1 PHYS230A Problem Set 6 Solutions

1.1 Problem 1

a) We consider the action,

$$S = \int d\tau d\sigma L = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\partial_\tau X \partial_\tau X - \partial_\sigma X \partial_\sigma X\right) \tag{1.1}$$

and plug in the solution

$$X(\tau, \sigma) = x(\tau) + wR\sigma$$

The action becomes,

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\dot{x}^2(\tau) - (wR)^2\right) \tag{1.2}$$

The Hamiltonian is given by,

$$\begin{aligned} H &= \int d\sigma \left(\frac{\partial L}{\partial \dot{x}} \dot{x} - L\right) && (1.3) \\ &= \frac{1}{4\pi\alpha'} \int d\sigma \left(\dot{x}^2 + (wR)^2\right) && (1.4) \\ &= \frac{1}{2\alpha'} \left(\dot{x}^2 + (wR)^2\right) && (1.5) \end{aligned}$$

Writing this in terms of the conjugate momentum,

$$p = \int d\sigma \frac{\partial L}{\partial \dot{x}} = \dot{x}/\alpha'$$

we obtain,

$$H = \frac{1}{2}\left(\alpha' p^2 + \frac{(wR)^2}{\alpha'}\right) \tag{1.6}$$

Now due to the periodicity of the field, $X \sim X + 2\pi R$ with the coordinate, $\sigma \sim \sigma + 2\pi$, the momentum becomes quantized. Explicitly this is,

$$e^{i2\pi Rp} = 1 \implies p = \frac{n}{R} \tag{1.7}$$

The spectrum is then

$$H = \frac{1}{2}\left(\alpha' \frac{n^2}{R^2} + \frac{(wR)^2}{\alpha'}\right) \tag{1.8}$$

Now to show that this is the zero mode of the sum $L_0 + \tilde{L}_0$. We first express X in lorentzian time,

$$\tau = \frac{1}{2i} \ln(z\bar{z}), \quad \sigma = \frac{1}{2i} \ln\left(\frac{z}{\bar{z}}\right) \tag{1.9}$$

The virasoro generators are then,

$$L_0 = -\frac{1}{\alpha'}\int \frac{dz}{2\pi i z} z^2 : \partial_z X \partial_z X : = \frac{1}{4\alpha'}(\dot{x} + wR)^2 \tag{1.10}$$

$$\tilde{L}_0 = -\frac{1}{\alpha'}\int \frac{dz}{2\pi i z} z^2 : \partial_{\bar{z}} X \partial_{\bar{z}} X : = \frac{1}{4\alpha'}(\dot{x} - wR)^2 \tag{1.11}$$

Now let's write this as the Hamiltonian,

$$H = L_0 + \tilde{L}_0 = \frac{1}{4\alpha'}\left((\dot{x} + wR)^2 + (\dot{x} - wR)^2\right) \tag{1.12}$$

$$= \frac{1}{4\alpha'}\left(2\dot{x}^2 + 2w^2R^2\right) \tag{1.13}$$

$$= \frac{1}{2}\left(\alpha'\frac{n^2}{R^2} + \frac{(wR)^2}{\alpha'}\right) \tag{1.14}$$

Which is identical to (1.8) as required.

b) Consider the system with two-dimensions, $X^{1,2}$. The coordinates satisfy,

$$X^1 \sim X^1 + 2\pi R_1 m_1 + 2\pi R_2' m_2, \quad X^2 \sim X^2 + 2\pi R_2 m_2 \tag{1.15}$$

We begin by finding the quantized momenta using,

$$1 = \exp\left(2\pi i(R_1 m_1 + R_2' m_2)p_1 + 2\pi i R_2 m_2 p_2\right) \tag{1.16}$$

$$= \exp\left(2\pi i\left(p_1 R_1 m_1 + m_2(p_1 R_2' + p_2 R_2)\right)\right) \tag{1.17}$$

Thus we pick $p_1 = \frac{n_1}{R_1}$ and $p_2 = \frac{n_2}{R_2} - \frac{n_1 R_2'}{R_2 R_1}$.

