Physics 221A: HW3 solutions

October 22, 2012

1. a) It will help to start things off by doing some gaussian integrals. Let x be a real vector of length N, and let's compute $\int dx e^{-\frac{1}{2}x^T A x}$, where A is some real $N \times N$ matrix. First write the exponent as as $A_{ij}x^i x^j$. In the upper indices this is symmetric in $i \leftrightarrow j$, so only the symmetric part of A will contribute and we can set $A = A^T$. This means we can diagonalize A, $A_{ij} = \lambda_i \delta_{ij}$ (no sum). So we get

$$I_1 = \int dx e^{-x^T A x} = \prod_{i=1}^{N} \left[\int dy_i e^{-\frac{1}{2}\lambda_i y_i^2} \right] = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = \sqrt{\frac{(2\pi)^N}{\det A}} \tag{1}$$

where y is an eigenbasis of A, obtained from x by an orthogonal coordinate transformation $y = Ox$. I'll write e^{-Ax^2} for $e^{-x^T A x}$.

What about integrals with a linear term, $e^{-\frac{1}{2}Ax^2 - Bx}$? These aren't much harder, just complete the square and shift the integration variable $x \mapsto x + A^{-1}B$ to get

$$\int dx e^{-\frac{1}{2}Ax^2 - Bx} = \int dx e^{-\frac{1}{2}A(x+A^{-1}B)^2 + \frac{1}{2}BA^{-1}B} = e^{\frac{1}{2}BA^{-1}B} I_1 \tag{2}$$

Now on to the problem. We want to evaluate the momentum integrals in the path integral expression giving the amplitude for the particle to travel from point q' at time t' to point q'' at time t'':

$$\langle q'', t'' | q', t' \rangle = \int_{q(t')=q',\ q(t'')=q''} \mathcal{D}q \mathcal{D}p e^{iS} \tag{3}$$

$$=_{N\to\infty} \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{-i\left(\frac{p_i p_i}{2m} - \frac{p_i \delta q_i}{\delta t} + V(q)\right)\delta t} \tag{4}$$

$$\tag{5}$$

where $\delta q_j = q_{j+1} - q_j$.

Comparing, we see that $A = i\delta t/m$ and $B = i\delta q_j$, which we'll write $i\dot{q}_j \delta t$. There are $N+1$ of these integrals, and we get a $\frac{1}{(2\pi)^{N+1}}$ from the $dp/2\pi$. This gives the result

$$\langle q'', t'' | q', t' \rangle =_{N\to\infty} \int \prod_{k=1}^{N} dq_k \left(\sqrt{\frac{m}{2\pi i \delta t}} \right)^{N+1} e^{i\delta t\left(\frac{1}{2}m\delta\dot{q}^2 + V(q)\right)} \tag{6}$$

b) Now setting V to zero, we want to calculate the integrals over the qs,

$$\int \prod_{k=1}^{N} dq_k \left(\sqrt{\frac{m}{2\pi i \delta t}}\right)^{N+1} e^{\frac{im}{2\delta t}\sum_{j=0}^{N}(q_{j+1}-q_j)^2} \tag{7}$$

Consider just the integral involving q_1:

$$I_{q_1} = \int dq_1 e^{\frac{a}{2}(q_1-q_0)^2} e^{\frac{a}{2}(q_2-q_1)^2} \tag{8}$$

$$= e^{\frac{1}{2}a(q_0^2+q_2^2)} \int dq_1 e^{aq_1^2 - aq_1(q_0+q_2)} \tag{9}$$

$$\Rightarrow B_1 = -a(q_0+q_2), \quad A_1 = -2a \tag{10}$$

$$\Rightarrow I_{q_1} = \sqrt{\frac{1}{2}(2\pi a^{-1})} e^{-\frac{1}{4}a(q_0+q_2)^2} e^{\frac{1}{2}a(q_0^2+q_2^2)} \tag{11}$$

