Physics 221A: HW7 solutions

December 7, 2012

1. Scale symmetry.

   Suppose you have a quantum field theory, for which you have written down the path integral $Z = \int \mathcal{D}\phi e^{iS[\phi]}$. The action can be written in terms of a lagrangian density (a functional of the fields ϕ labelled by spacetime points x) as $S[\phi] = \int dx \mathcal{L}[\phi(x)]$. When a transformation leaves $\mathcal{D}\phi e^{iS[\phi]}$ unchanged, call it a symmetry. For now we'll assume that $\mathcal{D}\phi$ is unchanged and restrict our attention to the action, although there are meaningful physical effects when $\mathcal{D}\phi$ changes.

   When the symmetry is infinitesimal, Nother's theorem guarantees a conserved current. In (22.7) Srednicki rewrites the chain rule (22.1) for the variation $\delta\mathcal{L}$ in a useful way:

$$\partial_\mu j^\mu = \delta\mathcal{L} - \frac{\delta S}{\delta\phi}\delta\phi \tag{1}$$

   where the quantity

$$j^\mu := \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi \tag{2}$$

   is the first thing he calls the current. He obtains this in the intervening lines by using the definition of the action to write the chain rule (22.1) as a total derivative plus stuff that vanishes when you use the equations of motion. The equations of motion are

$$\frac{\delta S}{\delta\phi} = 0 \tag{3}$$

   so when $\delta\mathcal{L} = 0$ the current j is conserved on shell, $\partial_\mu j^\mu = 0$. Now suppose $\delta\mathcal{L} \neq 0$, but instead it takes the general form

$$\delta\mathcal{L} = \partial_\mu K^\mu + \Delta \tag{4}$$

   Rewriting the chain rule (22.1) with our new symbols,

$$\delta\mathcal{L} = \partial_\mu j^\mu + \frac{\delta S}{\delta\phi}\delta\phi = \partial_\mu K^\mu + \Delta \tag{5}$$

   Now, if $\Delta \neq 0$, the action is not invariant, and we don't have a symmetry:

$$\delta S = \int dx\delta\mathcal{L} = \int dx(\partial_\mu K^\mu + \Delta) = \int dx\Delta \tag{6}$$

by our boundary condition on the fields. However, $\Delta = 0$ transformations with $K^\mu \neq 0$ are still symmetries, so we should be able to define a conserved current (this is indeed the case for poincare symmetries as discussed in the chapter.) Define

$$j^\mu_{new} = j^\mu - K^\mu \tag{7}$$

Then by construction

$$\partial_\mu j^\mu_{new} = \Delta \tag{8}$$

i.e. the current is conserved up to the term Δ that cannot be written as a total divergence; when Δ vanishes j^μ_{new} is conserved, $\partial_\mu j^\mu_\nu = 0$. This is the current of (22.27).

There are "internal" symmetry transformations that leave x unchanged and map $\phi \mapsto \phi'$ leaving the action unchanged, and also "spacetime" symmetries acting on the points x of spacetime as well as the fields ϕ which also have $\delta S = 0$. The ϕ_1, ϕ_2 rotational symmetry from the last homework was an example of internal symmetry; the poincare group and scale symmetry (and more generally conformal symmetry) are examples of spacetime symmetries. Let's see how this works for the lorentz group. We'd like our action to be lorentz-invariant, so we require

$$\int dx' \mathcal{L}(x') = \int dx \mathcal{L}(x) \tag{9}$$

when $x' = \Lambda x$ for some $\Lambda \in$ SO(3,1). The measure $d^d x$ is unchanged under $x \mapsto \Lambda x$ because $|\det \Lambda| = 1$. This means that we need $\mathcal{L}(x) = \mathcal{L}(\Lambda x)$: in other words, the lagrangian density must be lorentz invariant.

When there's a scale symmetry, it is no longer true that the measure dx appearing in $S = \int dx L$ remains unchanged. The measure transfroms as $d^d x \mapsto d^d x' = \ell^d d^d x$. In order for the action to remain invariant, we need the lagrangian density to transform as

$$\mathcal{L}(x') = \mathcal{L}(\ell x) = \ell^{-d} \mathcal{L}(x) \tag{10}$$

i.e. the lagrangian "must have scaling dimension $-d$." This means the lagrangian density is not scale invariant, except for d=0.

