Final exam solutions, Phys 221B, W15

1. *In HW5, #4 (Srednicki 48.5), you had the interaction*

$$\mathcal{L}_1 = 2g\overline{\mathcal{N}}\gamma^\mu P_{\rm L}\mathcal{M}\partial_\mu\phi + \text{h.c.}\,,$$

with $m_\phi > m_\mathcal{M}$ and $m_\mathcal{N} = 0$. Now let $m_\mathcal{M} > m_\phi$ and $m_\mathcal{N} = 0$.

a) Calculate the decay rate for $\mathcal{M} \to \phi\mathcal{N}$ when none of the spins is measured.

b) Calculate the decay rate with $\mathcal{N}$ having measured helicity, positive or negative.

c) Calculate the differential decay rate 11.48 when the initial $\mathcal{M}$ is known to have $s_z = +\frac{1}{2}$ and the spin of the $\mathcal{N}$ is not measured.

d) Which of P, T, and C is violated by this interaction? For each violation, can it be seen in the calculation you've done in (b)? in (c)? (Explain precisely).

a) Labeling the momenta $\mathcal{M}(p) \to \phi(k)\mathcal{N}(p')$, and $m \equiv m_\mathcal{M} > M \equiv m_\phi$ and $m_\mathcal{N} = 0$. Using the Dirac equation, we have

$$\begin{aligned}\mathcal{T} &= ig\bar{u}(p')\not{k}(1-\gamma_5)u(p)\\ &= ig\bar{u}(p')(\not{p}-\not{p}')(1-\gamma_5)u(p)\\ &= img\bar{u}(p')(1+\gamma_5)u(p)\\ .\end{aligned} \tag{1}$$

Using

$$(\bar{u}(p')(1+\gamma_5)u(p))^* = \bar{u}(p)(1+\overline{\gamma_5})u(p') = \bar{u}(p)(1-\gamma_5)u(p')\,.$$

this becomes

$$\begin{aligned}\sum_{\text{spins}}|\mathcal{T}|^2 &= -m^2g^2\text{Tr}[(m-\not{p})(1-\gamma_5)\not{p}'(1+\gamma_5)]\\ &= -2m^2g^2\text{Tr}[(m-\not{p})\not{p}'(1+\gamma_5)]\\ &= -2g^2m^2\text{Tr}[(m-\not{p})\not{p}']\\ &= -8g^2m^2p\cdot p' = 4g^2m^2(m^2-M^2)\,.\end{aligned}$$

Note the properties $(1\pm\gamma_5)\not{a} = \not{a}(1\mp\gamma_5)$, $(1+\gamma_5)(1-\gamma_5) = 0$, $(1\pm\gamma_5)\gamma_5 = (1\pm\gamma_5)(\pm1)$. These help to simplify things. Then

$$\begin{aligned}\Gamma &= \frac{1}{8\pi m^2}|\vec{p}'|\frac{1}{2}\sum_{\text{spins}}|\mathcal{T}|^2\\ &= \frac{g^2(m^2-M^2)^2}{8\pi m}\end{aligned}$$

Note the factor $\frac{1}{2}$ to average over the initial spin.

b) Now

$$\sum_{\substack{\text{initial} \\ \text{spin}}} |\mathcal{T}|^2 = -\frac{g^2 m^2}{2} \text{Tr}\big[(m - \not{p})(1-\gamma_5)(1+s'\gamma_5)\not{p}'(1+\gamma_5)\big] \,.$$

Noting that $(1-\gamma_5)(1+s'\gamma_5) = (1-\gamma_5)(1-s')$, we get immediately that

$$\Gamma(s'=+1) = 0\,, \quad \Gamma(s'=-1) = \frac{g^2(m^2-M^2)^2}{8\pi m}\,.$$

This is the property of the weak interaction that only left-handed fermions couple to it.