$$= \sqrt{\frac{1}{2}(2\pi a^{-1})} e^{-\frac{1}{4}a(q_2-q_0)^2} \tag{12}$$

$$\tag{13}$$

where $a = -im/\delta t$. Then

$$I_{q_2} = \int dq_2 I_{q_1} e^{-\frac{a}{2}(q_3-q_2)^2} \tag{14}$$

$$= \int dq_2 \sqrt{\frac{1}{2}(2\pi a^{-1})} e^{-\frac{a}{4}a(q_2-q_0)^2} e^{-\frac{a}{2}(q_3-q_2)^2} \tag{15}$$

$$= \sqrt{\frac{1}{2}(2\pi a^{-1})}\sqrt{\frac{2}{3}(2\pi a^{-1})} e^{-\frac{a}{6}(q_3-q_0)^2} \tag{16}$$

$$\tag{17}$$

At this point it's probably easiest to just guess the general form and show it by induction. If the $k-1$th integral term generates a factor $e^{-\frac{a}{2k}(q_k-q_0)^2}$ (times stuff that doesn't matter) then

$$I_{q_k} \sim \int dq_k e^{-\frac{a}{2}(q_{k+1}-q_k)^2} e^{-\frac{a}{2k}(q_k-q_0)^2} \tag{18}$$

$$= \sqrt{\frac{k}{k+1}(2\pi a^{-1})} e^{-\frac{a}{2(k+1)}(q_{k+1}-q_0)^2} \tag{19}$$

$$\tag{20}$$

We're intereted in the product of all these guys:

$$I_{q_N} = e^{-\frac{a}{2(N+1)}(q_{N+1}-q_0)^2} \prod_{k=1}^{N} \sqrt{\frac{k}{k+1}(2\pi a^{-1})} \tag{21}$$

$$= \sqrt{\frac{1}{N+1}} \sqrt{2\pi a^{-1}}^{N} e^{-\frac{a}{2(N+1)}(q_{N+1}-q_0)^2} \tag{22}$$

Bringing in the normalization from part a) gives the final result

$$\langle q'', t'' | q', t' \rangle = \left(\sqrt{\frac{m}{2\pi i \delta t}} \right)^{N+1} I_{q_N} \tag{23}$$

$$= \sqrt{\frac{m}{2\pi i (N+1)\delta t}} e^{\frac{im}{2(N+1)\delta t}(q''-q')^2} \tag{24}$$

$$= \sqrt{\frac{m}{2\pi i (t''-t')}} e^{\frac{im}{2(t''-t')}(q''-q')^2} \tag{25}$$

c) This is easy because for the free theory inserting a complete set of $|p\rangle$ diagonalizes the entire hamiltonian in one fell swoop. The computation proceeds straightforwardly from the definitions and the machinery from the first part.

$$\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iHt''} e^{iHt'} | q \rangle \tag{26}$$

$$= \int \frac{dp}{2\pi} \langle q'' | e^{-iH(t''-t')} | p \rangle \langle p | q' \rangle \tag{27}$$

$$= \int \frac{dp}{2\pi} e^{ip(q''-q')} e^{-i(t''-t')\frac{p^2}{2m}} \tag{28}$$

$$= \sqrt{\frac{m}{2\pi i (t''-t')}} e^{i\frac{m(q''-q')^2}{2(t''-t')}} \tag{29}$$

$$\tag{30}$$

2. Consider taylor expanding $\mathcal{L}(q, \dot{q})$ about the classical value:

$$\mathcal{L} = \mathcal{L}_{cl} + \left(\frac{\partial \mathcal{L}}{\partial q} \Delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \Delta \dot{q} \right)_{q_{cl}} + \frac{1}{2} \left(\frac{\partial^2 \mathcal{L}}{\partial q^2} \Delta q^2 + 2 \frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}} \Delta \dot{q} \Delta q + \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \Delta \dot{q}^2 \right)_{q_{cl}} + O(\Delta^3) \tag{31}$$

The first variation vanishes; this is the statement that we are expanding around a classical solution (saddle point of the action).

Now specialize to the free particle lagrangian, $\mathcal{L} = \frac{1}{2} m \dot{q}^2$. The first two terms in the second variation vanish in this case, and there are no terms of order δ^3, but this isn't necessary for the general conclusions of this problem.

