Physics 221A: HW2 solutions

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1. Srednicki 3.3. Before we start, a couple remarks on what we're showing in the problem. $U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ means that the unitary action implementing a lorentz transformation maps the field operator evaluated at one spacetime point to the field operator evaluated at the lorentz-transformed spacetime point $\Lambda^{-1}x$. This is the active picture of transformation. We have some definition of $a(k)$ and we'd like to make sure that this is true also for these operators. Alternatively, we can take the passive picture and transform the states rather than the operators. We define momentum eigenstates as in the problem; we'd like to see that the unitary action implementing the lorentz transformation in the passive picture, $U(\Lambda)|k\rangle$, is equal to $|\Lambda^{-1}k\rangle$.

   The proof involves a tricky step, so let's work the free field case first to see what's going on. For a free real scalar field, you can use equation (3.21) to write

$$a_k = \int dx e^{-ikx}(i\Pi + \omega\phi) \tag{1}$$

   We will conjugate both sides of the equation by $U(\Lambda)$, in other words we'll compute

$$U(\Lambda)^{-1}a_k U(\Lambda) = U(\Lambda)^{-1}\int dx e^{-ikx}\left(i\Pi(x) + \omega\phi(x)\right)U(\Lambda) \tag{2}$$

   Take a time derivative of both sides of this equation to see that Π transforms in the same way. This tells us

$$\begin{aligned}
U(\Lambda)^{-1}a_k U(\Lambda) &= \int dx e^{-ikx}\left(i\Pi(\Lambda^{-1}x) + \omega\phi(\Lambda^{-1}x)\right) && (3)\\
&= \int dy |\det\Lambda| e^{-ik\cdot(\Lambda y)}\left(i\Pi(y) + \omega\phi(y)\right) && (4)\\
&= \int dy e^{-i(\Lambda^{-1}k)\cdot y}\left(i\Pi(y) + \omega\phi(y)\right) && (5)\\
&= a_{\Lambda^{-1}k} && (6)
\end{aligned}$$

   In the second line I changed integration variable from x to $\Lambda^{-1}x = y$. In the next line I used $|\det\Lambda| = 1$ and $\Lambda^T = \Lambda^{-1}$, both conditions on an SO(3,1) matrix. a transforms the same way as ϕ under the unitary action implementing the lorentz transformation. This shows the claim for a free field.

Now, for a real scalar with arbitrary potential and maybe couplings to other particles, we still have the usual mode expansion, since that is a natural expansion to make so long as we have the symmetries of Minkowski space, but we don't know the creation/annihilation operators in terms of the fields because the a_k in general will be time-dependent.

First, write $\phi(x)$ in terms of its fourier expansion $\tilde{\phi}(k)$. The same logic as in the free field case shows that $\tilde{\phi}$ transforms the same way as ϕ under lorentz transformations. Here is the trick: if you plug the ansatz

$$\tilde{\phi}(k) = 2\pi\delta(k^2 + m^2)\left[\theta(k^0)a_k + \theta(-k^0)a^\dagger_{-k}\right] \tag{7}$$

into the fourier expansion

$$\phi(x) = \int (dk/2\pi)e^{ikx}\tilde{\phi}(k) \tag{8}$$

you get the usual mode expansion for a real scalar. Taking $k^0 > 0$, this tells us that a_k transforms the same way as $\tilde{\phi}(k)$ under Λ, namely the same way as $\phi(x)$. Note that we needed to use the fact that Λ is a proper, orthochronous lorentz transformation (an element of the connected component of SO(3,1)) to assume that $\theta(k^0)$ was invariant under $U(\Lambda)$.

Now, we want to check that (multiparticle) states also transform in the expected way under the action of the lorentz symmetry.

$$\begin{aligned} U(\Lambda)|k_1 \dots k_n\rangle &= U(\Lambda)a^\dagger_{k_1}\dots a^\dagger_{k_n}|0\rangle & (9)\\ &= U(\Lambda)a^\dagger_{k_1}U(\Lambda)^{-1}U(\Lambda)a^\dagger_{k_1}U(\Lambda)^{-1}\dots U(\Lambda)a^\dagger_{k_n}U(\Lambda)^{-1}|0\rangle & (10)\\ &= |\Lambda^{-1}k_1\dots\Lambda^{-1}k_n\rangle & (11) \end{aligned}$$

The second line follows from the unitarity of U and the U-invariance of the vacuum state. We have also showed that the operator products transform in the way that we expect.