Start off with the free massless real scalar lagrangian

$$\mathcal{L}(x) = -\frac{1}{2}\{d^\mu \phi \partial_\mu \phi = -\frac{1}{2}|\partial\phi|^2 \tag{11}$$

The derivatives transform $\frac{\partial}{\partial x'} = \frac{\partial}{\partial(\ell x)} = \ell^{-1}\partial$, so ϕ has to transform as

$$\phi'(x') = \phi(\ell x) = \ell^{-(d-2)/2}\phi(x) \tag{12}$$

(i.e. ϕ has scaling dimension $-(d-2)/2$) in order for the action to remain unchanged under the scale transformation. This gives us the transformation rule for ϕ under $x \mapsto x'$, which is enough to do everything else.

Taking $\ell = 1 + \epsilon$ for ϵ very small, the scale transformation becomes infinitesimal, so nother's theorem guarantees you a conserved current. Using the transformation rule for ϕ that we got by requiring the action to be invariant under scaling,

$$\phi(x') = \phi((1+\epsilon)x) = \frac{1}{(1+\epsilon)^{(d-2)/2}}\phi(x) \tag{13}$$

so that

$$\delta\phi = \phi(x') - \phi(x) = (1+\epsilon)^{(d-2)/2}\phi((1+\epsilon)x) - \phi(x) = \epsilon\left(\frac{d-2}{2} + x\cdot\partial\right)\phi + O(\epsilon^2) \tag{14}$$

As in (22.1) you can compute the change in the lagrangian density under an infinitesimal transformation using the chain rule: $\delta\mathcal{L} = \frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\partial\phi}\delta\partial\phi$. Now,

$$\partial_\mu\delta\phi = \epsilon(d/2 + x\cdot\partial)\partial_\mu\phi \tag{15}$$

so that

$$\delta\mathcal{L} = -\epsilon\partial^\mu\phi(d/2 + x\cdot\partial)\partial_\mu\phi = \epsilon\partial_\mu(x^\mu\mathcal{L}) \tag{16}$$

So

$$\delta S = \int dx\delta\mathcal{L} = \int dx\partial_\mu(x^\mu\mathcal{L}) = 0 \tag{17}$$

using the boundary condition on the fields. This shows that we indeed have a infinitesimal symmetry ($\Delta = 0$) of the action, with $K^\mu = x^\mu\mathcal{L}$.

Now let's do an example with nonzero Δ. If you add a mass term, $\mathcal{L} = -\frac{1}{2}|\partial\phi|^2 - \frac{1}{2}m^2\phi^2$, then it's a line of algebra to show that

$$\delta\mathcal{L} = \partial_\mu(x^\mu\mathcal{L}) + \left(\frac{d-2}{2}\right)m^2\phi^2 \tag{18}$$

from which you can read off

$$\Delta = \left(\frac{d-2}{2}\right)m^2\phi^2 \tag{19}$$

This vanishes when $m = 0$ or $d = 2$: mass terms (for scalars) are actually not forbidden by conformal invariance in $d = 2$. (Incidentally, you can have a conformal massive scalar theory in curved space, by coupling the field to the ricci scalar with a term $\sim R\phi^2$, and choosing the coefficient of this term to cancel the conformal transformation of the mass term. Also, if you're confused, this scale vs. conformal stuff is a little sloppy– basically, conformal combines poincare with scaling and inversions, but just read "scale" when you see "conformal" for now.)

Following the procedure above you might wonder what other types of terms are allowed by scale invariance, besides the kinetic term in any dimension and the mass term in $d = 2$. Let's restrict to self-interaction terms in a real scalar field theory that do not contain derivatives. A general interaction term looks like $-\frac{1}{n!}g_n\phi^n$ and under rescaling maps to $-\frac{1}{n!}g_n\phi^n\ell^{-n(d-2)/2}$, since each $\phi \mapsto \phi\ell^{-(d-2)/2}$. For scale invariance we need this term to go like ℓ^{-d}, which happens when $n(d-2)/2 = d$, or $\frac{1}{n} + \frac{1}{d} = \frac{1}{2}$. This gives us some simple possibilities: ϕ^6 in 3 dimensions, ϕ^4 in 4 dimensions, ϕ^3 in 6 dimensions.