Many people missed the s-dependence due to a subtlety. According to 38.28,

$$u_s \bar{u}_s = \frac{1}{2}(1 - s\gamma_5 \not{z})(m - \not{p})\,,$$

and then

$$\lim_{m\to 0} \frac{1}{2}(1 - s\gamma_5 \not{z})(m - \not{p}) = \frac{1}{2}(1 - s\gamma_5)(-\not{p})\,.$$

However, some people replaced this with

$$u_s \bar{u}_s = \frac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p})\,,$$

which is not the same. The point is that as $m \to 0$, $\not{z}$ is order $1/m$ (38.30), so there is another term.

c) Now

$$\begin{aligned}
\sum_{\substack{\text{final} \\ \text{spin}}} |\mathcal{T}|^2 &= -\frac{g^2 m^2}{2} \text{Tr}\big[(1 - s\gamma_5\not{z})(m - \not{p})(1-\gamma_5)\not{p}'(1+\gamma_5)\big] \\
&= -g^2 m^2 \text{Tr}\big[(1 - s\not{z})(m - \not{p})\not{p}'(1+\gamma_5)\big] \\
&= -g^2 m^2 \text{Tr}[(1 - s\not{z})(m - \not{p})\not{p}'] \\
&= -4g^2 m^2 (msz\cdot p' + p\cdot p') \\
&= 2g^2 m^2 (m^2 - M^2)(1 - s\cos\theta)\,,
\end{aligned}$$

where $\cos\theta$ is the angle between the z-axis and the $\mathcal{N}$ momentum, and the problem asks for $s=+1$. The differential decay rate is then

$$\frac{d\Gamma}{d\Omega} = \frac{g^2(m^2-M^2)^2}{32\pi^2 m}(1-\cos\theta)$$

We can understand the angle-dependence as follows. When $\theta = 0$ the momentum is in the z-direction, and the spin must also be in the z-direction by angular momentum conservation. But the neutrino has negative helicity (part b) so the amplitude must vanish.

d) From Srednicki 40.37, 40.41, and 40.47, we see that the γ^μ and $\gamma^\mu\gamma_5$ interactions have opposite P and C transformations, so these symmetries must be violated no matter how ϕ transforms. They have the same T transformation; to cancel the minus signs in 40.41 we need $T^{-1}\phi(x)T = -\phi(\mathcal{T}x)$.

For P, momentum is a vector and spin is a pseudovector, so the helicity changes sign. The helicity-dependent amplitude in (b) therefore implies violation of P. Similarly in (c) the correlation between initial spin and final direction implies violation of P. C.-S. Wu discovered parity violation by a correlation between spin and momentum in nuclear β decay.

We can't probe C in this one calculation. We would have to look also at $\overline{\mathcal{M}} \to \phi\overline{\mathcal{N}}$, and in fact we would find that antiparticles have the opposite correlation between spin and momentum, violating C but preserving CP. To probe T directly we would need to study the inverse process $\phi\mathcal{N} \to \mathcal{M}$.

2. *Consider a theory with two triplets of real scalars, ϕ_i and χ_i, so i runs from 1 to 3. Let*

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\cdot\partial^\mu\phi + \partial_\mu\chi\cdot\partial^\mu\chi + m^2\phi\cdot\phi + m^2\chi\cdot\chi) \\ -\frac{\lambda}{8}(\phi\cdot\phi\,\phi\cdot\phi + \chi\cdot\chi\,\chi\cdot\chi) - \frac{\lambda'}{2}\phi\cdot\chi\,\phi\cdot\chi\,,$$

where the dot products are with respect to i.

a) Find all continuous and discrete internal symmetries. Internal means acting trivially on spacetime, so excludes Lorentz, P, and T, which are obvious.

b) Is this the most general renormalizable Lagrangian in $d = 4$ with these symmetries? If not, what is missing?

c) For $m^2 < 0$ and $\lambda > 0$, find all classical minima of the potential. The coupling λ' is allowed to have either sign: consider all values of λ'.

d) Determine the unbroken continuous symmetry and the number of Goldstone bosons, as a function of λ'.

a) Since all the i indices are dotted, there is an obvious $SO(3)$,

$$\phi_i \to O_{ij}\phi_j\,, \quad \chi_i \to O_{ij}\chi_j\,.$$

The orthogonal matrix O must be the same for both, as there is a term where ϕ_i is dotted into χ_i. Since every term has an even number of ϕ's and and even number of χ's, there are both of

$$Z : (\phi_i, \chi_i) \to (-\phi_i, \chi_i)\,, \quad Z' : (\phi_i, \chi_i) \to (\phi_i, -\chi_i)\,.$$

By this notation I mean that these act on all i at once. i-dependent operations would be part of $SO(3)$. Finally, since the two m's are equal and the two λ's are equal, there is the swap

$$Z'' : (\phi_i, \chi_i) \to (\chi_i, \phi_i)\,.$$

Of course there are also products of these, for example $(\phi_i, \chi_i) \to (-\phi_i, -\chi_i)$ is ZZ'.

