**Physics 221B** **Quantum Field Theory** **Winter 2015**

## Selected Homework 1 solutions

**34.2** The all-spatial term was the messiest, lots of ways to organize it. Probably simplest is just to write the M commutator in terms of J's and K's, 33.11,.12,.13. Otherwise the space-space parts involve a mess of $\epsilon$-tensor manipulations, for example $\epsilon^{ijm}\epsilon^{kln} = \delta^{ik}\delta^{jl}\delta^{mn} + \delta^{jk}\delta^{ml}\delta^{in} + \delta^{mk}\delta^{jl}\delta^{in} - \delta^{ik}\delta^{ml}\delta^{jn} - \delta^{jk}\delta^{il}\delta^{mn} - \delta^{mk}\delta^{il}\delta^{jn}$ (all permutations of $ijm$, with minus sign for anti-cyclic). Or take shortcuts by showing that e.g. $[S^{12}, S^{13}]$ is the only nontrivial one, up to permutations.

**35.4** Various ways to organize this. I'd write the LHS as

$$\mathrm{Tr}(\epsilon\sigma^{\nu}\epsilon^{T}\sigma^{\mu T})$$

Now, easy to check that $\epsilon\sigma^{\nu} = \pm\sigma^{\nu T}\epsilon$, with a $+$ sign for $\nu = 0$ and a $-$ sign for $\nu$ spacelike. Also, $\epsilon\epsilon^{T} = I$. So the trace becomes

$$\pm\mathrm{Tr}(\sigma^{\nu T}\sigma^{\mu T}) = -2g^{\mu\nu}$$

a is easily checked. Some people derived the trace of 35.5 from 35.4. You would also need to argue (from Lorentz invariance) that the RHS is proportional to $g^{\mu\nu}$, and then 35.4 gives the normalization.

**36.5** a) The kinetic term is invariant for $U^{*}_{jk}U_{jl} = \delta_{kl}$, or $U^{\dagger}U = I$, $U$ unitary. But the mass term requires $U_{jk}U_{jl} = \delta_{kl}$, or $U^{T}U = I$, $U$ orthogonal. Together these mean that $U$ is orthogonal and real, $O(N)$.

d) The obvious symmetry is $U(N)$, but we know that $N$ massless Dirac $= 2N$ massless Weyl, so it must be $U(2N)$! Not so obvious in Dirac notation, the easiest way to show it is to write it in terms of Weyl fermions. Mark was throwing you a curve ball here! By the way, there is a 'chiral' $U(N) \times U(N) \subset U(2N)$ that does come up a lot:

$$(1+\gamma_5)\Psi_j \to (1+\gamma_5)U_{jk}\Psi_k\,, \quad (1-\gamma_5)\Psi_j \to (1-\gamma_5)V_{jk}\Psi_k$$

This is what you will get (later in the course) for a massless Dirac fermion coupled to a gauge field.

e) By the same logic as part d, $U(N)$ is obvious but $O(2N)$ is actually there.

**6.** Various ways to organize. I'd probably use $\gamma^{\mu}\gamma^{\nu} = -\gamma^{\nu}\gamma^{\mu}$ for $\mu \neq \nu$ and $g^{\mu\nu}$ for $\mu = \nu$ (from the basic anticommutation relation), and then commute through.

**7.** The adjoint of the Dirac equation $i\gamma^{\mu}\partial_{\mu}\Psi = m\Psi$ is $-i\partial_{\mu}\Psi^{\dagger}\gamma^{\mu\dagger} = m\Psi^{\dagger}$. Multiplying by $\beta$ on the right you get

$$-i\partial_{\mu}\Psi^{\dagger}\gamma^{\mu\dagger}\beta = -i\partial_{\mu}\Psi^{\dagger}\beta\gamma^{\mu} = -i\partial_{\mu}\overline{\Psi}\gamma^{\mu} = m\Psi^{\dagger}\beta = m\overline{\Psi}\,.$$

With this you can show conservation. (Some people just applied Noether's theorem, I had wanted you to practice the manipulations above).