2. a) Srednicki 3.4. In this problem we're checking that the unitary action of the translations $\subset$ the poincare group on the scalar field ϕ some properties that we'd expect from NRQM. This problem is all about the spacetime translations. I'll write ϵa to make explicit the infinite smallness. We'd like to check that the infinitesimal action reproduces the algebra of the symmetry generators with the fields. This is all in the position basis.

$$T(\epsilon a)\phi(x)T(-\epsilon a) = \phi(x) + ia\cdot[\phi, p] = \phi(x - \epsilon a) + O(\epsilon^2) \tag{12}$$

but

$$\phi(x - \epsilon a) = \phi(x) - \epsilon a \cdot \partial\phi + O(\epsilon^2) \tag{13}$$

Collecting terms of order epsilon on both sides,

$$[\phi, p^\mu] = \frac{1}{i}\partial^\mu\phi \tag{14}$$

This is just what you would expect in the position basis for the generator of spatial translations.

b) This follows immediately from (14). The time derivative $\dot{\phi}$ is $\partial_0 = -\partial^0$, not ∂^0 as we have on the right hand side of (14), and there's a sign flip from the metric factor g^{00}. If you don't have an intuition for why lower indices are natural for derivatives, think of the gradient $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ = the spatial components of $\partial_\mu f$.

c) Now we check that the Heisenberg equation of motion + canonical commutation relations imply the Klein-Gordon equation, bringing us back to QFT. The calculation is straightforward. I'll use $\Box$ and ∇ to distinguish between the laplacian on Minkowski space and just in the spatial indices, respectively.

$$\begin{aligned}
H &= \frac{1}{2}\int dy\left(\Pi(y)^2 + (\nabla\phi(y))^2 + m^2\phi^2\right) && (15)\\
&\Rightarrow i\dot{\Pi} = [\Pi(x), H] && (16)\\
&= \frac{1}{2}\int dy[\Pi(x), \nabla\phi(y)\cdot\nabla\phi(y)] + \frac{1}{2}\int dy[\Pi(x), \phi(y)^2]m^2 && (17)\\
&= -i(-\nabla^2 + m^2)\phi && (18)\\
&= i\dot{\Pi} && (19)
\end{aligned}$$

In the third line I integrated by parts before using the commutation relations. Since $\dot{\Pi} = \ddot{\phi}$, the last line gives

$$(-\partial_t^2 + \nabla^2 - m^2)\phi = 0 \tag{20}$$

This wouldn't break even if you added higher-order non-derivative terms in the potential. It's only a few lines of algebra to generalize to derivative potentials.

d) $[\phi(x), \vec{P}] = -[\phi(x)\int dy\Pi(y)\vec{\nabla}\phi(y)] = -i\vec{\nabla}\phi(x)$. The hamiltonian + CCR for a field with a non-derivative (velocity-independent) potential imply the usual relation for the commutator of a position-dependent operator with $\vec{P}$.

e)

$$P = -\int dx\tilde{d}k\tilde{d}k' \left(-i\omega a_k e^{ikx} + i\omega a_k^\dagger e^{-ikx}\right)\left(ik' a_{k'} e^{ik'x} - ik' a_{k'}^\dagger e^{-ik'x}\right) \tag{21}$$

$$= -(2\pi)^3 \int \tilde{d}k\tilde{d}k' [\delta(k-k')(-\omega k)\left(a_k^\dagger a_{k'} e^{-i(\omega-\omega')t} + a_{k'}^\dagger a_k e^{i(\omega-\omega')t}\right) \tag{22}$$

$$+ \delta(k+k')(\omega k)(\text{similar stuff})] \tag{23}$$

$$= \frac{1}{2}\int \tilde{d}kk\left(a_k^\dagger a_k + a_k a_k^\dagger\right) \tag{24}$$

$$= \int \tilde{d}kk\left(a_k^\dagger a_k + \text{more stuff}\right) \tag{25}$$

$$= \int \tilde{d}kka_k^\dagger a_k \tag{26}$$

To get the third equality I used the fact that the similar stuff is odd under $k \mapsto -k$ and so vanishes under integration. In the fourth equality more stuff is just a constant that comes from exchanges a and $a^\dagger$, which is odd under $k \mapsto -k$ when multiplied by k in the integrand.

