1 PHYS230A Problem Set 2 Solutions

1.1 Problem 1

a.) We consider the Polyakov action with the extra cosmological constant term,

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu - \mu\int d\tau d\sigma\sqrt{-\gamma} \tag{1.1}$$

We consider the variation with respect to γ^{ab}. We define $h_{ab} = \partial_a X^\mu \partial_b X_\mu$ to simplify the equations. Using the relation $\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{ab}\delta\gamma^{ab}$, we obtain the equations of motion,

$$-\frac{1}{4\pi\alpha'}\left(h_{ab} - \frac{1}{2}(\gamma^{cd}h_{ab})\gamma_{ab}\right) + \frac{\mu}{2}\gamma_{ab} = 0 \tag{1.2}$$

Contracting with γ^{ab} we obtain,

$$-\frac{1}{4\pi\alpha'}\left(\gamma^{ab}h_{ab} - \frac{1}{2}(\gamma^{cd}h_{cd})\gamma^2\right) + \frac{\mu}{2}\gamma^2 = 0 \tag{1.3}$$

$$\rightarrow (\gamma h)\left(1 - \frac{1}{2}\gamma^2\right) = 2\pi\alpha'\mu\gamma^2 \tag{1.4}$$

where $\gamma h = \gamma^{ab}h_{ab}, \gamma^2 = \gamma^{ab}\gamma_{ab}$. We notice from 1.4 that if we take $\gamma^2 = 2$ which must be the case (for an invertible matrix) we find that $\mu = 0$, which is not the desired solution. Thus we must take $\gamma_{ab} = 0$ which by equation 1.2 implies that $h_{ab} = 0$. This is the undesired trivial solution.

b.) Let's consider this action for a higher dimensional object. Equation 1.4 gives,

$$(\gamma h)\left(1 - \frac{1}{2}d\right) = 2\pi\alpha'\mu d \tag{1.5}$$

$$\rightarrow (\gamma h) = -4\pi\alpha'\mu\frac{d}{d-2} \tag{1.6}$$

Plugging this into 1.2 we obtain,

$$h_{ab} = \left(-\frac{4\pi\alpha'\mu}{d-2}\right)\gamma_{ab} \tag{1.7}$$

$$\sqrt{-h} = (-\frac{4\pi\alpha'\mu}{d-2})^{d/2}\sqrt{-\gamma} \tag{1.8}$$

Putting everything back into the action we find,

$$S = -\frac{1}{4\pi\alpha'}(-4\pi\alpha'\mu\frac{d}{d-2})(-\frac{4\pi\alpha'\mu}{d-2})^{-d/2}\int\sqrt{-h} - \mu(-\frac{4\pi\alpha'\mu}{d-2})^{-d/2}\int\sqrt{-h} \tag{1.9}$$

$$= -\mu(-\frac{4\pi\alpha'\mu}{d-2})^{-d/2}\left(-\frac{d}{d-2} + 1\right)\int\sqrt{-h} \tag{1.10}$$

$$= -\mu(-\frac{4\pi\alpha'\mu}{d-2})^{-d/2}(\frac{-2}{d-2})\int\sqrt{-h} \tag{1.11}$$

We find that $S \propto \int\sqrt{-h}$, the world volume.

1.2 Problem 2

We need to show that,

$$[L'_m, L'_n] = (m-n)L'_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \tag{1.12}$$

Where C is the central charge and

$$L'_m = L_m + (m+1)\nu_\mu \alpha^\mu_m$$

where L_m are the usual Virasoro generators with central charge D, the spacetime dimension, as seen in class. Without any evaluation we have,

$$[L'_m, L'_n] = [L_m, L_n] + (n+1)\nu_\mu[L_m, \alpha^\mu_n] + (m+1)\nu_\mu[\alpha^\mu_m, L_n] + (m+1)(n+1)\nu_\mu\nu_\sigma[\alpha^\mu_m, \alpha^\sigma_n] \tag{1.13}$$

Computing this term by term we have:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} && (1.14)\\
[\alpha^\mu_m, \alpha^\sigma_n] &= m\eta^{\mu\nu}\delta_{m+n,0} && (1.15)\\
[L_m, \alpha^\mu_n] &= \frac{1}{2}\sum_{p=-\infty}^{\infty}[\alpha^\nu_{n-p}\alpha_{\nu\ p}, \alpha^\mu_m] && (1.16)\\
&= \frac{1}{2}\sum_{p=-\infty}^{\infty}\left([\alpha^\nu_{n-p}, \alpha^\mu_m]\alpha_{\nu\ p} + \alpha^\nu_{n-p}[\alpha_{\nu\ p}, \alpha^\mu_m]\right) && (1.17)\\
&= \frac{1}{2}\sum_{p=-\infty}^{\infty}\left(-m\eta^{\mu\nu}\delta_{m+n-p,0}\alpha_{\nu\ p} - m\alpha^\nu_{n-p}\delta^\mu_\nu\delta_{p+m,0}\right) && (1.18)\\
&= -\frac{1}{2}\left(m\alpha^\mu_{m+n} + m\alpha^\mu_{m+n}\right) && (1.19)\\
&= -m\alpha^\mu_{m+n} && (1.20)
\end{aligned}$$

