Physics 221A: HW7 solutions

November 21, 2012

1. a) $[\bar{\psi}\gamma^\mu\partial_\mu\psi] = 2[\psi] - 1 = [\mathcal{L}] = d \Rightarrow [\psi] = (d-1)/2$ (compare a kinetic term with two derivatives, as for a boson, then you get $2[\phi] - 2$ instead.)

   b) $[g_n(\bar{\psi}\psi)^n] = d = [g_n] + n \cdot 2 \cdot (d-1)/2 \Rightarrow g_n = d - n(d-1)$

   c) $[g_{mn}\phi^m(\bar{\psi}\psi)^n] = d = [g_{mn}] + n \cdot 2 \cdot (d-1)/2 + m \cdot (d-2)/2 \rightarrow [g_{mn}] = d - m(d-2)/2 - n(d-1)$

   d) Using part b with $d = 4$ we see that $[g_n] = 4 - 3n \leq 0$ when $n \geq 2$. Using part c with $d = 4$, $[g_{mn}] = 4 - 3n - m$ which is marginal for $n = m = 1$.

   e) In d=2, $[g_n] = 2 - 3n$ and so no nonzero values are renormalizable. $[g_{mn}] = 2 - n$ and so $n = 1, 2$ are renormalizable, as are any number of powers of X.

2. Consider a lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi^i)^2 - \frac{1}{2}m_{ij}\phi_i\phi_j - \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{1}$$

where the ϕs are real scalar fields and the index i runs from 1 to N. We'll work out the real-space two and four-point functions. Ignoring the interactions for now, the matrix m_{ij} is symmetric and so we can find some orthogonal V such that m is diagonal in $V\phi$. Assuming we have already done so, the first two terms are just the sum of the free particle lagrangians for N real scalars, with masses m_i. If any of the masses m_i are equal, the corresponding ϕ_i can be rotated into each other, an internal $SO(n)$ symmetry, where $n \leq N$ is the number of m_i that are equal. There is also a $\mathbb{Z}_2$ symmetry sending all the $\phi_i \mapsto -\phi_i$, enlarging the symmetry group to O(n), but this is a discrete transformation, unlike the infinitesimal kind that give us conserved currents, so we usually call this an SO(n) symmetry. The $\mathbb{Z}_2$ reflection will have the usual unitary action on the operators required by quantum mechanics, but no current as it is not a continuous symmetry.

The internal symmetry is generically broken by the λ_{ijkl} term, but it may so happen that the couplings also respect the symmetry among some of the scalars. Luckily, this is the case in our problem, as you can see by inspecting (22.10). Life is much easier when you have symmetry.

First, let's think a bit more generally about the current $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i$. Write $\delta\phi_i = T_{ij}\phi_j$ where T_{ij} is an element of the lie algebra of so(N), which consists of antisymmetric

NxN matrices. T_{ij} can be expanded in a basis for so(N) as $T_{ij} = -i\omega_a T^a_{ij}$, where ij run from $1 \ldots N$ and $a = 1 \ldots$ dim so(N) where dim so(N) is the number of generators of so(N). The sign is chosen for convenience, and the i ensures that the T_{ij} are hermitian. (You can always choose a basis for any compact Lie algebra in which all the generators are hermitian. If the group is noncompact, you can always choose a basis in which all the generators are either hermitian or antihermitian, but this leads to a negative-definite term in the hamiltonian.)

So the current for a general SO(N) transformation looks like

$$\omega_a j^{\mu a} = -i\omega_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} T^a_{ij}\phi_j \tag{2}$$

Based on the general considerations above, if $\mathcal{L}$ does not involve derivative interactions this gives for the current

$$j^{\mu a} = iT^a_{ij}\partial^\mu \phi_i \phi_j \tag{3}$$

where again ij tell us which scalar field and a tells us which generator of so(N).

Specializing to our case of N=2, we know that SO(2) has only one generator, which if we want it to be hermitian we can write as $T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Plugging this into the general expression (3) gives the current (22.16) for the SO(2) symmetry. The label a is redundant because we have only one generator.

We want to verify the Ward identity (22.26) for this theory, for the special case $n = 2$; $a_1 = 1$, $a_2 = 2$. Since this is a path integral identity, it had better hold; the only assumption we made was that our SO(N) rotation left the path integral measure invariant, which turns out to be true in this case (though there was no a priori reason to assume so.)

Write the Lagrangian $\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi^i)^2 - \frac{1}{2}m\phi_i\phi_i + O(\lambda)$. Let's figure out $\langle 0|T\phi_a\phi_b|0\rangle \equiv < \phi_a\phi_b >$ in this theory, where I am representing the "contraction" of ϕ_a and ϕ_b with these funky angled brackets. *The contraction of an operator product is the product minus its normal-ordered part, i.e. the part with all the annihilation operators on the right. By construction, the normal-ordered part annihilates the interacting ground state* $|0\rangle$.