This is what you would have expected. You can check that it obeys $P|k\rangle = k|k\rangle$.

3. a)

$$\delta\mathcal{L} = -\partial^\mu \delta\phi \partial_\mu \phi^\dagger - \partial^\mu \phi \partial_\mu \delta\phi^\dagger - m^2\phi^\dagger\delta\phi - m^2\delta\phi^\dagger\phi \tag{27}$$

$$= (\Box - m^2)\phi^\dagger\delta\phi + (\Box - m^2)\phi\delta\phi^\dagger \tag{28}$$

$$\tag{29}$$

In the second line I integrated by parts to get the derivatives off the variations $\delta\phi$. Requiring this variation to vanish for arbitrary $\delta\phi$, $\delta\phi^\dagger$ gives the klein-gordon equations of motion for $\phi^\dagger$ and ϕ.

b)

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger, \quad \Pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger} = \dot{\phi} \tag{30}$$

The fields are each others' momentum conjugates. The hamiltonian density is

$$\mathcal{H} = \Pi\dot{\phi} + \Pi^\dagger\dot{\phi}^\dagger - \mathcal{L} \tag{31}$$

$$= 2\Pi\Pi^\dagger - \Pi\Pi^\dagger + (\nabla\phi)\cdot(\nabla\phi^\dagger) + m^2\phi^\dagger\phi - \Omega_0 \tag{32}$$

$$= \Pi\Pi^\dagger + (\nabla\phi)\cdot(\nabla\phi^\dagger) + m^2\phi^\dagger\phi - \Omega_0 \tag{33}$$

$$\tag{34}$$

c) Equation (3.21) gives a_k in terms of ϕ, when ϕ is a real field. However that form doesn't depend on the term multiplying the negative-frequency modes, so even when there's a $b^\dagger$ there instead of an $a^\dagger$, nothing changes. So that still gives us a_k. By conjugating the expansion of the complex scalar in (3.38) you can see that b_k is given by (3.21) with $\phi \to \phi^\dagger$.

d) From the last part, a is expressed in terms of ϕ and $\Pi^\dagger$, while b is expressed in terms of $\phi^\dagger$ and Π. This tells us that $[a, b^\dagger] = [a^{dagger}, b] = 0$, since each a and b^{dagger} contains only ϕ and $\Pi^\dagger$ terms, which commute. The same logic tells us that $[a, a]$ and $[b, b]$ must vanish. The only nontrivial ones are $[a, a^\dagger]$ and $[b, b^\dagger]$:

$$[a_k, a^\dagger_{k'}] = \int dx dx' e^{-ikx} e^{ik'x'} \left[\omega_x \phi(x) + i\Pi^\dagger(x), \omega_{x'} \phi^\dagger(x') - i\Pi(x')\right] \tag{35}$$

$$= \int dx dx' e^{-ikx} e^{ik'x'} \left(-i\omega_x [\phi(x), \Pi(x')] + i\omega_{x'} [\Pi^\dagger(x), \phi^\dagger(x')]\right) \tag{36}$$

$$= \int dx dx' e^{-i(k-k')x} 2\omega_x \tag{37}$$

$$= 2\omega\delta(k - k') \tag{38}$$

$$\tag{39}$$

e) This is a straightforward computation. There is literally nothing new here, you just have to follow the derivation of the 0-point energy in the text. You find that the result is the same, except that you get two terms $aa^\dagger$ and $b^\dagger b$ multiplying $\delta(k - k')$ instead of just one as in the scalar field case. This leads directly to the 0-pt energy being twice as large.

You can see this another way too. If you wrote $\phi(x) = f_1(x) + if_2(x)$ (this is just a redefinition of the field), where f_{12} are real scalar fields. Plugging this new ϕ into the complex scalar lagrangian gives the lagrangian for two uncoupled real scalar fields. Each of these has its own set of creation/annihilation operators and contributes $\frac{1}{2}\hbar\omega V$, where V is the volume of space, to the 0-point energy.