Physics 221A: HW5 solutions

October 31, 2012

1. Srednicki 11.1. We want to compute the decay rate Γ for $A \to BB$. Use equation (11.48), and specialize to $n = 2$. The square of the matrix element is trivial, just $4g^2$ (note the symmetry factor because two equivalent J_B derivatives act on the free generating functional $Z_0[J]$ when you do the perturbation expansion for the full $Z[J]$.) All we have to do is caluclate $\int dLIPS(k)$, which is actually done for you in Srednicki, with the result (11.30). For us, however, $|k_1'|$ is not given by (11.3). Our four-momenta satisfy $k_1' + k_2' = k_A$; in the A rest frame this reads $\vec{k}_1' + \vec{k}_2' = 0$ and so $E_1' = E_2' = \frac{1}{2}E_A = \frac{m_A}{2}$ (these are external particle states, so we're on shell.) So for (11.30) we get

$$dLIPS_2(k) = \frac{\sqrt{1 - \left(m_B/\frac{m_A}{2}\right)^2}}{8\pi^2} d\Omega \tag{1}$$

Now use expression (11.49) to get the total decay rate; we get a 4π from the $\int d\Omega$, $|T|^2/2E_1 = 4g^2/2m_A$, and finally divide by the symmetry factor of 2 for the indistinguishable outgoing Bs. The result is

$$\Gamma = \frac{g^2}{8\pi m_A}\sqrt{1 - \left(m_B/\frac{m_A}{2}\right)^2} \tag{2}$$

Part b follows from part a with very minor changes, you don't have to divide by the symmetry factor of 2 since χ and $\chi^\dagger$ are indistinguishable, but you have to divide by four because the square of the matrix element is g^2 not $4g^2$. So $\Gamma_{part\ b} = \frac{1}{2}\Gamma_{part\ a}$.

2. Srednicki 11.3.

   a) This is just a trivial rearrangement of (11.53), nothing going on here.

   b) (11.55) transforms as an (2,0) tensor under lorentz transformations, so it must be some linear combination of the (2,0) tensors in the problem. The only available tensors are $k^\mu k^\nu$ and $g^{\mu\nu}$. This is because $\vec{k}_{12}'$ are integrated over and k_{12}^0 prime are fixed by the mass-shell condition. To get the dimensions right, $g^{\mu\nu}$ must be multiplied by k^2; the squares of k_{12}' are fixed to zero, so they are of no use here.

   c) This follows straightforwardly from (11.30). We know $E_1' + E_2' = \sqrt{s}$ (this is not the $\sqrt{s}$ of our physical process, which is the muon mass, this doesn't matter because we're not trying to compute a matrix element yet.) In the massless limit, $|\vec{k}_1'| = E_1'$ so that

$$dLIPS_2(k) = \frac{\frac{\sqrt{s}}{2}}{16\pi^2\sqrt{s}} d\Omega \tag{3}$$

$$\Rightarrow \int dLIPS_2(k) = \frac{1}{8\pi} \tag{4}$$

d) Contracting both sides of (11.55) with the metric gives

$$\int k_1 \cdot k_2 dLIPS_2(k) = (4A + B)k^2 \tag{5}$$

while contracting with $k_\mu k_\nu$ instead gives

$$\int (k_1 \cdot k)(k_2 \cdot k) dLIPS_2(k) = (A + B)k^4 \tag{6}$$

From $k_1^2 = k_2^2 = 0$ we have $k_1 \cdot k_2 = k^2/2$. $k = k_1 + k_2$ in conjunction with $k_{12}^2 = 0$ gives $k_1 \cdot k = k_2 \cdot k = k_1 \cdot k_2$. So we just get $(k^2/2) \int dLIPS_2$ for the metric contraction, and $(k^2/2)^2 \int dLIPS_2$ for the other one. Some algebra then gives $A = \frac{1}{96\pi}$, $B = \frac{1}{48\pi}$.