b) In the potential, the $SO(3)$ requires that the quadratic terms be of one of the forms

$$\phi{\cdot}\phi\,, \quad \chi{\cdot}\chi\,, \quad \phi{\cdot}\chi\,.$$

The symmetry Z or Z' forbids $\phi{\cdot}\chi$, and Z'' requires that the remaining two have equal coefficients m^2. In the quartic term the $SO(3)$ invariants are

$$(\phi{\cdot}\phi)^2\,, \quad (\chi{\cdot}\chi)^2\,, \quad (\phi{\cdot}\chi)^2\,, \quad \phi{\cdot}\phi\,\chi{\cdot}\chi\,, \quad \phi{\cdot}\chi\,\phi{\cdot}\phi\,, \quad \phi{\cdot}\chi\,\chi{\cdot}\chi\,.$$

The last two are forbidden by Z and Z'. The first four are allowed, with the first two having equal coefficients due to Z''. So we see that a term $-\lambda''\phi{\cdot}\phi\,\chi{\cdot}\chi/4$ should have been included. In fact, there is a one-loop graph, with one λ vertex, and one λ' vertex, whose divergence requires a λ'' counterterm.

c) The only term that depends on the relative orientation ϕ_i and χ_i is the λ' term,

$$\phi{\cdot}\chi\,\phi{\cdot}\chi = \phi{\cdot}\phi\,\chi{\cdot}\chi\cos^2\theta$$

where θ is the angle between ϕ_i and χ_i.

For $\lambda' > 0$ it is favorable for them to be perpendicular,

$$\phi_i = vn_i\,, \quad \chi_i = v'n'_i\,, \quad n{\cdot}n = n'{\cdot}n' = 1\,, \quad n{\cdot}n' = 0\,. \tag{2}$$

The potential becomes

$$V = \frac{1}{2}m^2(v^2 + v'^2) + \frac{1}{8}\lambda(v^4 + v'^4)\,,$$

which is minimized at

$$v^2 = v'^2 = -2m^2/\lambda\,.$$

For $\lambda' < 0$ it is favorable for them to be parallel (or antiparallel),

$$\phi_i = vn_i\,, \quad \chi_i = v'n_i\,, \quad n{\cdot}n = 0\,. \tag{3}$$

The potential becomes

$$V = \frac{1}{2}m^2(v^2 + v'^2) + \frac{1}{8}\lambda(v^4 + v'^4) + \frac{1}{2}\lambda' v^2 v'^2\,,$$

which is minimized at

$$v^2 = v'^2 = -\frac{2m^2}{\lambda + 2\lambda'}\,.$$

However, for $\lambda + 2\lambda' < 0$ there is no minimum, we can decrease the energy without bound by increasing v and v'.

When $\lambda + 2\lambda' = 0$ the potential determines only $v^2 - v'^2$. This case thus has degenerate vacua not related by any symmetries. Such a situation is unnatural (fine-tuned) here, but can arise in supersymmetric theories.

d) For $\lambda' < -\frac{1}{2}\lambda$ there is no vacuum at all so the question is moot. For $-\frac{1}{2}\lambda < \lambda' < 0$, the vacuum (3) is invariant under the $SO(2)$ or $U(1)$ rotation around n_i, so $SO(3)$ is broken to $SO(2)$. There are $3 - 1 = 2$ Goldstone bosons, from the possible choices of n on the two-sphere. For $\lambda' > 0$, the symmetry is fully broken: the $SO(2)$ that leaves n_i invariant rotates n'_i around n_i. The breaking is $SO(3) \to I$ and there are three Goldstone bosons, two from the direction of n_i and one from the angle of n'_i around n_i. For completeness, when λ' is exactly zero there is an $O(3) \times O(3)$ symmetry and the relative orientation of ϕ_i and χ_i is undetermined. The breaking $O(3) \times O(3) \to O(2) \times O(2)$ leaves 4 Goldstone bosons.

