\frac{2}{3} \pi R^3}$$
$$= \frac{3}{8} R.$$

### Problems

**ns** 3.1 The density of a thin rod of length *l* varies with the distance *x* from one end as  $\rho = \rho_0 x^2/l^2$ . Find the position of the center of mass.

Ans. X = 3l/4

3.2 Find the center of mass of a thin uniform plate in the shape of an equilateral triangle with sides a.

3.3 Suppose that a system consists of several bodies, and that the position of the center of mass of each body is known. Prove that the center of mass of the system can be found by treating each body as a particle concentrated at its center of mass.

3.4 An instrument-carrying projectile accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the earth's curvature.

3.5 A circus acrobat of mass M leaps straight up with initial velocity  $v_0$  from a trampoline. As he rises up, he takes a trained monkey of mass m off a perch at a height h above the trampoline.

What is the maximum height attained by the pair?

3.6 A light plane weighing 2,500 lb makes an emergency landing on a short runway. With its engine off, it lands on the runway at 120 ft/s. A hook on the plane snags a cable attached to a 250-lb sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is 0.4, and if the plane's brakes give an additional retarding force of 300 lb, how far does the plane go before it comes to a stop?

3.7 A system is composed of two blocks of mass  $m_1$  and  $m_2$  connected by a massless spring with spring constant k. The blocks slide on a frictionless plane. The unstretched length of the spring is l. Initially  $m_2$ is held so that the spring is compressed to l/2 and  $m_1$  is forced against a stop, as shown.  $m_2$  is released at t = 0.

Find the motion of the center of mass of the system as a function of time.



3.8 A 50-kg woman jumps straight into the air, rising 0.8 m from the ground. What impulse does she receive from the ground to attain this height?

3.9 A freight car of mass M contains a mass of sand m. At t = 0 a constant horizontal force F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at constant rate dm/dt. Find the speed of the freight car when all the sand is gone. Assume the freight car is at rest at t = 0.

3.10 An empty freight car of mass M starts from rest under an applied force F. At the same time, sand begins to run into the car at steady rate b from a hopper at rest along the track.

Find the speed when a mass of sand, m, has been transferred. (*Hint:* There is a way to do this problem in one or two lines.)

Ans. clue. If M = 500 kg, b = 20 kg/s, F = 100 N, then v = 1.4 m/s at t = 10 s

3.11 Material is blown into cart A from cart B at a rate b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity as cart B, u. At the moment of interest, cart A has mass M and velocity v, as shown. Find dv/dt, the instantaneous acceleration of A.

3.12 A sand-spraying locomotive sprays sand horizontally into a freight car as shown in the sketch. The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate dm/dt = 10 kg/s with a velocity of 5 m/s relative to the locomotive. The car starts from rest with an initial mass of 2,000 kg. Find its speed after 100 s.

3.13 A ski tow consists of a long belt of rope around two pulleys, one at the bottom of a slope and the other at the top. The pulleys are driven by a husky electric motor so that the rope moves at a steady speed of 1.5 m/s. The pulleys are separated by a distance of 100 m, and the angle of the slope is 20°.

Skiers take hold of the rope and are pulled up to the top, where they release the rope and glide off. If a skier of mass 70 kg takes the tow every 5 s on the average, what is the average force required to pull the rope? Neglect friction between the skis and the snow.

3.14 N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

a. What is the final velocity of the flatcar if all the men jump at the same time?

b. What is the final velocity of the flatcar if they jump off one at a time? (The answer can be left in the form of a sum of terms.)







Constant