Use the usual trick of writing $\mathcal{L} \mapsto \mathcal{L} + J_i\phi_i$; then $Z_0[J] = e^{\frac{i}{2}\int dxdx' J_i(x)\Delta_{ij}(x'-x'')J_j(x')}$. Since the ϕ_i diagonalize the matrix m_{ij}, the Δ_{ij} which solve $(-\partial^2 + m_{ij})\Delta_{ij}(x - y) = \delta(x-y)$, will be diagonal $\Delta_{ij} = \delta_{ij}\Delta_i$, where Δ_i is the Feynman propagator for a scalar of mass m_i. This means that our $Z_0[J]$ breaks up into a product of $e^{\frac{i}{2}\int dxdx' J_i(x)\Delta_i(x'-x'')J_i(x')}$s, where i runs from 1 to N.

So the matrix element we want, $\langle 0|T\phi_a\phi_b|0\rangle = (\frac{1}{i}\frac{\delta}{\delta J_a})(\frac{1}{i}\frac{\delta}{\delta J_b})Z_0|_{J=0}$

$$= (\frac{1}{i}\frac{\delta}{\delta J_a})(\frac{1}{i}\frac{\delta}{\delta J_b})\prod_{i=1}^{N} e^{\frac{i}{2}\int dxdx' J_i(x)\Delta_i(x'-x'')J_i(x')} = \frac{1}{i}\Delta_b(x_a - x_b)\delta_{ab} =< \phi_a\phi_b > \tag{4}$$

where the δ_{ab} comes from taking say the J_b derivative first and realizing that your J_a derivative must hit the leftover J_b you pulled down along with Δ_b.

This is actually all we need, because this notion of contractions will do the rest of the work for us. *Wick's theorem tells us that the vacuum expectation value of a time-ordered product of fields is equal to the sum of all possible contractions of the fields in the product.* This follows straightforwardly from the definition of contraction given above, a time-ordered product separates out into its contraction and its normal-ordered part, and the normal-ordered part annihilates $|0\rangle$.

So the VEV we want to compute is

$$\langle 0|T(\phi_1(x)\partial\phi_2(x) - \phi_2(x)\partial\phi_1(x))\phi_1(x_1)\phi_2(x_2)|0\rangle \tag{5}$$

$$\tag{6}$$

$$=< \phi_1(x)\phi_1(x_1) > \partial < \phi_2(x)\phi_2(x_2) > -(1 \leftrightarrow 2) \tag{7}$$

$$= \left(\frac{1}{i}\right)^2 \Delta(x - x_1)\partial\Delta(x - x_2) - (1 \leftrightarrow 2) \tag{8}$$

This tells us that

$$\partial \cdot \langle 0|Tj(x)\phi_1\phi_2|0\rangle = \left(\frac{1}{i}\right)^2 \Delta(x - x_1)\partial^2\Delta(x - x_2) - (1 \leftrightarrow 2) \tag{9}$$

Now use $(-\partial^2 - m^2)\Delta(x) = \delta(x)$ to write this as

$$\partial \cdot \langle 0|Tj(x)\phi_1\phi_2|0\rangle = \left(\frac{1}{i}\right)^2 \Delta(x - x_2)\delta(x - x_2) - (1 \leftrightarrow 2) \tag{10}$$

You can immediately read off that this is equal to the right hand side of our ward identity.

3. The first matrix element was discussed above, it is not modified to order λ^0 by the presence of interactions.

   First, remember that when we calculate expectation values, we have to normalize by dividing by the full partition function without any operator insertions. That is, we want to divide by $\langle 0|0\rangle = Z_0[J]Z_1[J]|_{J=0}$, which is the exponential of the sum of disconnected diagrams. (Connected means that any point in the diagram can be obtained by tracing along the diagram from one of the external points; or, any point in the diagram is connected to one of the external legs.)

   The argument is not hard if you draw a picture, and can be found for example in section 4.4 of Peskin and Schroeder. A general diagram can be broken up into a connected piece and a bunch of disconnected pieces. Let's say n_i of these disconnected pieces are the same, where i labels the different types of disconnected subgraphs present in the diagram we care about; then the value of the full diagram is given by (connected)$\cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i}$, where V_i is the value of the disconnected piece and the prefactor is the symmetry factor. Then the time-ordered vev is given by the sum of all these diagrams $= \sum$ connected $\times \exp(\sum_i V_i)$. There are no connected diagrams when we don't have any operator insertions, so $\langle 0|0\rangle = \exp(\sum_i V_i)$, dividing out the annoying factor up top.

You already know the order-λ^0 contribution to the two point function from the last question.

