1 PHYS230A Problem Set 3 Solutions

1.1 Problem 1

We need to evaluate the OPE

$$T_{zz}(z) : e_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu e^{ikX(0,0)} : \tag{1.1}$$

where

$$T_{zz}(z) = -\frac{1}{\alpha'} : \partial_z X^\alpha \partial_z X_\alpha : \tag{1.2}$$

and find the weight of the operator and the conditions of the $e_{\mu\nu}$ for it to be a tensor.

We use the following evaluated forms for the contractions between the different operators appearing in the problem:

$$(X^\alpha(z), X^\mu(z')) = -\frac{\alpha'}{2} ln|z - z'|^2 \eta^{\alpha\mu} \tag{1.3}$$

which results in,

$$(\partial_z X^\alpha(z), \partial_z X^\mu(0)) = -\frac{\alpha'}{2}\frac{1}{z^2}\eta^{\alpha\mu} \tag{1.4}$$

$$(\partial_z X^\alpha(z), \partial_{\bar{z}} X^\mu(0)) = 0 \tag{1.5}$$

$$(\partial_z X^\alpha(z), e^{ikX(0,0)}) = -\frac{\alpha'}{2}\frac{1}{z} ik^\alpha : ...e^{ikX(0,0)} : \tag{1.6}$$

(1.6) comes from reducing one power of X from each term in the expansion of the exponential. The corresponding power of each term fixes the coefficient the term in the expansion which reproduces to the exponential.

Using these contractions in (1.1) we obtain,

$$\begin{aligned}
-\frac{e_{\mu\nu}}{\alpha'}[: \partial_z X^\alpha(z)\partial_z X_\alpha(z)\partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : +2(-\frac{\alpha'}{2}\frac{1}{z^2}\eta^{\alpha\mu}) : \partial_z X_\alpha(z)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : \\
+2(-\frac{\alpha'}{2}\frac{1}{z} ik^\alpha) : \partial_z X_\alpha(z)\partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : +(-\frac{\alpha'}{2}\frac{1}{z} ik^\alpha)(-\frac{\alpha'}{2}\frac{1}{z} ik_\alpha) : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : \\
+2(-\frac{\alpha'}{2}\frac{1}{z^2}\eta^{\alpha\mu})(-\frac{\alpha'}{2}\frac{1}{z} ik_\alpha) : \partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :]
\end{aligned} \tag{1.7}$$

Simplifying and keeping only the singular terms we obtain,

$$\begin{aligned}
-\frac{e_{\mu\nu}}{\alpha'}[-\frac{\alpha'}{z^2} : \partial_z X^\mu(z)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : -\frac{\alpha'}{z} ik^\alpha : \partial_z X_\alpha(z)\partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : \\
-\frac{\alpha'^2}{4}\frac{1}{z^2}k^2 : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : +\frac{\alpha'^2}{2}\frac{1}{z^3} ik^\mu : \partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :]
\end{aligned}$$

Next we expand about $z = 0$ as,

$$\partial_z X(z) = \partial_z X(0) + z\partial_z^2 X(0) + ... \tag{1.8}$$

and obtain,

$$-\frac{e_{\mu\nu}}{\alpha'}[-\frac{\alpha'}{z^2} : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : -\frac{\alpha'}{z} : \partial_z^2 X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :$$
$$-\frac{\alpha'}{z} ik^\alpha : \partial_z X_\alpha(0)\partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : -\frac{\alpha'^2}{4}\frac{1}{z^2} k^2 : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :$$
$$+\frac{\alpha'^2}{2}\frac{1}{z^3} ik^\mu : \partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :]$$

Which can be rearranged as,

$$\frac{(1+\frac{\alpha' k^2}{4})}{z^2} e_{\mu\nu} : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : + \frac{1}{z}\partial_z(e_{\mu\nu} : \partial_z X^\mu(0)\partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} :)$$
$$-\frac{\alpha'}{2}\frac{i}{z^3} e_{\mu\nu} k^\mu : \partial_{\bar{z}} X^\nu(0) e^{ikX(0,0)} : \tag{1.9}$$

which has the form,

$$\frac{h}{z^2}\mathcal{O} + \frac{1}{z}\partial_z \mathcal{O} + e_{\mu\nu}k^\mu(something) \tag{1.10}$$

