1 PHYS230A Problem Set 1 Solutions

1.1 Problem 1

a) One obtains the equations of motion by varying the action w.r.t. X^a.

$$\delta S = -m \int d\tau \delta(\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}) \tag{1.1}$$

$$= -m \int d\tau \frac{1}{2\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}} \left(-\delta G_{ab}\partial_\tau X^a \partial_\tau X^b - 2G_{ab}\partial_\tau X^a \partial_\tau \delta X^b\right) \tag{1.2}$$

$$= -m \int d\tau \frac{1}{2\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}} \left(-\partial_c G_{ab}\delta X^c \partial_\tau X^a \partial_\tau X^b - 2G_{ab}\partial_\tau X^a \partial_\tau \delta X^b\right) \tag{1.3}$$

We make the following definition to simplify the equations,

$$\sqrt{-g} \equiv \sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}$$

Integrating the second term in the δS we obtain

$$\frac{-2G_{ab}\partial_\tau X^a \partial_\tau \delta X^b}{\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}} \to \frac{2\partial_\tau (G_{ab}\partial_\tau X^a)\delta X^b}{\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}} + 2G_{ab}\partial_\tau X^a \delta X^b \partial_\tau \left(\frac{1}{\sqrt{-G_{ab}\partial_\tau X^a \partial_\tau X^b}}\right) \tag{1.4}$$

$$\to \frac{1}{\sqrt{-g}} \left(2G_{ab}\partial_\tau^2 X^a \delta X^b + 2\partial_c G_{ab}\partial_\tau X^c \partial_\tau X^a \delta X^b - 2G_{ab}\partial_\tau X^a \delta X^b \frac{\partial_\tau g}{2g}\right) \tag{1.5}$$

Plugging 1.5 back into 1.3 we obtain the condition for $\delta S = 0$ to be,

$$-\partial_c G_{ab}\partial_\tau X^a \partial_\tau X^b + 2G_{ac}\partial_\tau^2 X^a + 2\partial_b G_{ac}\partial_\tau X^b \partial_\tau X^a - 2G_{ac}\partial_\tau X^a \frac{\partial_\tau g}{2g} = 0 \tag{1.6}$$

Dividing by 2 and multiplying by G^{cd} we get,

$$\partial_\tau^2 X^d + \frac{1}{2}G^{dc}(2\partial_b G_{ac} - \partial_c G_{ab})\partial_\tau X^a \partial_\tau X^b = \partial_\tau X^d \frac{\partial_\tau g}{2g} \tag{1.7}$$

$$\to \partial_\tau^2 X^d + \frac{1}{2}G^{dc}(\partial_b G_{ac} + \partial_a G_{bc} - \partial_c G_{ab})\partial_\tau X^a \partial_\tau X^b = \partial_\tau X^d \frac{\partial_\tau g}{2g} \tag{1.8}$$

$$\to \partial_\tau^2 X^d + \Gamma^d_{ab}\partial_\tau X^a \partial_\tau X^b = \partial_\tau X^d \frac{\partial_\tau g}{2g} \tag{1.9}$$

where we used the symmetry in a, b in 1.8. This is the form of the geodesic equation or better the parallel transport equation of transporting the tangent vector of a line along itself. The geodesic equation is defined for the specific parameterization where $\partial g = 0$ which is true for any affinely parameterized curve, where an affine parameter is defined as any λ s.t. $\lambda = a\tau + b$ where τ is the proper time.

b) The variation of the kinetic term is already done in part a. Let's look at the gauge field term.

$$\delta(A_a\partial_\tau X^a) = \delta A_a\partial_\tau X^a + A_a\partial_\tau\delta X^a \tag{1.10}$$
$$= \partial_b A_a\delta X^b\partial_\tau X^a - \partial_\tau A_a\delta X^a \tag{1.11}$$
$$= \partial_b A_a\delta X^b\partial_\tau X^a - \partial_b A_a\partial_\tau X^b\delta X^a \tag{1.12}$$
$$= (\partial_a A_b - \partial_b A_a)\partial_\tau X^b\delta X^a \tag{1.13}$$
$$= F_{ab}\partial_\tau X^b\delta X^a \tag{1.14}$$

Combining this with the previous variation we obtain,

$$-m\frac{\partial_\tau^2 X^a}{\sqrt{-\partial_\tau X^a\partial_\tau X_a}} + qF^a{}_b\partial_\tau X^b = 0 \tag{1.15}$$

Choosing to parameterize with proper time, we set $\sqrt{-\partial_\tau X^a\partial_\tau X_a} = 1$. The equation of motion becomes,

$$m\partial_\tau^2 X^a = qF^a{}_b\partial_\tau X^b \tag{1.16}$$

Which is the expected equation of motion of a charged particle.