Because of the periodicity on the X's we write,

$$X^1 = x_1(\tau) + w_1 R_1 \sigma + w_2 R_2' \sigma, \quad X^2 = x_2(\tau) + w_2 R_2 \sigma \tag{1.18}$$

The action becomes,

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(\dot{x}_1^2 + \dot{x}_2^2 - (w_1 R_1 + w_2 R_2')^2 - (w_2 R_2)^2\right) \tag{1.19}$$

As before, we obtain the hamiltonian,

$$H = \frac{1}{2\alpha'}\left(\alpha'^2 p_1^2 + \alpha'^2 p_2^2 + (w_1 R_1 + w_2 R_2')^2 + (w_2 R_2)^2\right) \tag{1.20}$$

$$= \frac{1}{2\alpha'}\left(\alpha'^2\frac{n_1^2}{R_1^2} + \alpha'^2\left(\frac{n_2}{R_2} - \frac{n_1 R_2'}{R_2 R_1}\right)^2 + (w_1 R_1 + w_2 R_2')^2 + (w_2 R_2)^2\right) \tag{1.21}$$

Now we turn to computing $L_0 + \tilde{L}_0$. Since the steps are just as before, we can guess the forms of the generators to be,

$$L_0 = \frac{1}{4\alpha'}\left(\alpha' p_1 + w_1 R_1 + w_2 R_2'\right)^2 + \frac{1}{4\alpha'}\left(\alpha' p_2 + w_2 R_2\right)^2 \equiv \frac{1}{4\alpha'}\left((\alpha' p_L^1)^2 + (\alpha' p_L^2)^2\right) \quad (1.22)$$

$$\tilde{L}_0 = \frac{1}{4\alpha'}\left(\alpha' p_1 - (w_1 R_1 + w_2 R_2')\right)^2 + \frac{1}{4\alpha'}\left(\alpha' p_2 - w_2 R_2\right)^2 \equiv \frac{1}{4\alpha'}\left((\alpha' p_R^1)^2 + (\alpha' p_R^2)^2\right) \quad (1.23)$$

Adding those two terms we obtain,

$$L_0 + \tilde{L_0} = \frac{1}{4\alpha'}\left((\alpha' p_1 + w_1 R_1 + w_2 R_2')^2 + (\alpha' p_1 - (w_1 R_1 + w_2 R_2'))^2 + (\alpha' p_2 + w_2 R_2)^2 + (\alpha' p_2 - w_2 R_2)^2\right) \quad (1.24)$$

Which after canceling the cross terms is identical to eqn (1.20) as required.

c) In this part we include the background field B_{12}. This appears in the action as,

$$S_B = \frac{1}{4\pi\alpha'}\int d\tau d\sigma \epsilon^{ab} B_{mn}\partial_a X^m \partial_b X^n = \frac{1}{4\pi\alpha'}\int d\tau d\sigma 2 B_{12}(\partial_\tau X^1 \partial_\sigma X^2 - \partial_\sigma X^1 \partial_\tau X^2) \quad (1.25)$$

$$= \frac{1}{4\pi\alpha'}\int d\tau d\sigma 2\left(\dot{x}_1 w_2 B_{12} R_2 - \dot{x}_2 B_{12}(w_1 R_1 + w_2 R_2')\right) \quad (1.26)$$

The momenta get an extra contribution from this extra term,

$$p_1 = \frac{\dot{x}_1}{\alpha'} + \frac{w_2 B_{12} R_2}{\alpha'} \quad (1.27)$$

$$p_2 = \frac{\dot{x}_2}{\alpha'} - B_{12}\frac{w_1 R_1 + w_2 R_2'}{\alpha'} \quad (1.28)$$