Thus we find that the weight and the tensor condition are

$$h = (1 + \frac{\alpha' k^2}{4}),\ e_{\mu\nu}k^\mu = 0$$

To find the other weight, $\tilde{h}$, of this this operator, and further conditions on $e_{\mu\nu}$, we must compute the OPE of this operator with $T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} : \partial_{\bar{z}} X^\alpha \partial_{\bar{z}} X_\alpha :$. Due to the symmetry in the problem, we can see that the weights are equal, $h = \tilde{h}$, and that the condition on $e_{\mu\nu}$ becomes $e_{\mu\nu}k^\nu = 0$. The reason for the shift in the index from $\mu \to \nu$ is because if one was to rearrange the order the of the operators to have $\partial_{\bar{z}}$ first, the problem would be identical with $z \to \bar{z}$ with the μ and ν switched places.

1.2 Problem 2

We have to work out the covariant quantization on the state,

$$|f, e, k\rangle = (f_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu)|0, k\rangle \tag{1.11}$$

This state has $D(D+1)/2 + D$ states, not all of which are physical.

Consider first the norm of the state,

$$\langle f, e, k'|f, e, k\rangle = \langle 0, k'| \left(f_{ab}^* f_{\mu\nu}\alpha_1^a \alpha_1^b \alpha_{-1}^\mu \alpha_{-1}^\nu + e_a^* e_\mu \alpha_2^a \alpha_{-2}^\mu\right)|0, k\rangle \tag{1.12}$$

Commuting the lowering operators to the right gives a factor of two on both terms. This is obvious on the second term. As for the first one, we would commute twice which will result in two identical terms that sum up to two. Thus,

$$\langle f,e,k'|f,e,k\rangle = 2(f^*_{\mu\nu}f^{\mu\nu}+e^*_\mu e^\mu)\langle 0,k'|0,k\rangle \tag{1.13}$$

The Virasoro generators we need are (up to relevant terms),

$$L_0 = \alpha' p^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2 + ... \tag{1.14}$$
$$L_1 = \sqrt{2\alpha'}p\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + ... \tag{1.15}$$
$$L_2 = \sqrt{2\alpha'}p\cdot\alpha_2 + \frac{1}{2}\alpha_1\cdot\alpha_1 + ... \tag{1.16}$$
$$L_{-1} = \sqrt{2\alpha'}p\cdot\alpha_{-1} + \alpha_{-2}\cdot\alpha_1 + ... \tag{1.17}$$
$$L_{-2} = \sqrt{2\alpha'}p\cdot\alpha_{-2} + \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1} + ... \tag{1.18}$$

We begin with the mass shell condition,

$$0 = (L_0 - 1)|f,e,k\rangle \tag{1.19}$$

We first evaluate,

$$L_0|f,e,k\rangle = (\alpha' p^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2)(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2})|0,k\rangle \tag{1.20}$$
$$= \alpha' k^2|f,e,k\rangle + \alpha_{-1}\cdot\alpha_1 f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + \alpha_{-2}\cdot\alpha_2 e_\mu\alpha^\mu_{-2}|0,k\rangle \tag{1.21}$$

The other terms do not contribute. Commuting α_1 twice gives a factor of 2, and so does commuting α_2. The result is,

$$L_0|f,e,k\rangle = \alpha' k^2|f,e,k\rangle + 2f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + 2e_\mu\alpha^\mu_{-2}|0,k\rangle \tag{1.22}$$
$$= (\alpha' k^2 + 2)|f,e,k\rangle \tag{1.23}$$

and thus,

$$0 = (L_0 - 1)|f,e,k\rangle = (\alpha' k^2 + 1)|f,e,k\rangle \implies m^2 = -k^2 = \frac{1}{\alpha'} \tag{1.24}$$

