Physics 221A: HW4 solutions

October 22, 2012

1. Srednicki 8.5. We want Δ_{ret} to vanish for $x^0 \geq y^0$. When $x^0 \geq y^0$ we close the contour in the lower half-plane and so we want to put the poles in the upper half plane. Similarly for the advanced green's function we should put the poles in the lower half plane. There's a discrepancy between this explanation and the discussion following my solution to Srednicki 7.1, but this is just a question of definition. I call Δ_{ret} the green's function that vanishes for $x^0 \leq y^0$.

2. Srednicki 8.7. There is no trickery here. Fourier transform ϕ and $\phi^\dagger$ to perform the x-integral in $S_0 = \int dx \mathcal{L}_0$. Couple to sources as in the hint, $J^\dagger \phi + J\phi^\dagger$. This leads to a version of (8.7) with our new coupling to sources and replacing the term quadratic in ϕ by $\phi^\dagger(k)(k^2 + m^2)\phi(-k)$. Define χ exactly as in (8.8) and its conjugate $\chi^\dagger$. This leads to (8.9) with $J(k)J(-k) \mapsto J^\dagger(k)J(-k)$ and the same for χ, which gives the free complex scalar generating functional Z_0 generalizing (8.10):

$$Z_0[J, J^\dagger] = \exp\left[i \int dx\ dx'\ J^\dagger(x)\Delta(x - x')J(x')\right] \tag{1}$$

   where Δ is the feynman propagator.

   Doing these manipulations allows us to compute J-functional derivatives of Z_0, which we know from the path integral form of Z_0 gives us the time-ordered product of ϕs. The only new bit is that we can take a J derivative to pull down a $\phi^\dagger$ and a $J^\dagger$ derivative to pull down a ϕ. Explicitly

$$\langle 0|T\phi_{x_1} \dots \phi_{x_n}\phi^\dagger_{y_1} \dots \phi^\dagger_{y_m}|0\rangle = \frac{1}{i}\delta_{J^\dagger(x_1)} \dots \frac{1}{i}\delta_{J^\dagger(x_n)}\frac{1}{i}\delta_{J(y_1)} \dots \frac{1}{i}\delta_{J(y_m)}Z_0[J, J^\dagger|_{J,J^\dagger=0} \tag{2}$$

   Now we want to compute objects like $\langle 0|T\phi_1\phi_2|0\rangle$. If you take a look at the form of $Z_0[J, J^\dagger]$ you'll see that you need to take the same number of J derivatives as $J^\dagger$ derivatives to get anything that survives when you take the sources to zero. For example, if you start with a $\delta_{J^\dagger}$ acting on Z_0, you'll pull down a ΔJ, so you'll have to take a δ_J as well before taking the sources to zero. This tells us immediately that $\langle 0|T\phi_1\phi_2|0\rangle = \langle 0|T\phi_1^\dagger\phi_2^\dagger|0\rangle = 0$ and quick math gives $\langle 0|T\phi_1\phi_2^\dagger|0\rangle = \frac{1}{i}\Delta(x_2 - x_1)$.

   Now let's think about what happens when we have a bunch of operators in the time ordered product. This means taking a bunch of J and $J^\dagger$ derivatives of Z_0 and setting sources to zero. Consider acting with some δ_J late in the sequence. It acts on a product like $e^{iJ\phi+\cdots}\Delta J_1 \ldots$, so it can either pair up with one of the previous J_is that have been pulled down via $\delta_{J^\dagger}$, or on the exponent, in which case it'll have to be paired with a later $\delta_{J^\dagger}$. By the product rule, this means we have to sum over ways of matching up the ϕs and $\phi^\dagger$s, giving us the only nontrivial matrix elements of free complex scalar field theory:

$$\langle 0|T\phi_{x_1}\ldots\phi_{x_n}\phi^\dagger_{y_1}\ldots\phi^\dagger_{y_n}|0\rangle = \left(\frac{1}{i}\right)^{2n} \sum_{\sigma\in\mathrm{perm}(n)} \Delta(y_1 - x_{\sigma(1)})\ldots\Delta(y_n - x_{\sigma(n)}) \tag{3}$$

This is also wick's theorem, because the contraction of any two ϕs $=< T\phi\phi >= 0$.

There isn't much to be gained by working through the second-to-last sentence, it's a lot of algebra of a sort that you have already suffered through. Basically in the terms you expect to vanish, creation operators annihilate $\langle 0|$ while annihilation operators annihilate $|0\rangle$, while $< T\phi\phi^\dagger >$ involves computing a commutator, performing the integral and using the definition of the feynman propagator in terms of θ functions.

3. Srednicki 9.2.

   a) There is a 4-point vertex from the ϕ^4 term, and the associated vertex factor is $4!\,(i)(-\lambda/24) = -i\lambda$. The factor of 4! comes from the 4! arrangements of the 4 ϕs, the i from $\exp^{i\mathcal{L}}$ and the $-\lambda/24$ from $\mathcal{L}$.

   b) See attached sheet for drawings and explanation of the symmetry factors.

   c) The lagrangian has a $\phi \mapsto -\phi$ symmetry, so no terms that violate this symmetry will be generated in the sum over diagrams, and we need not introduce counterterms to cancel them. Since a linear, tadpole-cancelling term is not symmetric under $\phi \mapsto -\phi$, we don't need to introduce a linear counterterm.

      Another way to see it is, the ϕ's always come in pairs in the lagrangian, so there's never just one left over to hook up to a source.