With the same quantization as above. The hamiltonian in this case becomes,

$$H = \frac{\dot{x}_1^2}{\alpha'} + \dot{x}_1\frac{w_2 B_{12} R_2}{\alpha'} + \frac{\dot{x}_2^2}{\alpha'} - \dot{x}_2 B_{12}\frac{w_1 R_1 + w_2 R_2'}{\alpha'} - \frac{\dot{x}_1^2}{2\alpha'} - \frac{\dot{x}_2^2}{2\alpha'} + \frac{(w_1 R_1 + w_2 R_2')^2}{2\alpha'} + \frac{(w_2 R_2)^2}{2\alpha'} - \frac{(\dot{x}_1 w_2 B_{12} R_2 - \dot{x}_2 B_{12}(w_1 R_1 + w_2 R_2'))}{\alpha'} \quad (1.29)$$

$$= \frac{\dot{x}_1^2}{2\alpha'} + \frac{\dot{x}_2^2}{2\alpha'} + \frac{(w_1 R_1 + w_2 R_2')^2}{2\alpha'} + \frac{(w_2 R_2)^2}{2\alpha'} \quad (1.30)$$

$$= \frac{1}{2\alpha'}\left((\alpha' p_1 - w_2 B_{12} R_2)^2 + (\alpha' p_2 + B_{12}(w_1 R_1 + w_2 R_2'))^2 + (w_1 R_1 + w_2 R_2')^2 + (w_2 R_2)^2\right) \quad (1.31)$$

In terms of the quantized momenta this is,

$$H = \frac{1}{2\alpha'}\left((\alpha'\frac{n_1}{R_1} - w_2 B_{12} R_2)^2 + (\alpha'\left(\frac{n_2}{R_2} - \frac{n_1 R_2'}{R_2 R_1}\right) + B_{12}(w_1 R_1 + w_2 R_2'))^2 + (w_1 R_1 + w_2 R_2')^2 + (w_2 R_2)^2\right) \quad (1.32)$$

As for the virasoro generators, we just have to modify the expressions in eqns (1.22, 1.23) by adding the B_{12} contribution as in H,

$$L_0 = \frac{1}{4\alpha'}\left(\alpha' p_1 - w_2 B_{12} R_2 + w_1 R_1 + w_2 R_2'\right)^2 + \frac{1}{4\alpha'}\left(\alpha' p_2 + B_{12}(w_1 R_1 + w_2 R_2') + w_2 R_2\right)^2 \tag{1.33}$$

$$\equiv \frac{1}{4\alpha'}\left((\alpha' p_L^1)^2 + (\alpha' p_L^2)^2\right) \tag{1.34}$$

$$\tilde{L}_0 = \frac{1}{4\alpha'}\left(\alpha' p_1 - w_2 B_{12} R_2 - (w_1 R_1 + w_2 R_2')\right)^2 + \frac{1}{4\alpha'}\left(\alpha' p_2 + B_{12}(w_1 R_1 + w_2 R_2') - w_2 R_2\right)^2 \tag{1.35}$$

$$\equiv \frac{1}{4\alpha'}\left((\alpha' p_R^1)^2 + (\alpha' p_R^2)^2\right) \tag{1.36}$$

And it is clear that $H = L_0 + \tilde{L}_0$ since the only modification here is that $p_i \to p_i + B_{12}(...)$.

d) We begin by listing the left and right moving momenta,

$$p_L^1 = \frac{n_1}{R_1} - \frac{w_2 B_{12} R_2}{\alpha'} + \frac{(w_1 R_1 + w_2 R_2')}{\alpha'} \tag{1.37}$$

$$p_R^1 = \frac{n_1}{R_1} - \frac{w_2 B_{12} R_2}{\alpha'} - \frac{(w_1 R_1 + w_2 R_2')}{\alpha'} \tag{1.38}$$

$$p_L^2 = \left(\frac{n_2}{R_2} - \frac{n_1 R_2'}{R_2 R_1}\right) + B_{12}\frac{(w_1 R_1 + w_2 R_2')}{\alpha'} + \frac{w_2 R_2}{\alpha'} \tag{1.39}$$

$$p_R^2 = \left(\frac{n_2}{R_2} - \frac{n_1 R_2'}{R_2 R_1}\right) + B_{12}\frac{(w_1 R_1 + w_2 R_2')}{\alpha'} - \frac{w_2 R_2}{\alpha'} \tag{1.40}$$