Next we look for states that satisfy the other physical condition $L_{n>0}|f,e,k\rangle = 1$. First, since our particle is massive, ($m^2 > 0$), we go to its rest frame given by $k_0 = \frac{1}{\sqrt{\alpha'}}, k_i = 0$. At the first level, the condition is,

$$0 = L_1|f,e,k\rangle = (\sqrt{2\alpha'}p\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2)(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2})|0,k\rangle \tag{1.25}$$
$$= (2\sqrt{2\alpha'}f_{\mu\nu}k^\mu\alpha^\nu_{-1} + 2e_\mu\alpha^\mu_{-1}) \tag{1.26}$$

thus,

$$0 = (\sqrt{2\alpha'} f_{\mu\nu} k^\mu + e_\nu) = (-\sqrt{2\alpha'} f_{0\nu} \frac{1}{\sqrt{\alpha'}} + e_\nu) \implies e_\nu = \sqrt{2} f_{0\nu} \tag{1.27}$$

Which gets rid of D degrees of freedom. Next we look at,

$$0 = L_2|f, e, k\rangle = (\sqrt{2\alpha'} p \cdot \alpha_2 + \frac{1}{2}\alpha_1 \cdot \alpha_1)(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu \alpha^\mu_{-2})|0, k\rangle \tag{1.28}$$

$$= (2\sqrt{2\alpha'} e_\mu k^\mu + f^\mu_\mu)|0, k\rangle \tag{1.29}$$

This gives

$$f^\mu_\mu = -2\sqrt{2\alpha'} e_\mu k^\mu \implies -f_{00} + f_{ii} = 2\sqrt{2} e_0 \tag{1.30}$$

From the previous constraint we find that

$$f_{ii} = 5 f_{00}$$

This removes one more degree of freedom.

Now we consider the spurious states given by,

$$|g, w, k\rangle = (L_{-1} g_\mu \alpha^\mu_{-1} + L_{-2} w)|0, k\rangle \tag{1.31}$$

$$= (g_\mu \sqrt{2\alpha'} p \cdot \alpha_{-1}\alpha^\mu_{-1} + g_\mu \alpha^\mu_{-2} + w\sqrt{2\alpha'} p \cdot \alpha_{-2} + \frac{1}{2} w \alpha_{-1} \cdot \alpha_{-1})|0, k\rangle \tag{1.32}$$

$$= \left((\sqrt{2\alpha'} k_{(\mu} g_{\nu)} + \frac{w}{2}\eta_{\mu\nu})\alpha^\mu_{-1}\alpha^\nu_{-1} + (w\sqrt{\alpha'} k_\mu + g_\mu)\alpha^\mu_{-2}\right)|0, k\rangle \tag{1.33}$$

In terms of our physical state, this is,

$$f_{\mu\nu} = \sqrt{2\alpha'} k_{(\mu} g_{\nu)} + \frac{w}{2}\eta_{\mu\nu} \tag{1.34}$$

$$e_\mu = w\sqrt{2\alpha'} k_\mu + g_\mu \tag{1.35}$$

Plugging these into the physical state conditions we get the constraints,

$$\frac{w}{2}(D - 8) = 3\sqrt{2} g_0 \tag{1.36}$$

$$\frac{3}{2} w\sqrt{2} = g_0 \tag{1.37}$$

with no constraints on g_i. This removes $D - 1$ null states. We can remove another state which corresponds to $g_0 = w = 0$ in arbitrary D. Choosing $D = 26$ satisfies the constraints and makes g_0 and w independent. Thus in 26 dimensions there are a total of $D + 1$ null states to remove. Doing the math we find that the number of physical states that we obtain are:

$$D(D + 1)/2 + D - D - 1 - (D + 1) = D(D - 1)/2 - 1$$

As required. This Verifies one of the claims on p30 of the notes. We need to show that there is a negative norm state for $D > 26$ and that the issue $D < 26$ is that we have more states than expected. This last claim we already showed by showing that we need $D = 26$ to remove one more extra state.

Negative norm state in $D > 26$: Consider that trace state corresponding to $f_{ij} = f\delta_{ij}$, $f_{00} = \frac{1}{5}f(D-1)$, and $e_0 = \frac{\sqrt{2}(D-1)}{5}f$. Plugging this into the norm (1.13),

$$\langle f, e, k'|f, e, k\rangle = 2(f^*_{\mu\nu}f^{\mu\nu} + e^*_\mu e^\mu)\langle 0, k'|0, k\rangle \tag{1.38}$$

$$= 2(f^2\delta^i_i - e_0^2 + f_0 0^2)\langle 0, k'|0, k\rangle \tag{1.39}$$

$$= 2f^2((D-1) + (D-1)^2/25 - 2(D-1)^2/25)\langle 0, k'|0, k\rangle \tag{1.40}$$

$$= 2f^2\frac{(D-1)(26-D)}{25} \tag{1.41}$$

Which indeed has negative norm for $D > 26$.