1.2 Problem 2

a) NG action is given by

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-det(\partial_a X^\mu\partial_b X_\mu)} \tag{1.17}$$

Working in static gauge:

$$\partial_\tau X^\mu\partial_\tau X_\tau = -1 + \partial_\tau X^i\partial_\tau X_i, \qquad \partial_\sigma X^\mu\partial_\sigma X_\tau = 1 + \partial_\sigma X^i\partial_\sigma X_i \tag{1.18}$$
$$\partial_\tau X^\mu\partial_\sigma X_\mu = \partial_\tau X^i\partial_\sigma X_i \tag{1.19}$$

In this gauge the action takes the form,

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{1 - \partial_\tau X^i\partial_\tau X_i + \partial_\sigma X^i\partial_\sigma X_i - (\partial_\tau X^i\partial_\tau X_i)(\partial_\sigma X^j\partial_\sigma X_j) + (\partial_\sigma X^i\partial_\tau X_i)^2} \tag{1.20}$$

b) Expanding to second order we obtain,

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\left(1 - \frac{1}{2}\partial_\tau X^i\partial_\tau X_i + \frac{1}{2}\partial_\sigma X^i\partial_\sigma X_i\right) \tag{1.21}$$

after taking $\partial_\tau X \ll 1$ and $\partial_\sigma X \ll 1$. It is clear that the condition $\partial_\tau X \ll 1$ is that of a nonrelativistic velocity. As for the condition $\partial_\sigma X \ll 1$, one views the fields X^μ as fields which have excitations as harmonic oscillators whose energy sloshes between kinetic and potential terms. Thus one expects the potential term $\partial_\sigma X \partial_\sigma X$ to be of the same order as $\partial_\tau X \partial_\tau X$ and so we take $\partial_\sigma X \ll 1$.

c) The rest mass per unit length is the coefficient of the '1' in the action.

$$m/l = \frac{1}{2\pi\alpha'}$$

d) The mass per unit length from the kinetic term is the coefficient of $\frac{1}{2}\partial_\tau X^i \partial_\tau X_i$ and thus

$$m_k/l = \frac{1}{2\pi\alpha'}$$

1.3 Problem 3

We need to regularize the sum,

$$\sum_{n=1}^{\infty}(n-\lambda)e^{-\epsilon(n-\lambda)} = -\frac{d}{d\epsilon}e^{\epsilon\lambda}\sum_{n=1}^{\infty}e^{-\epsilon n} \tag{1.22}$$

$$= -\frac{d}{d\epsilon}e^{\epsilon\lambda}\left(\frac{1}{e^{\epsilon}-1}\right) \tag{1.23}$$

$$= -\frac{d}{d\epsilon}\frac{1}{\epsilon}(1+\epsilon\lambda+\frac{1}{2}\epsilon^2\lambda^2)(1-\epsilon/2+\epsilon^2/12) \tag{1.24}$$

$$= -\frac{d}{d\epsilon}\frac{1}{\epsilon}(1-\epsilon/2+\epsilon^2/12+\epsilon\lambda-\epsilon^2\lambda/2+\epsilon^2\lambda^2/2+O(\epsilon^3)) \tag{1.25}$$

$$= \frac{1}{\epsilon}-\frac{1}{12}+\frac{\lambda}{2}-\frac{\lambda^2}{2} \tag{1.26}$$

Dropping the divergent term because of symmetry considerations, we get

$$\sum_{n=1}^{\infty}(n-\lambda) = -\frac{1}{12}+\frac{\lambda}{2}-\frac{\lambda^2}{2} \tag{1.27}$$

1.4 Problem 4

We consider an open string connecting two Dp-branes y distance apart in the x^25 direction. The mode expansions are,

$$X^i = x^i + 2\alpha' p^i \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\tau} cos(n\sigma) \tag{1.28}$$

$$X^\mu = y\frac{\sigma}{\pi}\delta^{25,\mu} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau} sin(n\sigma) \tag{1.29}$$

where the mode expansion 1.28 is that for the Neumann-Neumann boundary conditions. As for the Dirichlet-Dirichlet conditions the mode expansions take the form 1.29. Note the i difference between the mode expansions to ensure reality.