Putting everything together we get,

$$\begin{aligned}
[L'_m, L'_n] &= (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} + m(m+1)(n+1)\nu^2\delta_{m+n,0} && (1.21)\\
&\qquad + (m(m+1) - n(n+1))\nu_\mu\alpha^\mu_{m+n} && (1.22)\\
&= (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} - (m^3 - m)\nu^2\delta_{m+n,0} && (1.23)\\
&\qquad + (m-n)(m+n+1)\nu_\mu\alpha^\mu_{m+n} && (1.24)\\
&= (m-n)L'_{m+m} + \frac{D - 12\nu^2}{12}(m^3 - m)\delta_{m+n,0} && (1.25)
\end{aligned}$$

We find that the L'_m do indeed satisfy the Virasoro algebra with central charge $D - 12\nu^2$.

1.3 Problem 3

We need to show that the following operators satisfy the Virasoro Algebra, and find the central charge:

$$L_m = \sum_{n=-\infty}^{\infty} (2m-n) b_n c_{m-n}, \quad m \neq 0 \tag{1.26}$$

$$L_0 = -1 + \sum_{n=1}^{\infty} n(b_{-n}c_n + c_{-n}b_n) \tag{1.27}$$

where b_n and c_n satisfy,

$$\{b_m, c_n\} = \delta_{m+n,0}, \quad \{b_m, b_n\} = \{c_m, c_n\} = 0 \tag{1.28}$$

First we consider $[L_m, L_n]$ for $m, n \neq 0$,

$$[L_m, L_n] = \sum_{q,p} (2m-q)(2n-p)[b_q c_{m-q}, b_p c_{n-p}] \tag{1.29}$$

$$= \sum_{q,p} (2m-q)(2n-p)\left(b_q c_{m-q} b_p c_{n-p} - b_p c_{n-p} b_q c_{m-q}\right) \tag{1.30}$$

$$= \sum_{q,p} (2m-q)(2n-p)\left(b_q c_{n-p}\delta_{p+m-q,0} - b_q b_p c_{m-q} c_{n-p} - b_p c_{n-p} b_q c_{m-q}\right) \tag{1.31}$$

$$= \sum_{q,p} (2m-q)(2n-p)\left(b_q c_{n-p}\delta_{p+m-q,0} - b_p b_q c_{n-p} c_{m-q} - b_p c_{n-p} b_q c_{m-q}\right) \tag{1.32}$$

$$= \sum_{q,p} (2m-q)(2n-p)\left(b_q c_{n-p}\delta_{p+m-q,0} - b_p c_{m-q}\delta_{q+n-p,0} + b_p c_{n-p} b_q c_{m-q} - b_p c_{n-p} b_q c_{m-q}\right) \tag{1.33}$$

$$= \sum_{q,p} (2m-q)(2n-p)\left(b_q c_{n-p}\delta_{p+m-q,0} - b_p c_{m-q}\delta_{q+n-p,0}\right) \tag{1.34}$$

$$= \sum_{p} (2m-p-m)(2n-p)\left(b_{p+m} c_{n-p}\right) + (2m-p+n)(2n-p)\left(-b_p c_{m+n-p}\right) \tag{1.35}$$

$$= \sum_{p} (2m-p)(2n+m-p)\left(b_p c_{n-p+m}\right) + (2m-p+n)(2n-p)\left(-b_p c_{m+n-p}\right) \tag{1.36}$$

$$= \sum_{p} \left((2m-p)(2n+m-p) - (2m-p+n)(2n-p)\right) b_p c_{n-p+m} \tag{1.37}$$

$$= (m-n)\sum_{p} (m+n-p) b_p c_{n-p+m} \tag{1.38}$$

$$= (m-n) L_{m+n} \tag{1.39}$$

As required. Next we compute $[L_m, L_0]$,

$$[L_m, L_0] = \sum_{n=-\infty, q=1}^{\infty} (2m-n)q\left([b_n c_{m-n}, b_{-q}c_q] + [b_n c_{m-n}, c_{-q}b_q]\right) \tag{1.40}$$