1.3 Problem 3

The Virasoro-Shapiro amplitude is given by,

$$VS = 2\pi\frac{\Gamma(-1-\frac{\alpha' s}{4})\Gamma(-1-\frac{\alpha' t}{4})\Gamma(-1-\frac{\alpha' u}{4})}{\Gamma(2+\frac{\alpha' s}{4})\Gamma(2+\frac{\alpha' t}{4})\Gamma(2+\frac{\alpha' u}{4})} \tag{1.42}$$

At $s = M^2 = \frac{4}{\alpha'}(N-1)$, t and u satisfy

$$-\frac{\alpha' t}{4} - \frac{\alpha' t}{4} = \frac{12+4N}{4}$$

Under this the Gamma functions take the form,

$$\Gamma(-1-\frac{\alpha' s}{4}) = \Gamma(-N) \tag{1.43}$$

$$\Gamma(2+\frac{\alpha' s}{4}) = \Gamma(N+1) = N! \tag{1.44}$$

$$\Gamma(-1-\frac{\alpha' t}{4}) = \Gamma(\frac{\alpha' u}{4} + 2 + N) \tag{1.45}$$

$$\Gamma(-1-\frac{\alpha' u}{4}) = \Gamma(\frac{\alpha' t}{4} + 2 + N) \tag{1.46}$$

Since we need to take the pole of $\Gamma(-1-\frac{\alpha' s}{4})$ at $s = M^2 = \frac{4}{\alpha'}(N-1)$, we look for it's behavior close to the pole. We find,

$$\Gamma(-1-\frac{\alpha'(s-\epsilon)}{4}) = \frac{(-1)^{N+1}}{\Gamma(N+1)}\frac{4}{\alpha'}\frac{1}{\epsilon} \tag{1.47}$$

Thus the pole is

$$\frac{(-1)^N}{\Gamma(N+1)}\frac{4}{\alpha'}$$

The amplitude then takes the form,

$$VS = 2\pi\left(\frac{(-1)^N}{\Gamma(N+1)}\frac{4}{\alpha'}\right)\left(\frac{\Gamma(\frac{\alpha' u}{4}+2+N)}{\Gamma(\frac{\alpha' u}{4}+2)}\right)\left(\frac{\Gamma(\frac{\alpha' t}{4}+2+N)}{\Gamma(\frac{\alpha' t}{4}+2)}\right) \quad (1.48)$$

$$= 2\pi\left(\frac{(-1)^N}{\Gamma(N+1)}\frac{4}{\alpha'}\right)\left(\prod_{i=0}^{N}(\frac{\alpha' u}{4}+2+i)(\frac{\alpha' t}{4}+2+i)\right) \quad (1.49)$$

To show that this is a polynomial in $t-u$ we note that we can write,

$$t = (\frac{-12-4N}{2\alpha'}) + \frac{1}{2}(t-u) \quad (1.50)$$

$$u = (\frac{-12-4N}{2\alpha'}) - \frac{1}{2}(t-u) \quad (1.51)$$

Plugging this into VS we see that to leading order the amplitude takes the form,

$$VS \sim (t-u)^{2N} \quad (1.52)$$

Thus completing the proof that VS is a polynomial in $(t-u)$ of order $2N$.

Next we compare this to the maximum spin of the state at level N. The spin operator has the form,

$$S^{ij} = -i\sum_{n=1}\frac{1}{n}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n - \tilde{\alpha}^j_{-n}\tilde{\alpha}^i_n) \quad (1.53)$$

A maximum level spin closed string state at level $2N$ has the form,

$$\sim (\alpha^2_{-1} + i\alpha^3_{-1})^N(\tilde{\alpha}^2_{-1} + i\tilde{\alpha}^3_{-1})^N|0,k\rangle \quad (1.54)$$