It's not hard to verify that the additional variation of the action when we include the interaction term is a total divergence. Write $\mathcal{L}_{int} = -\frac{1}{n!}g_n\phi^n$, then $\partial_\mu(x^\mu \mathcal{L}_{int}) = -\frac{d}{n!}g_n\phi^n - \frac{1}{(n-1)!}g_n\phi^{n-1}(x\cdot\partial)\phi$, which you can check is equal to $\delta\mathcal{L}_{int} = \frac{\delta\mathcal{L}_{int}}{\delta\phi}\delta\phi = -\frac{n}{n!}g_n\phi^n\left(\frac{d-2}{2} + (x\cdot\partial)\right)\phi$ iff $n(d-2)/2 = d$, which we derived by power counting in the previous paragraph. This includes the ϕ^4 theory in 4 dimensions in the problem.

From the considerations above you can read off the current:

$$j^\mu_{new} = -\partial^\mu\phi\left(\frac{d-2}{2} + x\cdot\partial\right)\phi - x^\mu\mathcal{L} \tag{20}$$

We already computed $\Delta = \frac{d-2}{2}m^2\phi^2$, but the physical interpretation is that the failure of the scale current to be conserved is proportionate to the mass of the field, which violates scale invariance.

You can also compute the current by letting ϵ be spacetime-dependent in the infinitesimal scale transformation; after you use the equations of motion you get $\delta S = \int dx f^\mu\partial_\mu\epsilon$ so $f^\mu = -j^\mu$ is conserved for variations under which the action vanishes.

2. a) In $d=3$, $[\phi] = 1/2$ and we can have terms up to 6th order in the field without losing renormalizability. The $\mathbb{Z}_2$ reflection symmetry requires that all powers be even. $\mathcal{L} = -\frac{1}{2}|\partial\phi|^2 - \frac{1}{2}m^2\phi^2 - \frac{a}{4!}\phi^4 - \frac{b}{6!}\phi^6$

   b) We need terms of up to 6th order, but now ϕ and $\phi^\dagger$ must be paired. The most general renormalizable lagrangian is $\mathcal{L} = -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi - \frac{g_4}{(2!)^2}(\phi^\dagger\phi)^2 - \frac{g_6}{(3!)^2}(\phi^\dagger\phi)^3$.

   c) In addition to the stuff in part b) (which is $\mathbb{Z}_6$ invariant because the $\mathbb{Z}_6$ lives inside the original U(1)) we can also have terms $a\phi^6 + b(\phi^\dagger)^6$; the $\mathbb{Z}_6$ does not mix ϕ and $\phi^\dagger$ so the couplings are independent.

   d) In $d=4$, $[\phi]=1$ and we can have terms up to quartic. The $\mathbb{Z}_2\otimes\mathbb{Z}_2$ action switches the fields and switches their signs, but it doesn't rotate them into each other. Each ϕ_1 term needs to have a corresponding ϕ_2 term with the same coupling, but the cross terms don't have to be related to these couplings. $\mathcal{L} = -\frac{1}{2}|\partial\phi_1|^2 - \frac{1}{2}|\partial\phi_2|^2 - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) - \frac{g_{2,2}}{(2!)^2}\phi_1^2\phi_2^2 - \frac{g_4}{4!}(\phi_1^4 + \phi_2^4)$.

3. Write $U \in Sp(2N,\mathbb{R}) = e^{i\theta_a T^a}$ where θ_a is infinitesimal and T^a generates the algebra sp(2N). Since $U \in Sp(2N)$ it satisfies $UJU^T = J$ where J is the symplectic form $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, I_N being the NxN identity matrix. In other words U is a member of Sp(2N) is if it preserves the symplectic form on 2N-dimensional euclidean space, just like the lorentz group SO(3,1) consists of those matrices whose action by conjugation preserves the minkowski metric $\eta_{\mu\nu}$. Using the taylor expansion of U and keeping only terms linear in θ gives $TJ + JT^T = 0$.