The order-λ^0 contribution to the four-point function has no internal ϕs, so

$$\langle 0|T\phi_a\phi_b\phi_c\phi_d|0\rangle =< ab >< cd > +2 \text{ other pairings} + O(\lambda) \tag{11}$$

To order λ^0 this is

$$\langle 0|T\phi_a\phi_b\phi_c\phi_d|0\rangle = \left(\frac{1}{i}\right)^2 (\delta_{ab}\delta_{cd}\Delta(x_a - x_b)\Delta(x_c - x_d) + 2 \text{ other pairings}) \tag{12}$$

Note that this is a disconnected contribution, so it will disappear when we divide by the norm $\langle 0|0\rangle$ to calculate the correlation function $\langle \phi_a\phi_b\phi_c\phi_d\rangle$. Diagrammatically, this is just four points $x_a \ldots x_d$ given corresponding colors $a \ldots d$, with two propagators connecting different pairs of points, which get a $\delta_{ab}\Delta(x_a - x_b)$ (say) if ab are linked by a propagator.

Now we want to know the order λ contribution. Let's focus on the connected part, which is the only stuff that survives division by the norm of the vacuum state.

I will break up our interaction term into two pieces, corresponding to a four-point, single-color interaction, or a four-point, two-color interaction. To make the counting more transparent I'll write the interaction lagrangian as $\mathcal{L}_{int} = -\lambda_{1c}|\phi|^4 - \lambda_{2c}\sum_{pairs}\phi_i^2\phi_j^2$. To enforce our SO(N) symmetry we must have $\lambda_{2c} = 2\lambda_{1c}$, just consider rotating ϕ_1 into ϕ_2. Then we can write it as $\mathcal{L}_{int} = -\lambda(\sum_i \phi_i^2)^2 = -\lambda(\sum_i \phi_i^2)(\sum_j \phi_j^2)$.

Let's figure out the two-point correction. There are four internal ϕs, all from the first-order expansion of $e^{iS_{int}}$, along with a $\lambda \int dz$. One of internal ϕs must contract with ϕ_a, another with ϕ_b, which leaves two internal ϕs to contract amongst themselves forming a loop. The diagram is just a loop inserted tangent to the propagator (so that the interaction is four-point.)

So our time-ordered product is $-i\lambda \int dz < az >< bz >< zz >$ times some counting factor, where z has an index i or j we'll have to be careful about. There are two different contributions: we can either contract a and b to the same type of index (i or j), or we can contract one with i and the other with j. Supposing we contract a with a ϕ_i, there is one remaining way we could connect b to an i (giving a $\delta_{ai}\delta_{ib}\delta_{jj} = N\delta_{ab}$), where we got N=number of scalars terms from the sum of contractions of the ϕ_j. There are four ways to contract a with any of the four internal ϕs, after which we are forced to contract b with the adjacent ϕ. This is a diagram where a and b are the same color, but the loop can be of any color, so we must sum over colors in the internal loop.

Now let's say we contract a with i and b with j, then we get a $\delta_{ai}\delta_{bi}\delta_{ij} = \delta_{ab}$ with no sum over the colors. There are 8 ways to do this: if a contracts with i, there are two choices for which a contracts with and two for which j b contracts with, and another factor of two from $i \leftrightarrow j$.

Putting it all together,

$$\langle\phi_a\phi_b\rangle = -4(N+2)i\lambda\delta_{ab}\int dz\Delta(x_a-z)\Delta(x_b-z)\Delta(z-z) \tag{13}$$

Now for the four-point function we have four external ϕs. Now the order λ term in $\langle\phi_a\phi_b\phi_c\phi_d\rangle$ is

$$\langle 0|T\phi_a\phi_b\phi_c\phi_d\left[\int dz(-i\lambda)(\sum_i \phi_i(z)^2)(\sum_j \phi_j(z)^2)\right]|0\rangle_C \tag{14}$$

where the C means that every vertex is connected eventually to an external leg.

So let's sit down and figure this guy out. It's instructive to work out one term and use counting to do the rest.

Let's say a is contracted with the first ϕ_i. Then b, c, d can be contracted with either i or j. There is also a factor of 2 from the symmetry in j. So this term schematically is $\int dz < ai > (< bi >< cj >< dj > + < bj >< ci >< dj > + < bi >< cj >< dj >) \cdot 2$, where the contractions $<>$ were defined above.

Now, we could have initially contracted a with any of the ϕs in the interaction term, so we have four of these guys, and they're all equal. Another way to do the counting would be to note that we have two choices for the a contraction (i or j), a factor of 2 from the remaining symmetry in whichever of i or j we didn't contract, and another factor of 2 from contracting a with the other index (j or i), an overall factor of 8.

Noting that $\delta_{ai}\delta_{bi} = \delta_{ab}$, etc, we can just read off the correlator:

$$\langle\phi_a\phi_b\phi_c\phi_d\rangle = -8i\lambda(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\int dz\Delta(x_a-z)\Delta(x_b-z)\Delta(x_c-z)\Delta(x_d-z) \tag{15}$$

This is our usual four-point vertex, with some extra indices taking care of the conservation of color charge. The amplitude for all colors being equal is three times the amplitude when there are two different colors in the diagram.