Acting by S^{23} one obtains the eigenvalue $2N$. Thus the relation between the spin and mass is,

$$\alpha' M^2 = 2(2N) - 4$$

which has slope $\alpha'/2$

1.4 Problem 4

a) The relevant term in the Veneziano amplitude is,

$$V = \frac{2ig_o^2}{\alpha'}\frac{\Gamma(-1-\alpha' t)\Gamma(-1-\alpha' s)}{\Gamma(2+\alpha' u)} \quad (1.55)$$

Obtaining the pole at $t = 0$ as before, we find,

$$\Gamma(-1 - \alpha' t) \to -\frac{1}{\alpha'} \tag{1.56}$$

From the relation, $s + t + u = -\frac{4}{\alpha'}$, we have

$$-\alpha' s = 4 + \alpha' u$$

The veneziano amplitude becomes,

$$V = -\frac{2ig_o^2}{\alpha'^2}\frac{\Gamma(3 + \alpha' u)}{\Gamma(2 + \alpha' u)} = -\frac{2ig_o^2}{\alpha'^2}(2 + \alpha' u) = -\frac{ig_o^2}{\alpha'}(\frac{4}{\alpha'} + 2u) \tag{1.57}$$

Next we compare this amplitude with the exchange of a photon between two scalar charged particles. We use the convention that all momenta ingoing. The amplitude is,

$$V = (ie(k_1 - k_3)_\mu)(ie(k_2 - k_4)_\nu)(\frac{-ig^{\mu\nu}}{(k_1 + k_3)^2}) \tag{1.58}$$

$$= ie^2\frac{(k_1 \cdot k_2 + k_3 \cdot k_4 - k_1 \cdot k_4 - k_3 \cdot k_2)}{(k_1 + k_3)^2} \tag{1.59}$$

In terms of s, t, u, these products are,

$$k_1 \cdot k_2 = k_3 \cdot k_4 = m^2 - s/2 \tag{1.60}$$

$$k_1 \cdot k_3 = k_2 \cdot k_4 = m^2 - t/2 \tag{1.61}$$

$$k_1 \cdot k_4 = k_2 \cdot k_3 = m^2 - u/2 \tag{1.62}$$

Plugging this in we get,

$$V = ie^2(u - s)/(-t) \to V = ie^2(s - u) = ie^2(-2u + 4m^2) \tag{1.63}$$

Using $m^2 = -1/\alpha'$ we get,

$$V = -ie^2(\frac{4}{\alpha'} + 2u) \tag{1.64}$$

Comparing the two expressions we find that,

$$e^2 = \frac{g_o^2}{\alpha'}$$

b) The term that has poles in u and t differs from part a only in that $u \leftrightarrow s$. Thus,

$$V_{ut} = -\frac{ig_o^2}{\alpha'}(\frac{4}{\alpha'} + 2s) \tag{1.65}$$

$$= -\frac{ig_o^2}{\alpha'}(\frac{4}{\alpha'} + 2(-u - \frac{4}{\alpha'})) \tag{1.66}$$

$$= \frac{ig_o^2}{\alpha'}(\frac{4}{\alpha'} + 2u) \tag{1.67}$$

$$= -V_{st} \tag{1.68}$$

As required.

c) In this part we include the Chan-Paton factors. Since we are looking at the pole $t \to 0$ we only consider the contributions from the last two parts. This gives (we define $\lambda^{ijkl} \equiv \lambda^i \lambda^j \lambda^k \lambda^l$),

$$V = \frac{1}{2}\left(Tr(\lambda^{1243} + \lambda^{1342})V_{st} + Tr(\lambda^{1324} + \lambda^{1423})V_{ut}\right) \tag{1.69}$$

$$= \frac{1}{2}V_{st}\left(Tr(\lambda^{1243} + \lambda^{1342} - \lambda^{1324} - \lambda^{1423})\right) \tag{1.70}$$

$$= \frac{1}{2}V_{st}Tr\left(\lambda^1[\lambda^2, \lambda^4]\lambda^3 + \lambda^1\lambda^3[\lambda^4, \lambda^2]\right) \tag{1.71}$$

$$= \frac{1}{2}V_{st}Tr\left([\lambda^1, \lambda^3][\lambda^4, \lambda^2]\right) \neq 0 \tag{1.72}$$

As required.