4. Srednicki 9.3.

   a) The vertex contracts 2 ϕs and 2 $\phi^\dagger$s. The factor is $(2!\,)^2(-1)(i\lambda/4) = -\lambda$, the two 2!s from the 2! ways of permuting the two ϕs and two $\phi^\dagger$s amongs themselves. Draw arrows away from the vertex for the $\phi^\dagger$s and towards the vertex for the ϕs; in this convention, all momenta are incoming, so the momentum of the $\phi^\dagger$ goes in the opposite direction of the arrow.

      The fact that these arrows "flow" through the diagram such that incoming - outgoing arrows = 0 at every vertex reflects the existence of a U(1) symmetry under which $\phi \mapsto e^{i\theta}\phi$, $\phi^\dagger \mapsto e^{-i\theta}\phi$. By nother's theorem there is a corresponding conservation law, let's call it particle number conservation. When an operator O goes to $e^{iq\theta}O$ under a symmetry, the operator is said to have charge q; in our example ϕ has charge (particle number) +1, while $\phi^\dagger$ has particle number -1. Any state in which the particle number is initially zero can only evolve to a state in which the particle number also vanishes, unless the symmetry is broken.

b) See attached sheet for drawings.

9.2b/

E=0

V=1

S = 2^3: flipping propagators ≡ flipping derivs @ vertex (2)
flipping ends of each prop ≡ flipping derivs @ vertex (2^2)

V=2

S = 2^4: in addition to the above, the props in the middle circle can be flipped ≡ flipping derivs (2)

S = 2 × 4! : the 4! rearrangements of propagators can be achieved by an appropriate rearrangement of derivs
also, flipping the vertices is the same as flipping the ends of all props (2)

E=2

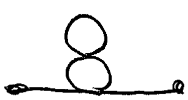
V=0

S=2: exchanging ends of prop ≡ exchanging derivs

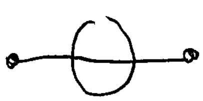
V=1

S = 2^2: now we can also exchange the ends of the internal propagator

V=2



S=2^3: same as E=2, V=1, except now in the middle circle we can exchange the two props ≡ exchange derivs

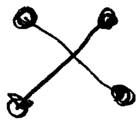


S = 3!·2 : 3! ways to switch the derivs for internal lines at both vertices ≡ switching up the internal props
flipping the vertices ≡ flipping the ends of internal props & exchanging the external guys

S = 2^3 (flip ends of each loop prop; flip vertices)

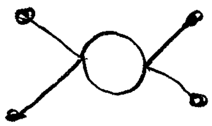
E=4

V=1

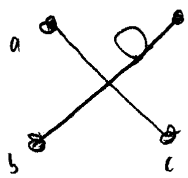


$S = 4!$: rearrangements of props $\equiv$ rearrangements of derivs

V=2



$S = 2^4$: exchanging top & bottom external on either side $\equiv$
exchanging derivs on same side $\Rightarrow 2^2$
flipping internal props $\equiv$ flipping derivs $\Rightarrow 2$
flipping vertices $\equiv$ flip internal and, L&R externals $\Rightarrow 2$



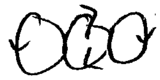
$S = 2 \times 3!$: 2 for exchanging the ends of the propagator in the loop
3! for exchanging a, b, c $\equiv$ exchanging derivatives

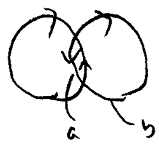
9.3b

The counting rules are modified from 9.2b: the two ends of a propagator are no longer equivalent. With some care this allows us to obtain the complex scalar symmetry factors from 9.2b without too much work.

E = 0

V=1  S = 2 (we can no longer flip the ends of each prop)

V=2:  S=2: flipping vertices ≡ flipping L & R, T & B props



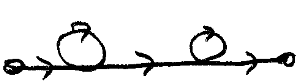
$S=2^3$: flip all vertices ≡ flop all props (2)
flop a & b ≡ flop derivs @ both vertices (2)
same for the other two propagators.

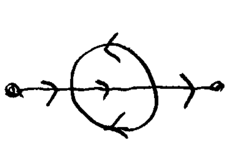
E = 2

V=0  S = 1 (we can no longer flip the propagator ends)

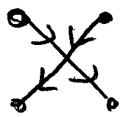
V=1  S = 1 (ditto)

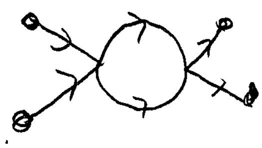
V=2  S = 1 $(\text{ditto})^2$

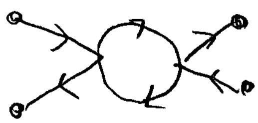
 S = 1 $(\text{ditto})^3$

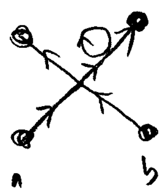
 S = 2: the two internal props pointed in the same direction can be exchanged ≡ exchanging derivs @ both vertices

E = 4

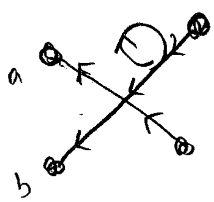
V=1  $S = 2^2$: we can only rearrange amongst incoming/outgoing lines

V=2  $S = 2^3$: we can exchange top & bottom propagators pairwise ≡ exchange derivs

 $S = 2$: exchanging the vertices ≡ exchanging propagators diagonally



$S = 2$: we can exchange props a & b, or the derivs @ the central vertex



$S = 2$ (ditto)