Physics 221A: QFT I
w/ Prof. Joe Polchinski

Solution Set #1

0/ I'll use $a/a^\dagger$ instead of $\chi/\chi^\dagger$ to avoid confusion with x.

First we want to show

$$|x_1 \ldots x_n\rangle = a^\dagger_{x_1} \ldots a^\dagger_{x_n}|0\rangle, \text{ where } a^\dagger_x \equiv a^\dagger(x).$$

This is just the definition of the state with particles at $x_1 \ldots x_n$, but we can check that it has the properties we'd like it to have when defined this way.

First, if the particles are bosonic (resp. fermionic), $|x_1 \ldots x_n\rangle$ should be (anti) symmetric under exchanging any of the x_i. This follows immediately from the canonical (anti) commutation relations for the $a^\dagger_{x_i}$. Let σ be a map permuting the numbers $1 \ldots n$, i.e. $\sigma \in \mathrm{Perm}(n)$. Then $|x_{\sigma(1)} \ldots x_{\sigma(n)}\rangle = (\det\sigma)^F |x_1 \ldots x_n\rangle$, (*) where F is the fermion # $= \begin{cases} 0 & \text{bosons} \\ 1 & \text{fermions} \end{cases}$, and $\det\sigma$ is $+1$ if $\sigma(1) \ldots \sigma(n)$ is an even permutation of $1 \ldots n$, -1 if it's odd.

If the particle is bosonic, symmetry under particle exchange follows immediately because the $a^\dagger$'s commute. If we have a fermion, we get the more complicated action (*) from the anticommutation relations, since if σ is odd, we have to anticommute an odd # of times, picking up a -1.

$\underline{\text{① cont'd}}$

We also want $a_x|x_1 \ldots x_n\rangle \sim \sum_j \delta(x-x_j)|x_1 \ldots \cancel{x_j} \ldots x_n\rangle$

So that $a_x|x_1 \ldots x_n\rangle$ is the state with $n-1$ particles at $x_1 \ldots \cancel{x_j} \ldots x_n$,

and $a^\dagger_x a_x|x_1 \ldots x_n\rangle = \sum_j \delta(x-x_j)|x_1 \ldots x_n\rangle$, so that $|x_1 \ldots x_n\rangle$

is an eigenstate of the number operator, vanishing away from the x_i and returning the state (with the usual δ-function normalization) when $x = x_i$.

All that remains is to compute. The upper sign is always for bosons.

$$
\begin{aligned}
a^\dagger_x a_x|x_1 \ldots x_n\rangle &= a^\dagger_x a_x a^\dagger_{x_1} \ldots a^\dagger_{x_n}|0\rangle \\
&= a^\dagger_x [a_x, a^\dagger_{x_1}]_\mp a^\dagger_{x_2} \ldots a^\dagger_{x_n}|0\rangle \pm a^\dagger_x a^\dagger_{x_1} a_x a^\dagger_{x_2} \ldots a^\dagger_{x_n}|0\rangle \\
&= a^\dagger_x [a_x, a^\dagger_{x_1}]_\mp a^\dagger_{x_2} \ldots a^\dagger_{x_n}|0\rangle \pm a^\dagger_x a^\dagger_{x_1} [a_x, a^\dagger_{x_2}]_\mp a^\dagger_{x_3} \ldots a^\dagger_{x_n}|0\rangle \\
&\quad + \ldots (\pm)^{n-1} a^\dagger_x a^\dagger_{x_1} \ldots [a_x, a^\dagger_{x_n}]|0\rangle
\end{aligned}
$$

(where the series terminates because a annihilates the vacuum)

$$
\begin{aligned}
&= a^\dagger_x \sum_{j=1}^n \delta(x-x_j) a^\dagger_{x_1} \ldots \cancel{a^\dagger_{x_j}} \ldots a^\dagger_{x_n}|0\rangle \cdot (\pm)^{j-1} \\
&= \sum_{j=1}^n \delta(x-x_j) a^\dagger_{x_1} \ldots a^\dagger_{x_j} \ldots a^\dagger_{x_n}|0\rangle \\
&= \sum_{j=1}^n \delta(x-x_j)|x_1 \ldots x_n\rangle \checkmark
\end{aligned}
$$

In the second to last line, I (anti) commuted $a^\dagger_x$ $j-1$ times to get it in the j^{th} position, which killed the $(\pm 1)^{j-1}$ in the previous line. I also used the properties of the δ function in setting $a^\dagger_x = a^\dagger_{x_j}$.