Next write $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are (for the moment) arbitrary NxN matrices. The symplectic condition reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} + \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = 0 \tag{21}$$

from which some matrix multiplication gives $D^T = A, B = B^T, C = C^T$. These are our conditions: T consists of those matrices with an arbitrary Gl(N) matrix in the upper left block, its transpose in the lower right block, and two independent symmetric Gl(N) matrices, one in the upper right block and the other in the lower left. We started with $4N^2$ degrees of freedom; $D^T = A$ removes N^2 while e.g. $B = B^T$ is equivalent to requiring that the antisymmetric part of B vanishes, $N(N-1)/2$ conditions on the matrix. Thus we are left with $4N^2 - N^2 - 2N(N-1)/2 = 2N^2 + N$ as the number of independent generators of Sp(2N).

Last, show that the algebra of sp(2N) closes: we want to show that if $TJ = -JT^T$ and $T'J = -J(T')^T$, then $[T, T']J = -J[T, T']^T$. This isn't hard:

$$[T, T']J = TT'J - T'TJ \tag{22}$$
$$= -TJ(T')^T + T'JT^T \tag{23}$$
$$= JT^T(T')^T - J(T')^T T^T \tag{24}$$
$$= -J[T, T']^T \tag{25}$$

4. This is mostly a matter of fitting Z_ϕ, Z_m, Z_λ (field strength renomralization, mass renormalization, 4-point vertex renormalization) for ϕ^4 in 4 dimensions into the scheme of chapter 28. I pulled the Zs out of chapter 31, where Srednicki uses the MSbar scheme: $Z_\phi = 1 + \sum \frac{a_n(\lambda)}{\epsilon^n} = 1 + O(\lambda^2)$, $Z_m = 1 + \sum \frac{b_n(\lambda)}{\epsilon^n} = 1 + \frac{\lambda}{16\pi^2}\frac{1}{\epsilon} + O(\lambda^2)$, $Z_\lambda = 1 + \sum \frac{c_n(\lambda)}{\epsilon^n} = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon} + O(\lambda^2)$.

   In the language of section 28, we have $\lambda_0 = Z_\lambda Z_\phi^{-4} \mu^\epsilon \lambda$ and $G = \log Z_\lambda Z_\phi^{-4}$ which gives $G_1 = \frac{3\lambda}{16\pi^2}$. The rest of the steps from (28.18) to (28.21) go through unchanged. Calculate $\frac{d \log \lambda_0}{d \log \mu}$ using the definition above and write $\frac{d\lambda}{d \log \mu} = -\epsilon\lambda + \beta(\lambda)$. Now require $\frac{d \log \lambda_0}{d \log \mu}$ to vanish as it must, because λ_0 was defined without reference to μ whatsoever. This gives us $\beta(\lambda) = \lambda^2 G_1'(\lambda) + \cdots = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$.

   Next, mass. $\log m_0 = M + \log m$ where $M = \log Z_m^{1/2} Z_\phi^{-1/2}$ which gives $M_1 = \frac{\lambda}{32\pi^2}$. Then use (28.26): the vanishing $\log \mu$ derivative of $\log m_0$ gives $\gamma_m = \frac{d \log m}{d \log \mu} = \lambda M_1' + \cdots = \frac{\lambda}{32\pi^2} + O(\lambda^2)$.

   Last, field strength. The bare propagator is $\Delta_0 = Z_\phi \Delta$. So $\gamma_\phi = \frac{d \log \sqrt{Z_\phi}}{d \log \mu} = \frac{\partial \log \sqrt{Z_\phi}}{\partial \lambda} \frac{d\lambda}{d \log \mu} = -\frac{1}{2}\lambda a_1' + \cdots = O(\lambda^2)$. The square root is convenient as $\sqrt{Z_\phi}\phi = \phi_0$.