1 PHYS230A Problem Set 4 Solutions

1.1 Problem 1

a) We need to evaluate

$$: e_1 \cdot \dot{X}(y_1) e^{ik_1 \cdot X(y_1)} :: e_2 \cdot \dot{X}(y_2) e^{ik_2 \cdot X(y_2)} :: e^{ik_3 \cdot X(y_3)} : \tag{1.1}$$

where I mean by : the open string normal ordering. We use the following computed forms for the contractions,

$$e_i \cdot (\dot{X}_i, e^j) = -2i\alpha' \frac{e_i \cdot k_j}{y_{ij}} \tag{1.2}$$

$$(e_1 \cdot \dot{X}_1, e_2 \cdot \dot{X}_2) = e_1 \cdot e_2 (-2\alpha') \frac{1}{y_{12}^2} \tag{1.3}$$

Where I have defined: $X(y_i) = X_i$, $e^{ik_j \cdot X(y_j)} = e^j$, and $y_{ij} = (y_i - y_j)$.

The relevant parts of the OPE become,

$$(e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} + (2i\alpha')^2 \left(-\frac{(e_1 \cdot k_2)(e_2 \cdot k_1)}{y_{12} y_{12}} + \frac{(e_1 \cdot k_2)(e_2 \cdot k_3)}{y_{12} y_{23}} - \frac{(e_1 \cdot k_3)(e_2 \cdot k_1)}{y_{13} y_{12}} + \frac{(e_1 \cdot k_3)(e_2 \cdot k_3)}{y_{13} y_{23}} \right) \tag{1.4}$$

Using the conservation of momentum, $k_1 + k_2 + k_3 = 0$ and $e_i \cdot k_i = 0$ we simplify the above expression.

$$(e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} + (2i\alpha')^2 (e_1 \cdot k_2)(e_2 \cdot k_1) \left(-\frac{1}{y_{12} y_{12}} - \frac{1}{y_{12} y_{23}} + \frac{1}{y_{13} y_{12}} + \frac{1}{y_{13} y_{23}} \right) \tag{1.5}$$

$$= (e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} + \frac{(2i\alpha')^2 (e_1 \cdot k_2)(e_2 \cdot k_1)}{y_{12}^2} \left(-1 - \frac{y_{12}}{y_{23}} + \frac{y_{12}}{y_{13}} + \frac{y_{12}^2}{y_{13} y_{23}} \right) \tag{1.6}$$

$$= (e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} + \frac{(2i\alpha')^2 (e_1 \cdot k_2)(e_2 \cdot k_1)}{y_{12}^2} \left(-1 + (y_{23} - y_{13}) \frac{y_{12}}{y_{23} y_{13}} + \frac{y_{12}^2}{y_{13} y_{23}} \right) \tag{1.7}$$

$$= (e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} + \frac{(2i\alpha')^2 (e_1 \cdot k_2)(e_2 \cdot k_1)}{y_{12}^2} \left(-1 - \frac{y_{12}^2}{y_{23} y_{13}} + \frac{y_{12}^2}{y_{13} y_{23}} \right) \tag{1.8}$$

$$= (e_1 \cdot e_2)(-2\alpha') \frac{1}{y_{12}^2} - \frac{(2i\alpha')^2 (e_1 \cdot k_2)(e_2 \cdot k_1)}{y_{12}^2} \tag{1.9}$$

$$= \frac{-2\alpha'}{y_{12}^2} \left[(e_1 \cdot e_2) - 2\alpha' (e_1 \cdot k_2)(e_2 \cdot k_1) \right] \tag{1.10}$$

The other cyclic order produces the same factor. Next we need to add the other amplitude factors. There is the delta function, $(2\pi)^D \delta^D(\sum k)$ and the Jacobian $|y_{12} y_{13} y_{23}|$. Also the constant factors: $(\frac{-ig_o}{\sqrt{2\alpha'}})^2$ from the massless vectors, (g_o) for the tachyon, and the overall factor $(\frac{i}{\alpha' g_o^2})$. Also, there is the amplitude from the exponentials $\prod |y_{ij}|^{2\alpha' k_i \cdot k_j} = |y_{12}||y_{23}|^{-1}|y_{13}|^{-1}$.

Putting everything together we obtain,

$$\frac{2ig_o}{\alpha'}(2\pi)^D \delta^D(\sum k)\,[(e_1 \cdot e_2) - 2\alpha'(e_1 \cdot k_2)(e_2 \cdot k_1)] \tag{1.11}$$

To include the Chan-Paton factor we multiply by $\frac{1}{2}Tr[\lambda_a[\lambda_b, \lambda_c]]$ to get,

$$\frac{ig_o}{\alpha'}(2\pi)^D \delta^D(\sum k)\,[(e_1 \cdot e_2) - 2\alpha'(e_1 \cdot k_2)(e_2 \cdot k_1)]\,Tr[\lambda_a[\lambda_b, \lambda_c]] \tag{1.12}$$

b) From the form of the amplitude we make a guess and proceed to verify it. We guess the lagrangian,

$$L = Tr\left(-\frac{1}{2}D_\mu \phi D^\mu \phi + \frac{1}{2\alpha'}\phi^2 - \frac{1}{4}F^2 + g_o F^2 \phi\right) \tag{1.13}$$

Expanding the interaction term,

$$\begin{aligned} Tr[g_o F^2 \phi] &= g_o F^a_{\mu\nu} F^{b\ \mu\nu} \phi^c Tr[\lambda^a \lambda^b \lambda^c] && (1.14)\\ &= \frac{1}{2} g_o F^a_{\mu\nu} F^{b\ \mu\nu} \phi^c Tr[\{\lambda^a, \lambda^b\}\lambda^c] && (1.15)\\ &= g_o(\partial_\mu A^a_\nu \partial^\mu A^{b\ \nu} - \partial_\mu A^a_\nu \partial^\nu A^{b\ \mu} + ...)Tr[\{\lambda^a, \lambda^b\}\lambda^c] && (1.16) \end{aligned}$$

Where the ... corresponds to terms that are not of interest and arise from the form of maxwell stress tensor, $F = dA + [A, A]$. Labeling momenta of the photons as k_1, k_2 and working with the convention that all momenta are ingoing we obtain the vertex factor,

$$\begin{aligned} & 2ig_o((k_1 \cdot k_2)(e_1 \cdot e_2) - (k_1 \cdot e_2)(k_2 \cdot e_1))Tr[\{\lambda^a, \lambda^b\}\lambda^c] && (1.17)\\ &= \frac{ig_o}{\alpha'}\left((e_1 \cdot e_2) - 2\alpha'(k_1 \cdot e_2)(k_2 \cdot e_1)\right) Tr[\{\lambda^a, \lambda^b\}\lambda^c] && (1.18) \end{aligned}$$

where the initial factor of 2 comes from permuting the legs of the diagram. Also we have used $k_1 \cdot k_2 = \frac{1}{2\alpha'}$ from before. This is the required form that matches the previous result.