3. *Consider the renormalizable $d = 2$ theory*

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=2}^{\infty}\frac{\lambda_n}{(2n)!}\phi^{2n}\,.$$

a) Considering $\mathbf{V}_{2k}$, show that there is a divergent one-loop graph that is first-order in couplings (just one vertex). Calculate Z_{λ_k} and Z_m to this order and then calculate β_{λ_k} and γ_m in $\overline{MS}$.

b) Defining

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{n=2}^{\infty}\frac{\lambda_n}{(2n)!}\phi^{2n}\,,$$

write the equation for the running $m(\mu)$ and $\lambda_n(\mu)$ in terms of $V(\phi, \mu)$.

c) If $V(\phi, \mu_1) = g\cos\beta\phi$ for constants g and β, what is $V(\phi, \mu_2)$? If $V(\phi, \mu_1) = g\cosh\beta\phi$ for constants g and β, what is $V(\phi, \mu_2)$?

d) This also provides a nice example of Wilsonian renormalization. Consider the theory with bare potential $V_1(\phi) = g_1\cos\beta_1\phi$, and a UV cutoff on the propagator momenta,

$k^2 < \Lambda_1^2$. *Consider the contribution to* $\mathbf{V}_{2k}$ *from all graphs that have just one vertex but any number of loops. Show that you get the same result with a lower cutoff* Λ_2 *and a new potential* $g_2 \cos \beta_2 \phi$. *Give* g_2, β_2 *in terms of* g_1, β_1. *Assume that* $m \ll \Lambda_1, \Lambda_2$.

a) There is a divergent contribution to $\mathbf{V}_{2k}$ in which two of the lines from a λ_{k+1} vertex join to form a loop. Then

$$\begin{aligned}
\mathbf{V}_{2k} &= -\lambda_k Z_k - \frac{1}{2}\lambda_{k+1} \int \frac{d^{2-\epsilon}\bar{q}}{(2\pi)^{2-\epsilon}} \frac{1}{\bar{q}^2 + m^2} \\
&= -\lambda_k Z_k - \frac{1}{8\pi}\lambda_{k+1}\Gamma(\epsilon/2)(4\pi/D)^{\epsilon/2} \\
&= -\lambda_k Z_k - \frac{1}{4\pi\epsilon}\lambda_{k+1} + \text{finite}\,.
\end{aligned}$$

So in $\overline{\mathrm{MS}}$,

$$Z_k = 1 - \frac{\lambda_{k+1}}{4\pi\lambda_k\epsilon}$$

to this order. A similar calculation yields

$$Z_m = 1 - \frac{\lambda_2}{4\pi m^2 \epsilon}$$

from the usual graph.

Now,

$$\begin{aligned}
0 &= \mu\partial_\mu\big|_{\text{bare}} \ln \lambda_{k0} \\
&= \mu\partial_\mu\big|_{\text{bare}} \ln(\lambda_k Z_k \tilde{\mu}^{(k-1)\epsilon})
\end{aligned} \tag{4}$$

In the second line, we have used that the dimension of ϕ is $\frac{1}{2}(d-2) = -\epsilon/2$, so the dimension of λ_k is 2 while that of λ_{k0} is $2 + (k-1)\epsilon$. Also, I've set $Z_\phi = 1$ to this order. This becomes

$$\begin{aligned}
0 &= (k-1)\epsilon + \sum_{j=2}^{\infty} \hat{\beta}_{\lambda_j}\partial_{\lambda_j} \ln(\lambda_k Z_k) \\
&= (k-1)\epsilon + \frac{\hat{\beta}_k}{\lambda_k} - \frac{\hat{\beta}_{k+1}}{4\pi\lambda_k\epsilon} + \frac{\hat{\beta}_k \lambda_{k+1}}{4\pi\lambda_k^2\epsilon}\,.
\end{aligned} \tag{5}$$

Expanding in ϵ, we have from order ϵ^1

$$\hat{\beta}_k = -(k-1)\epsilon\lambda_k + \beta_k\,.$$

Order ϵ^0 then gives

$$0 = \frac{\beta_k}{\lambda_k} + \frac{k\lambda_{k+1}\epsilon}{4\pi\lambda_k\epsilon} - \frac{(k-1)\lambda_k\epsilon\lambda_{k+1}}{4\pi\lambda_k^2\epsilon}\,. \tag{6}$$

and so

$$\beta_k = -\frac{\lambda_{k+1}}{4\pi}\,.$$

Similarly one finds

$$\gamma_m = -\frac{\lambda_2}{8\pi m^2}\,.$$

b) We have

$$\mu\partial_\mu m^2 = 2\gamma_m m^2\,, \quad \mu\partial_\mu\lambda_k = \beta_k\,.$$

