1 PHYS230A Problem Set 5 Solutions

1.1 Problem 1

We are asked to obtain the annulus amplitude from a closed string calculation. Choosing the cylinder to have a circumference of 2π, $\sigma^1 \sim \sigma^1 + 2\pi$, and height π/t, $0 \leq \sigma^2 \leq \pi/t$,we obtain the amplitude by propagating the closed string along the length of the cylinder by using,

$$\langle B|e^{-\pi H/t}|B\rangle \tag{1.1}$$

where $|B\rangle$ is determined by the string boundary condition:

$$\partial_2 X^\mu |B\rangle = 0 \tag{1.2}$$

at $\sigma^2 = 0$. The X mode expansion is given by,

$$X^\mu(\sigma^1, \sigma^2) = x^\mu + i\frac{p^\mu}{p^+}\sigma^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{i(\sigma^1+i\sigma^2)} + \tilde{\alpha}_n^\mu e^{-i(\sigma^1-i\sigma^2)}\right) \tag{1.3}$$

The boundary condition is given by,

$$\left(i\frac{p^\mu}{p^+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(-\alpha_n^\mu e^{in(\sigma^1+i\sigma^2)} - \tilde{\alpha}_n^\mu e^{-in(\sigma^1-i\sigma^2)}\right)\right)|B\rangle = 0 \tag{1.4}$$

Changing $n \to -n$ in the third term we obtain and setting $\sigma^2 = 0$,

$$\left(i\frac{p^\mu}{p^+} - i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} e^{in\sigma^1}\left(-\alpha_n^\mu - \tilde{\alpha}_{-n}^\mu\right)\right)|B\rangle = 0 \tag{1.5}$$

Including the p^μ/p^+ as $\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$ in the sum we can write,

$$\sum_n e^{in\sigma^1}\left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right)|B\rangle = 0 \tag{1.6}$$

This must vanish term by term and thus we have the condition,

$$\left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right)|B\rangle = 0, \tag{1.7}$$

for all n.

Since oscillators at different levels commute, we can write the state $|B\rangle$ as,

$$|B\rangle = \prod_{n>0} F_n(\alpha_{-n}^\mu, \tilde{\alpha}_{-n}^\nu)\ |0,0\rangle \tag{1.8}$$

From the commutation relation of the $[\alpha_n^\mu, \alpha_{-m}^\nu] = n\eta^{\mu\nu}\delta_{n-m,0}$, we can write,

$$[\alpha_n^\mu, \alpha_{-m}^\nu] = n\frac{\partial}{\partial \alpha_{-n}^\mu}(\alpha_{-m}^\nu) = n\eta^{\mu\nu}\delta_{m,n} \tag{1.9}$$

and thus define,

$$[\alpha_n^\mu, f] \equiv n\frac{\partial}{\partial \alpha_{-n}^\mu} f \tag{1.10}$$

So now we have to conditions on the functions, F_n,

$$(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu)F_n|0,0\rangle = 0, \tag{1.11}$$

$$(\alpha_{-n}^\mu + \tilde{\alpha}_n^\mu)F_n|0,0\rangle = 0 \tag{1.12}$$

where $n > 0$. Which by using the fact that lowering operators kill the vacuum, we get,

$$[\alpha_n^\mu, F_n] + \tilde{\alpha}_{-n}^\mu F_n = 0, \tag{1.13}$$

$$\alpha_{-n}^\mu F_n + [\tilde{\alpha}_n^\mu, F_n] = 0 \tag{1.14}$$

Using our newly defined derivative operator, we write this as,

$$n\frac{\partial}{\partial \alpha_{-n}^\mu} F_n + \tilde{\alpha}_{-n}^\mu F_n = 0 \tag{1.15}$$

$$n\frac{\partial}{\partial \tilde{\alpha}_{-n}^\mu} F_n + \alpha_{-n}^\mu F_n = 0 \tag{1.16}$$

the solution of which is clearly an exponential. Adding in all the levels we obtain,

$$\prod_n F_n \propto \exp\left(-\sum_{n>0}\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}\right) \tag{1.17}$$

Thus,

$$|B\rangle \propto \exp\left(-\sum_{n>0}\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}\right)|0,0\rangle \tag{1.18}$$

Which we use to compute the matrix element (1.1). The hamiltonian is given by,

$$H = L_0 + \tilde{L}_0 - 2\frac{26}{24} = \sum_{n>0}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) - 2\frac{26}{24} \tag{1.19}$$

Thus the matrix element becomes,

$$\langle B|e^{-\pi H/t}|B\rangle = e^{2\pi/t\frac{26}{24}}\langle 0,0|\prod_{n>0}\exp\left(-\frac{1}{n}\alpha_n\cdot\tilde{\alpha}_n\right)\exp\left(-\frac{\pi}{t}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)\right)\exp\left(-\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}\right)|0,0\rangle \tag{1.20}$$

We are allowed to remove the sums from the arguments of the exponentials because the α's at different levels commute. We need to evaluate,

$$\exp\left(-\frac{\pi}{t}\alpha_{-n}\cdot\alpha_n\right)\exp\left(-\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}\right)= \tag{1.21}$$

$$=\exp\left(\frac{\pi}{t}\alpha^0_{-n}\alpha^0_n\right)\exp\left(\frac{1}{n}\alpha^0_{-n}\tilde{\alpha}^0_{-n}\right)\prod_i\exp\left(-\frac{\pi}{t}\alpha^i_{-n}\alpha^i_n\right)\exp\left(-\frac{1}{n}\alpha^i_{-n}\tilde{\alpha}^i_{-n}\right) \tag{1.22}$$

$$=\sum_a\exp\left(\frac{\pi}{t}\alpha^0_{-n}\alpha^0_n\right)(\frac{1}{n})^a\frac{1}{a!}(\alpha^0_{-n}\tilde{\alpha}^0_{-n})^a\prod_i^D\sum_a\exp\left(-\frac{\pi}{t}\alpha^i_{-n}\alpha^i_n\right)(-\frac{1}{n})^a\frac{1}{a!}(\alpha^i_{-n}\tilde{\alpha}^i_{-n})^a \tag{1.23}$$