Calculating the M^2 for the NN case is already done in the notes/book which produces(for the ground state),

$$M_{NN}^2 = -\frac{(\#\text{ of NN directions})}{24\alpha'} = -\frac{p-1}{24\alpha'} \tag{1.30}$$

We compute the relative quantities required for the M^2,

$$\partial_\tau X^i = 2\alpha' p^i + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^i e^{-in\tau}cos(n\sigma) \tag{1.31}$$

$$\partial_\sigma X^i = -i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^i e^{-in\tau}sin(n\sigma) \tag{1.32}$$

$$\partial_\tau X^\mu = -i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}sin(n\sigma) \tag{1.33}$$

$$\partial_\sigma X^\mu = y\frac{1}{\pi}\delta^{25,\mu} + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}cos(n\sigma) \tag{1.34}$$

Notice that 1.33 = 1.32 and that 1.31 = 1.34 provided that $2\alpha' p^i \leftrightarrow \frac{y}{\pi}$ for the 25 direction. Ignoring the y term for a moment, we see that the contribution from the DD modes will be,

$$M_{DD}^2 = -\frac{(\#\text{ of DD directions})}{24\alpha'} = -\frac{25-p}{24\alpha'} \tag{1.35}$$

Now we focus on the y term. We know that in the NN case, as done in the notes, the term $2p^+p^-$ contributes the term $(p^i)^2$ which gets subtracted away in the full equation of M^2 as

$$M^2 = 2p^+p^- - (p^i)^2$$

In the case of the y there is nothing to subtract it off, and thus we add it as a contribution to the M^2 to get (after combining it with eqs 1.30 & 1.35),

$$M^2 = \frac{y^2}{4\pi^2(\alpha')^2} - \frac{1}{\alpha'} \tag{1.36}$$

We note that there is no p^μ to include in the definition of M^2 since the strings are stuck in those directions. If one calculates these momenta one finds

$$p^\mu \sim e^{-i\tau}$$

which time averages to zero. Any instantaneous value of the momentum is just the wiggling of the string back and forth normal to the brane.

Finally, he condition for $M^2 \geq 0$ is

$$y^2 \geq 4\pi^2\alpha'$$

1.5 Problem 5

a) We want the mode expansion with the conditions

$$X(\tau,0) = \partial_\sigma X(\tau,\pi) = 0$$

This is given by,

$$X^i = \sqrt{2\alpha'} \sum_{n\in Z/2,\neq 0} \frac{1}{n}\alpha^i_n sin(n\sigma) \tag{1.37}$$

b) We consider the set-up of Dp and Dp' branes placed on top of each other, with $p' < p$. We are interested in the strings that connect the two branes. We have $p' - 1$ coordinates with NN boundary conditions, $25 - p$ coordinates with DD conditions, and $p - p'$ coordinates with ND mixed boundary conditions. Their respective mode expansions are

$$X^i_{NN} = x^i + 2\alpha' p^i \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha^i_n e^{-in\tau} cos(n\sigma) \tag{1.38}$$

$$X^i_{DD} = \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha^i_n e^{-in\tau} sin(n\sigma) \tag{1.39}$$

$$X^i_{ND} = \sqrt{2\alpha'} \sum_{n\in Z_{odd}/2} \frac{1}{n}\alpha^i_n sin(n\sigma) \tag{1.40}$$

We see that from the equivalence of the forms of the DD and ND mode expansions that the only difference in M^2 calculations will be the sum $\sum n$ which will be over the positive half integers,

$$\sum_{n\in Z_{odd}/2>0} n = \frac{1}{24} \tag{1.41}$$

We find that the M^2 is given by

$$M^2 = -\frac{25-p}{24\alpha'} - \frac{p'-1}{24\alpha'} + \frac{p-p'}{48\alpha'} \tag{1.42}$$

$$= \frac{-24 + 3/2(p-p')}{24\alpha'} \tag{1.43}$$

Thus the condition for $M^2 \geq 0$ translates into

$$p - p' \geq 16$$