$$\tag{1.41}$$

From the mass shell conditions $(L_0 - 1) = (\tilde{L}_0 - 1) = 0$, we obtain,

$$0 = m^2 = \frac{1}{4\alpha'}\left((\alpha' p_L^1)^2 + (\alpha' p_L^2)^2\right) + (N - 1) \tag{1.42}$$

$$0 = m^2 = \frac{1}{4\alpha'}\left((\alpha' p_R^1)^2 + (\alpha' p_R^2)^2\right) + (\tilde{N} - 1) \tag{1.43}$$

We are looking for 12 massless gauge bosons. We expect that they split into two groups satisfying $N = 1, \tilde{N} = 0$ and $N = 0, \tilde{N} = 1$. As given in the hint of the problem, the eigenvalues are expected to sit on the root lattice of SU(3), which is hexagonal. The symmetry of this lattice requires the space lattice to also by symmetric: we pick R_1, R_2, and R_2' that correspond to making the space lattice a rhombus. This corresponds to $R_1 = \sqrt{\alpha'}$, $R_2 = (\sqrt{3}/2)\sqrt{\alpha'}$, and $R_2' = \sqrt{\alpha'}/2$. This over all scale is a special one and thus picked. In doing this, one finds that in order to find solutions, the background field, B_{12}, must be set as $B_{12} = 1/\sqrt{3}$. Under these choices the momenta become,

$$p_L^1 = \frac{1}{\alpha'}(n_1 + w_1) \tag{1.44}$$

$$p_R^1 = \frac{1}{\alpha'}(n_1 - w_1 - w_2) \tag{1.45}$$

$$p_L^2 = \frac{1}{\sqrt{3}\sqrt{\alpha'}}(-n_1 + w_1 + 2n_2 + 2w_2) \tag{1.46}$$

$$p_R^2 = \frac{1}{\sqrt{3}\sqrt{\alpha'}}(-n_1 + w_1 + 2n_2 - w_2) \tag{1.47}$$

$$\tag{1.48}$$

Considering the case where $N = 1, \tilde{N} = 0$, we find that the momenta must satisfy,

$$(p_L^1)^2 + (p_L^2)^2 = 0 \tag{1.49}$$

$$(p_R^1)^2 + (p_R^2)^2 = \frac{4}{\alpha'} \tag{1.50}$$

Thus these states correspond to $p_L^i = 0$ and the p_R^j correspond to the 6 off-center points on the root lattice.These correspond to,

$$\sqrt{\alpha'}(p_R^1, p_R^2) \sim (n1, w1, n2, w2) \tag{1.51}$$

$$(2, 0), \quad (1, -1, 1, 0) \tag{1.52}$$

$$(-2, 0), \quad (-1, 1, -1, 0) \tag{1.53}$$

$$(1, \sqrt{3}), \quad (0, 0, 1, -1) \tag{1.54}$$

$$(1, -\sqrt{3}), \quad (1, -1, -1, 1) \tag{1.55}$$

$$(-1, \sqrt{3}), \quad (-1, 1, 1, -1) \tag{1.56}$$

$$(-1, -\sqrt{3}) \quad (0, 0, -1, -1) \tag{1.57}$$

And for the other case we have,

$$\sqrt{\alpha'}(p_L^1, p_L^2) \sim (n1, w1, n2, w2) \tag{1.58}$$

$$(2, 0), \quad (1, 1, 0, 0) \tag{1.59}$$

$$(-2, 0), \quad (-1, -1, 0, 0) \tag{1.60}$$

$$(1, \sqrt{3}), \quad (1, 0, 1, 1) \tag{1.61}$$

$$(1, -\sqrt{3}), \quad (0, 1, -1, -1) \tag{1.62}$$

$$(-1, \sqrt{3}), \quad (0, -1, 1, 1) \tag{1.63}$$

$$(-1, -\sqrt{3}) \quad (-1, 0, -1, -1) \tag{1.64}$$

These are the 12 massless gauge bosons as required.