It is easy to see that

$$\alpha^0_{-n}\alpha^0_n\ \alpha^0_{-n}\tilde{\alpha}^0_{-n}=-n\alpha^0_{-n}\tilde{\alpha}^0_{-n}$$

$$\alpha^i_{-n}\alpha^i_n\ \alpha^i_{-n}\tilde{\alpha}^i_{-n}=n\alpha^i_{-n}\tilde{\alpha}^i_{-n}$$

and thus we can write (1.23) as,

$$\prod_{\mu=0}^D\exp\left(-\frac{\pi}{t}n\right)(-\frac{1}{n})^a\frac{1}{a!}(\eta_{\mu\mu}\alpha^\mu_{-n}\tilde{\alpha}^\mu_{-n})^a \tag{1.24}$$

We also obtain the same contribution from the $\tilde{\alpha}$ oscillators which doubles the argument of the first exponential. Next, we write the left most exponential inside the matrix element (1.20) as,

$$\exp\left(-\frac{1}{n}\alpha_n\cdot\tilde{\alpha}_n\right)=\prod_{\mu=0}^D\exp\left(-\frac{1}{n}\alpha^\mu_n\tilde{\alpha}^\mu_n\eta_{\mu\mu}\right) \tag{1.25}$$

$$=\prod_{\mu=0}^D\sum_a(-\frac{1}{n})^a\frac{1}{a!}(\eta_{\mu\mu}\alpha^\mu_n\tilde{\alpha}^\mu_n)^a \tag{1.26}$$

Using the fact that the levels between the bra and ket must match, we can write the matrix element as,

$$\langle B|e^{-\pi H/t}|B\rangle=e^{2\pi/t\frac{26}{24}}\langle 0,0|\prod_{n>0}\prod_{\mu=0}^D\sum_a\exp\left(-\frac{\pi}{t}(2na)\right)(-\frac{1}{n})^{2a}(\frac{1}{a!})^2(\eta_{\mu\mu}\alpha^\mu_n\tilde{\alpha}^\mu_n)^a(\eta_{\mu\mu}\alpha^\mu_{-n}\tilde{\alpha}^\mu_{-n})^a|0,0\rangle \tag{1.27}$$

$$=e^{2\pi/t\frac{26}{24}}\langle 0,0|\prod_{n>0}\prod_{\mu=0}^D\sum_a\exp\left(-\frac{\pi}{t}(2na)\right)(-\frac{1}{n})^{2a}(\frac{1}{a!})^2(\alpha^\mu_n\tilde{\alpha}^\mu_n)^a(\alpha^\mu_{-n}\tilde{\alpha}^\mu_{-n})^a|0,0\rangle \tag{1.28}$$

and we have,

$$(\alpha_n^\mu)^a(\alpha_{-n}^\mu)^a = n\ a\ (\alpha_n^\mu)^{a-1}(\alpha_{-n}^\mu)^{a-1} \tag{1.29}$$
$$= n^a a! \tag{1.30}$$

Keeping only non-vanishing terms. The matrix element becomes,

$$\langle B|e^{-\pi H/t}|B\rangle = e^{2\pi/t\frac{26}{24}}\langle 0,0|\prod_{n>0}\prod_{\mu=0}^{D}\sum_a \exp\left(-\frac{\pi}{t}(2na)\right)(-\frac{1}{n})^{2a}(\frac{1}{a!})^2(n^a a!)^2|0,0\rangle \tag{1.31}$$
$$= e^{2\pi/t\frac{26}{24}}\langle 0,0|\prod_{n>0}\prod_{\mu=0}^{D}\sum_a \exp\left(-\frac{\pi}{t}(2na)\right)|0,0\rangle \tag{1.32}$$
$$= e^{2\pi/t\frac{26}{24}}\prod_{n>0}\prod_{\mu=0}^{D}\sum_a \exp\left(-\frac{\pi}{t}(2na)\right) \tag{1.33}$$
$$= e^{2\pi/t\frac{26}{24}}\prod_{n>0}\prod_{\mu=0}^{D}(1-\exp\left(-\frac{\pi}{t}(2n)\right))^{-1} \tag{1.34}$$

Which from the definition of the dedekind eta function,

$$\eta(i/t) = e^{\frac{-2\pi}{24t}}\prod_{n>0}(1-e^{\frac{-2n\pi}{t}})$$

obtains the form,

$$\langle B|e^{-\pi H/t}|B\rangle = e^{2\pi/t\frac{26}{24}}\prod_{\mu=0}^{D=26} e^{-2\pi/t\frac{1}{24}}\eta(i/t)^{-1} \tag{1.35}$$
$$= \eta(i/t)^{-26} \tag{1.36}$$
$$= t^{-13}\eta(it)^{-26} \tag{1.37}$$

The last step follows from an η function identity. Finally, the form of the amplitude is given by integrating over t,

$$\int \frac{dt}{t}J(t)t^{-13}\eta(it)^{-26} \tag{1.38}$$

To get this to match the annulus amplitude (JBB eqn (7.4.1)) we need

$$J(t) = \eta(it)^2$$

to get

$$Amp. \sim \int \frac{dt}{t}t^{-13}\eta(it)^{-24} \tag{1.39}$$