e) With all we've done this is now just some algebra; substitute (11.55) into (11.54) with the A and B given above. You need to compute $k_1 \cdot k_3' = -mE_3$ and $(k_1 - k_3')^2 = -m^2 + 2mE_3$. After you do some algebra and $\int d\Omega$, the dust shakes down and you find

$$\Gamma = \frac{G_F^2 m}{4\pi^3} \int dE_3 (m - \frac{4}{3}E_3) E_3^2 \tag{7}$$

$$\frac{d\Gamma}{dE_3} = \frac{G_F^2 m}{4\pi^3} (m - \frac{4}{3}E_3) E_3^2 \tag{8}$$

f) Now we need to figure out the bounds on E_3, but this isn't too hard. The minimum is zero, because the electron can be sitting at rest if the neutrinos are emitted in exactly opposite directions, and then $E_3 = m_3 \approx 0$. The maximum occurs when $|k_3|$ is maximized, which means the electron going in one direction, all the other particles in the other direction. Then $(E_3, \vec{k}_3) + (E_{other\ guys}, -\vec{k}_3) = (m, 0)$, but the other guys are massless and so $E_3 = E_{other\ guys} = m/2$. Performing the integral you get the famous result

$$\Gamma = \frac{G_F^2 m^5}{192\pi^3} \tag{9}$$

which allows you to determine Fermi's constant experimentally from the measured lifetime of the muon.

g) Plug in numbers to find $G_F = 1.164 \times 10^{-5}\mathrm{GeV}^{-2}$, plus corrections from loops of virtual particles.

3. Srednicki 11.4. All you have in this theory is a vertex where A, B and C all meet. You want to draw the diagrams for $2 \to 2$ scattering at tree level.

   Focus on the incoming particles; they can either be the same, or different. If they're different, say AB, then we can contract them into an ABC vertex, with the internal C propagator going to the other vertex, where the outgoing particles are contracted. This forces the outgoing particles to be AB as well. This rules out $AB \to AC$ and tells us that $AB \to AB$ is allowed.

   If they're the same, say AA, then the incoming particles can't be contracted together, they have to be contracted with one of the outgoing particles; since we then string a propagator between the two contractions, to get the tree level diagram, the outgoing particles must be of the same type as each other, but not as the incoming particles, e.g. BB or CC. This rules out everything remaining besides $AA \to BB$.

   So the two nontrivial matrix elements at tree level, at least of the list we're given, are $AA \to BB$ and $AB \to AB$. These are related by a symmetry called "crossing" symmetry, which is easy to see from LSZ. LSZ distinguishes incoming states from outgoing states only by the sign of the momentum in the fourier transform. Say we have a process $12 \to 34$, which I'll write $\langle 34|12\rangle$, this tells us that $\langle 34|12\rangle = \langle 4|3^\dagger 12\rangle$ where on the right hand side we've replaced the asymptotic creation/annihilation operator $a_3(k_3)$ acting on the $T \to +\infty(1 - i\epsilon)$ vacuum with $a_3^\dagger(-k_3)$ acting on the vacuum at $T \to -\infty(1 - i\epsilon)$. Likewise we have $\langle 34|12\rangle = \langle 2^\dagger 4|3^\dagger 1\rangle$. LSZ makes no use of perturbation theory, so this is a non-perturbative symmetry of the S-matrix.

   Let $1 = 2 = A$, and $3 = 4 = B$. Then $AA \to BB$ is the same as $AB \to AB$ with $k_2 \leftrightarrow -k_3$, i.e. with $s \leftrightarrow t$ and u the same. (see page 80 of srednicki for definitions of the mandelstam variables.)

   Now we just need to compute $AA \to BB$. We can contract either A with either B, so the t and u channel diagrams contribute. These were computed for you in chapter 10, ignoring the momentum-conserving delta function, the t and u channel diagrams are the last two terms of (10.13). The relevant mass is that of C, which connects the two contractions. So $T_{AA\to BB}$ has $c_s = 0$, $c_t = c_u = 1$. By the crossing symmetry described above $T_{AB\to AB}$ has $c_t = 0$, $c_s = c_u = 1$.