Using the steps from the previous calculation we obtain,

$$[L_m, L_0] = \sum_{n=-\infty, q=1}^{\infty} (2m-n)q\left([b_n c_{m-n}, b_{-q}c_q] + [b_n c_{m-n}, c_{-q}b_q]\right) \tag{1.41}$$

$$= \sum_{n,q}(2m-n)q\left(b_n c_q \delta_{m-n-q,0} - b_{-q}c_{m-n}\delta_{n+q,0} - b_n c_{-q}\delta_{m-n+q,0} + b_q c_{m-n}\delta_{n-q,0}\right) \tag{1.42}$$

$$= \sum_{n,q=-\infty}^{\infty} (2m-n)q\left(b_n c_q \delta_{m-n-q,0} - b_{-q}c_{m-n}\delta_{n+q,0}\right) \tag{1.43}$$

$$= \sum_{n=-\infty}^{\infty} (2m-n)\left((m-n)b_n c_{m-n} + n b_n c_{m-n}\right) \tag{1.44}$$

$$= \sum_{n=-\infty}^{\infty} (2m-n)\left((m)b_n c_{m-n}\right) \tag{1.45}$$

$$= m\sum_{n=-\infty}^{\infty} (2m-n)\left(b_n c_{m-n}\right) \tag{1.46}$$

$$= mL_m \tag{1.47}$$

As required. So far we have found that

$$[L_m, L_n] = (m-n)L_{m+n} + (\propto \text{Central Charge})$$

and we need to find the central charge. We do this by acting the operator $[L_m, L_{-m}]$, for $m > 0$, on the state $|0,0\rangle$. We pick the particular state that,

$$b_0|0,0\rangle = b_q|0,0\rangle = c_q|0,0\rangle = 0, \quad q > 0$$

Firstly consider $L_m|0,0\rangle$,

$$L_m|0,0\rangle = \sum_{q=-\infty}^{\infty} (2m-q)b_q c_{m-q}|0,0\rangle \tag{1.48}$$

$$= -\sum_{q=-\infty}^{\infty} (2m-q)c_{m-q}b_q|0,0\rangle \tag{1.49}$$

We require $m - q \leq 0$, from the first relation, and $q < 0$ from the second to obtain a non vanishing answer. Since there is no q that satisfies this, we can deduce that,

$$L_m|0,0\rangle = 0 \tag{1.50}$$

Next, we look at $L_{-m}|0,0\rangle$,

$$L_{-m}|0,0\rangle = \sum_{q=-\infty}^{\infty} (-2m-q) b_q c_{-m-q} |0,0\rangle \tag{1.51}$$

$$= -\sum_{q=-\infty}^{\infty} (2m-q) c_{-m-q} b_q |0,0\rangle \tag{1.52}$$

The conditions in q now are: $q < 0$ and $q \geq -m$. So,

$$L_{-m}|0,0\rangle = \sum_{q=-m}^{-1} (-2m-q) b_q c_{-m-q} |0,0\rangle \tag{1.53}$$

Now lets go back and compute $[L_m, L_{-m}]|0,0\rangle$,

$$[L_m, L_{-m}]|0,0\rangle = L_m L_{-m} |0,0\rangle \tag{1.54}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{q=-m}^{-1} (2m-p)(-2m-q) b_p c_{m-p} b_q c_{-m-q} |0,0\rangle \tag{1.55}$$

We notice here that if $m \neq q-p$ we can commute $b_p c_{m-p}$ through to act directly on the state, which we showed in 1.49 that this would vanish. We thus set $m = p-q$ and obtain,

$$[L_m, L_{-m}]|0,0\rangle = \sum_{q=-m}^{-1} (m-q)(-2m-q) b_{q+m} c_{-q} b_q c_{-m-q} |0,0\rangle \tag{1.56}$$

$$= \sum_{q=-m}^{-1} (m-q)(-2m-q)(b_{q+m} c_{-(m+q)} + b_{q+m} b_q c_{-m-q} c_{-q})|0,0\rangle \tag{1.57}$$

$$= \sum_{q=-m}^{-1} (m-q)(-2m-q)(1 - c_{-(m+q)} b_{q+m} + b_{q+m} b_q c_{-m-q} c_{-q})|0,0\rangle \tag{1.58}$$

The last two terms annihilate the state by the rules mentioned above. The remaining result is,

$$[L_m, L_{-m}]|0,0\rangle = \sum_{q=-m}^{-1} (m-q)(-2m-q)|0,0\rangle \tag{1.59}$$

$$= \sum_{q=-m}^{-1} \left(-2m^2 + qm + q^2\right)|0,0\rangle \tag{1.60}$$

$$= -\frac{1}{6} m(-1 + 13m^2)|0,0\rangle \tag{1.61}$$

We compare this to the action of

$$[L_m, L_-m] = (2m)L_0 + k$$

on the same state (also for $m > 0$). From the form of L_0 we deduce that $L_0|0,0\rangle = -1|0,0\rangle$. We thus obtain,

$$[L_m, L_-m]|0,0\rangle = (2m)L_0|0,0\rangle + k|0,0\rangle \tag{1.62}$$

$$= (-2m + k)|0,0\rangle \tag{1.63}$$

Equation the expressions in equations 1.61 and1.63 we find that

$$k = \frac{-26}{12}(m^3 - m)$$

and thus

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{-26}{12}(m^3 - m) \tag{1.64}$$

We find that the $L's$ satisfy a Virasoro algebra with central charge -26.