Then

$$\begin{aligned}
\mu\partial_\mu V(\phi,\mu) &= \gamma_m m^2\phi^2 + \sum_{n=2}^{\infty}\frac{\beta_n}{(2n)!}\phi^{2n} \\
&= -\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{\lambda_{n+1}}{(2n)!}\phi^{2n} \\
&\overset{m=n+1}{=} -\frac{1}{4\pi}\frac{\partial^2}{\partial\phi^2}\sum_{m=1}^{\infty}\frac{\lambda_m}{(2m)!}\phi^{2m} + \text{constant} \\
&= -\frac{1}{4\pi}\frac{\partial^2}{\partial\phi^2}V(\phi,\mu) + \text{constant}\,.
\end{aligned} \tag{7}$$

We have defined $\lambda_1 = m^2$ for uniformity. As usual, the constant is not important. If you did not find this exact form, that's fine, but this makes the next part easier.

c) Using part (b), we get for $g\cos b\phi$ (I're renamed β to b to avoid confusion)

$$\mu\partial_\mu g = \frac{b^2}{4\pi}g\,, \quad \mu\partial_\mu b = 0\,.$$

So

$$g(\mu_2) = (\mu_2/\mu_1)^{b^2/4\pi}g(\mu_1)\,, \quad V(\phi,\mu_2) = (\mu_2/\mu_1)^{b^2/4\pi}V(\phi,\mu_1)\,.$$

for the cosine potential. Similarly

$$g(\mu_2) = (\mu_2/\mu_1)^{-b^2/4\pi}g(\mu_1)\,, \quad V(\phi,\mu_2) = (\mu_2/\mu_1)^{-b^2/4\pi}V(\phi,\mu_1)\,.$$

for the cosh potential.

Note, by the way, that the dimensionless coupling $\hat{g}(\mu) = g(\mu)/\mu^2$ satisfies

$$\hat{g}(\mu_2) = (\mu_2/\mu_1)^{b^2/4\pi - 2}\hat{g}(\mu_1)$$

in the cosine case. This coupling is *relevant*, meaning that the dimensionless size grows at low energy, only for $b^2 < 8\pi$, and it is irrelevant for larger values. The point $b^2 = 8\pi$ is known as the Kosterlitz-Thouless phase transition.

d) A vertex λ_{2k+2l} gives a divergent contribution to $\mathbf{V}_{2k}$ in which $2l$ of the external lines pair up into l loops, each a copy of the one-loop divergence above:

$$\mathbf{V}_{2k} = -\sum_{l=0}^{\infty} \lambda_{k+l} \frac{1}{2^l l!} \left(\int \frac{d^2\bar{q}}{(2\pi)^2} \frac{1}{\bar{q}^2 + m^2} \right)^l .$$

The symmetry factor 2^l comes from interchanging the two ends of each loop. The symmetry factor $l!$ comes from permuting the loops.

Now, the Wilsonian idea is that we focus on the effect of modes between Λ_1 and Λ_2. If we only include those in the integral, we get

$$\int_{\Lambda_2}^{\Lambda_1} \frac{d^2\bar{q}}{(2\pi)^2} \frac{1}{\bar{q}^2 + m^2} = \frac{1}{2\pi} \ln(\Lambda_1/\Lambda_2)$$

for negligible m^2. We can then write

$$\mathbf{V}_{2k} = -\lambda_{k,2}$$

where

$$\lambda_{k,2} = \sum_{l=0}^{\infty} \lambda_{k+l,1} \frac{1}{l!} \left(\frac{1}{4\pi} \ln(\Lambda_1/\Lambda_2) \right)^l .$$

That is, we absorb the virtual effects in this range into the effective coupling at lower scale Λ_2. Now, for $V_1(\phi) = g_1 \cos b_1\phi$, $\lambda_{k,1} = (-b_1)^{2k}$. Then we can do the sum,

$$\lambda_{k,2} = (-b_1)^{2k} (\Lambda_1/\Lambda_2)^{-b_1^2/4\pi^2}$$

as found above.

It's a bit different because here we're talking about cutoffs Λ, and before about the reference scale μ. But we can think about setting the cutoff just slightly above the scale of interest, so as to integrate out all higher energy effects, so it comes to the same thing.