1/ Let $\psi(x_1 \dots x_n; t)$ be the quantum-mechanics position-space wave function.

Here we will take $\psi \in \mathbb{C}$, but more generally ψ could be valued in an arbitrary group manifold in some representation of the gauge symmetry group.

We want to show that if $i\hbar \frac{\partial}{\partial t}\psi = \underbrace{\left[\sum_j \left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(x_j)\right) + \sum_{ij} V(x_i - x_j)\right]}_{H_{QM}}\psi$,

then the QFT state $|\psi\rangle \equiv \int \prod_i (dx_i\, a^\dagger_{x_i})\, \psi(x_1 \dots x_n; t)|0\rangle$

obeys $i\hbar\, \partial/\partial t\, |\psi\rangle = \underbrace{\left[\int dx\, a^\dagger_x \left(-\frac{\hbar^2}{2m}\nabla_x^2 + U(x)\right)a_x + \frac{1}{2}\int dx\,dy\, a^\dagger_x a^\dagger_y V(x-y) a_y a_x\right]}_{H_{QFT}}|\psi\rangle$

I'll do this piece by piece.

$$\int dx\, a^\dagger_x \nabla_x^2 a_x|\psi\rangle = \int dx \prod_i dx_i\, a^\dagger_x \nabla_x^2 a_x a^\dagger_{x_1} \dots a^\dagger_{x_n} \psi(x_1 \dots x_n, t)|0\rangle$$

$$= \sum_j \int dx \prod_i dx_i\, \psi(x_1 \dots x_n, t)\, a^\dagger_x \nabla_x^2 \delta(x - x_j)\, a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n}|0\rangle \cdot (\pm 1)^{j-1}$$

bosons / fermions

by (anti)commuting a_x to the right in steps

$$= \sum_j \int dx \prod_i dx_i\, \nabla_x^2 \psi(x_1 \dots x_n, t)\, a^\dagger_x \delta(x - x_j)\, a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n}|0\rangle \cdot (\pm 1)^{j-1}$$

under integrating by parts and using the boundary condition requiring ψ to fall off sufficiently fast @ ∞

$$= \sum_j \int \prod_i dx_i\, \nabla_{x_j}^2 \psi(x_1 \dots x_n, t)\, a^\dagger_{x_j} a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n}|0\rangle (\pm)^{j-1}$$

by performing the x-integral

1 cont'd

$$= \sum_j \int \prod_i dx_i \, \nabla_{x_j}^2 \psi(x_1 \dots x_n, t)\, a^\dagger_{x_1} \dots a^\dagger_{x_n} |0\rangle$$

upon (anti)commuting $a^\dagger_{x_j}$ to the right $j-1$ times. Next,

$$\int dx\, a^\dagger_x U(x) a_x |\psi\rangle = \sum_j \int \prod_i dx_i\, U(x_j) \psi(x_1 \dots x_n, t)\, a^\dagger_{x_1} \dots a^\dagger_{x_n} |0\rangle$$

by identical logic, only we don't have to integrate by parts to get ∇^2 off the δ-fxn.

Even the V term is not so bad if we keep our heads on straight:

$$\frac{1}{2} \int dx\,dy\, V(x-y)\, a^\dagger_x a^\dagger_y a_y a_x \int \prod_i dx_i\, a^\dagger_{x_1} \dots a^\dagger_{x_n} \psi(x_1 \dots x_n, t) |0\rangle$$

$$= \frac{1}{2} \sum_j \int dx\,dy \prod_i dx_i\, a^\dagger_x a^\dagger_y a_y V(x-y) \delta(x - x_j)\, a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n} |0\rangle (\pm 1)^{j-1}$$

$$= \sum_{k<j} \int dx\,dy \prod_i dx_i\, a^\dagger_x a^\dagger_y V(x-y)\, \delta(x - x_j)\, \delta(y - x_k)\, a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_k}} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n} |0\rangle (\pm)^{j+k}$$