Let's think about $\langle q'', t'' | q', t' \rangle$. We're integrating over all paths taking us between the two points, which is the same as integrating over all variations Δq about the classical path. As for the boundary conditions, we're fixing q on the endpoints, which is equivalent to fixing $\Delta q = 0$ on the endpoints since we're taking q_{cl} to hit our desired endpoints at the appropriate times.

$$\langle q'', t''|q', t'\rangle \sim \int_{q(t')=q',\ q(t'')=q''} \mathcal{D}q e^{i\mathcal{L}} \tag{32}$$

$$= e^{i\mathcal{L}_{cl}} \int_{\Delta q(t')=0, \Delta q(t'')=0} \mathcal{D}\Delta q e^{i(\Delta^2 + \dots)} \tag{33}$$

$$\tag{34}$$

In the free theory this is

$$= e^{i\mathcal{L}_{cl}} \int_{\Delta q(t')=0, \Delta q(t'')=0} \mathcal{D}\Delta q e^{i\frac{\Delta \dot{q}^2}{2m}} \tag{35}$$

$$\tag{36}$$

The only dependence on the endpoints appears in the prefactor $\mathcal{L}_{cl}$, the action of the classical trajectory from (q', t') to (q'', t'').

The second factor in this expression represents all the quantum physics correcting the classical action for particle propagation.

3. $|q, t\rangle$ is defined to be the instantaneous eigenstate of the operator $Q(t)$ in the heisenberg picture, $Q(t)|q, t\rangle = q|q, t\rangle$, which evolves according to $Q(t) = e^{iHt}Q(0)e^{-iHt}$. Computing $Q(t)|q, t\rangle = e^{iHt}Q(0)e^{-iHt}|q, t\rangle$, we see that the state $|q, t\rangle = e^{iHt}|q\rangle$ will satisfy $Q(t)|q, t\rangle = q|q, t\rangle$. The active transformation of the operators Q is compatible with the passive transformation of the basis states $|q\rangle$.

4. Srednicki 7.1. Factor the denominator in (7.12):

$$G(t - t') = -\int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(E - (\omega - i\epsilon))(E + (\omega - i\epsilon))} \tag{37}$$

To evaluate this via the method of contour integration, promote E to a complex variable; then the function G has simple poles as $E = \pm(\omega - i\epsilon)$, i.e. just below the real axis at $+\omega$ and just above at $-\omega$. When $t > t'$, evaluate the integral by closing the contour in the lower half-plane, where the integrand remains finite if we take the arc to infinity. This picks up the pole at $+\omega$, and cauchy's integral formula gives

$$G(t - t') = -(-2\pi i)\frac{e^{-i\omega(t-t')}}{2\omega(2\pi)} = \frac{i}{2\omega}e^{-i\omega(t-t')} \tag{38}$$

where the extra minus sign in the integral formula comes from choosing a clockwise contour.

For $t < t'$, close the contour counterclockwise in the upper half-plane and pick up the $-\omega$ pole to find

$$G(t - t') = \frac{i}{2\omega}e^{+i\omega(t-t')} \tag{39}$$

which shows the claim.

Consider the effect of a different choice of $i\epsilon$s. For example, we could have taken the poles to lie below the real axis. Then whenever $t < t'$ and we close the contour in the upper half-plane the green's function vanishes; this is the retarded green's function, appropriate for describing the effect of a disturbance at time t' on objects at a later time t. Similarly choosing the poles to lie above the real axis gives the advanced green's function.

5. Srednicki 7.2. The only tricky part is evaluating $\partial_t^2|t-t'|$, but this isn't bad. Consider acting with ∂_t on $|t-t'|$ when $t < t'$ and $t > t'$, this gives $+1$ in the first case and -1 in the latter, so $\partial_t|t-t'|$=sign$(t-t')$. The derivative of the sign function is twice the dirac delta, which follows from writing the sign function as $\theta(t-t') - \theta(t'-t)$. So $\partial_t^2 = 2\delta(t-t')$, and simple algebra gives the result.