↑ Make sure you understand this part – I saved a lot of work here

$$= \sum_{k<j} \int \prod_i dx_i\, a^\dagger_{x_j} a^\dagger_{x_k} V(x_j - x_k)\, a^\dagger_{x_1} \dots \cancel{a^\dagger_{x_k}} \dots \cancel{a^\dagger_{x_j}} \dots a^\dagger_{x_n} |0\rangle (\pm)^{j+k}$$

$$= \sum_{k<j} \int \prod_i dx_i\, V(x_j - x_k)\, a^\dagger_{x_1} \dots a^\dagger_{x_n} \psi(x_1 \dots x_n, t)$$

$$\Rightarrow H_{QFT} |\psi\rangle = \int \prod_i dx_i\, H_{QM} \psi(x_1 \dots x_n, t)\, a^\dagger_{x_1} \dots a^\dagger_{x_n} |0\rangle$$

$$= \int \prod_i dx_i \left(i\hbar \partial_t \psi(x_1 \dots x_n, t) \right) a^\dagger_{x_1} \dots a^\dagger_{x_n} |0\rangle$$

$$= i\hbar \partial_t |\psi\rangle$$

Since the creation/annihilation operators have no time dependence in the Schrödinger picture.

2/ $[A, BC] = ABC - BCA$

$$= ABC - BAC - BCA + BAC$$

$$= [A,B]C + B[A,C]$$

$[A, BCDE] = [A,BC]DE + BC[A,DE]$ (using the previous result, replacing $B \to BC$, $C \to DE$)

$$= [A,B]CDE + B[A,C]DE + BC[A,D]E + BCD[A,E]$$

(again).

Now, let's calculate $[N, a_x]$ and $[N, a_x^\dagger]$.

$$[N, a_x] = \left[\int dy\, a^\dagger_y a_y, a_x\right] = -\int dy\, [a_x, a^\dagger_y a_y] = -\int dy\, [a_x, a^\dagger_y] a_y - \int dy\, a^\dagger_y \overset{0}{\cancel{[a_x, a_y]}}$$

$$= -a_x$$

$[N, a_x^\dagger] =$ (previous with $a_x \to a_x^\dagger$) $= -\int dy\, \overset{0}{\cancel{[a^\dagger_x, a^\dagger_y]}} a_y - \int dy\, a^\dagger_y [a^\dagger_x, a_y]$

$$= +a_x$$

The clever approach below is due to Srednicki.

$$[N, a^\dagger_{x_1} \dots a^\dagger_{x_n}] = [N, a^\dagger_{x_1}] a^\dagger_{x_2} \dots a^\dagger_{x_n} + a^\dagger_{x_1} [N, a^\dagger_{x_2}] a^\dagger_{x_3} \dots a^\dagger_{x_n} + \dots + a^\dagger_{x_1} \dots a^\dagger_{x_{n-1}} [N, a^\dagger_{x_n}]$$

$= n\, a^\dagger_{x_1} \dots a^\dagger_{x_n}$ (of course, since there are n particles)

Similarly $[N, a_{x_1} \dots a_{x_m}] = -m\, a_{x_1} \dots a_{x_m}$ (" since $a_{x_1} \dots a_{x_m}$ removes m particles from the state)

$$\Rightarrow [N, a^\dagger_{x_1} \dots a^\dagger_{x_n} a_{y_1} \dots a_{y_m}] = [N, a^\dagger_{x_1} \dots a^\dagger_{x_n}] a_{y_1} \dots a_{y_m} + a^\dagger_{x_1} \dots a^\dagger_{x_n} [N, a_{y_1} \dots a_{y_m}]$$

$$= (n-m)\, a^\dagger_{x_1} \dots a^\dagger_{x_n} a_{y_1} \dots a_{y_m}$$

Since H only contains terms with $n=m$ (either $n=m=1$, or 2 in the case of the V term), its commutator $[N,H] = 0 \Rightarrow N$ is constant in time. In general, $[N,H] \neq 0$.