Physics 221B **Quantum Field Theory** **Winter 2015**

Selected Homework 2 solutions

1. Verify 38.15.

First you can show that $\overline{\gamma^\mu} = \gamma^\mu$ either from an explicit basis, or from the general properties $\gamma^{\mu\dagger} = -\gamma_\mu$, $\beta = \gamma^0$. Then note that

$$\overline{AB} = \beta(AB)^\dagger\beta = \beta B^\dagger A^\dagger \beta = \beta B^\dagger \beta\beta A^\dagger\beta = \overline{B}\,\overline{A}\,.$$

Using $\overline{\gamma^\mu} = \gamma^\mu$ and the anticommutation relations, you can derive all the rest without referring back to the basis.

2. Verify the Gordon identities.

Steps given on p. 240 of Srednicki.

3. a) Several people solved the problem by turning on a parity-breaking external field. Let me clarify what I was looking for. We can break any symmetry by turning on a external field; for example, the physics near the Earth's surface is not rotationally invariant due to the gravitational field of the Earth. To be more precise, what I was asking about is the symmetries of the basic theory, not of the particular background we are in. So to answer that, you would want to include the symmetry operation (rotation or parity) acting on the external field as well. In practice, this means that we are looking for a Lagrangian built only out of fields and their derivatives, with no explicit dependence on x^μ.

So, to break parity, we need an odd number of spatial derivatives. We can do this by including the Levi-Civita tensor $\epsilon^{\mu\nu\sigma\rho}$. Each index must contract with a derivative, $\epsilon^{\mu\nu\sigma\rho}\partial_\mu\partial_\nu\partial_\sigma\partial_\rho$. Each derivative must hit a different kind of field, or else the term must vanish by antisymmetry. Finally we need another field in front or else it will be a total derivative. So let's try

$$\mathcal{L}_? = -\frac{1}{2}\sum_{i=1}^{5}(\partial_\mu\phi_i\partial^\mu\phi_i + m_i^2\phi_i^2) + g\epsilon^{\mu\nu\sigma\rho}\phi_1\partial_\mu\phi_2\partial_\nu\phi_3\partial_\sigma\phi_4\partial_\rho\phi_5\,.$$

This is not invariant under $P^{-1}\phi_i(x)P = \phi_i(\mathcal{P}x)$.

However, it is invariant under

$$P^{-1}\phi_i(x)P = -\phi_i(\mathcal{P}x)\,,$$

meaning that the ϕ_i are pseudoscalars. If the theory is invariant under *any* symmetry that relates x to $\mathcal{P}x$, we would say that it is parity invariant. So we haven't succeeded yet. Actually there are 16 *different* parity operations,

$$P_{\mathbf{s}}^{-1}\phi_i(x)P_{\mathbf{s}} = s_i\phi_i(\mathcal{P}x)\,,$$

where any odd number of the s_i are -1 and the remainder are $+1$. There are also 16 internal (not acting on x) symmetries

$$Z_{\mathbf{s}}^{-1}\phi_i(x)Z_{\mathbf{s}} = s_i\phi_i(x)\,,$$

where now any even number of the s_i are -1 and the remainder are $+1$. These 32 operations are closed under multiplication. (I made the masses different to get rid of additional symmetries that mix up different fields.)

The four-derivative term is known as a Wess-Zumino term. It shows up in the effective low energy action of the pions, kaons, and eta, where it is the simplest term that violates the internal symmetry $\phi_i \to -\phi_i$ (all i at once), and allows the number of particles to change mod 2. In two dimensions it would become $\epsilon^{\mu\nu}\phi_1\partial_\mu\phi_2\partial_\nu\phi_3$, which is renormalizable, and so especially interesting.

By the way, several people tried $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\phi^3$ and noted that it was not invariant under $P^{-1}\phi(x)P = -\phi(\mathcal{P}x)$. But it is invariant under $P^{-1}\phi(x)P = \phi(\mathcal{P}x)$, so we would say that there is a parity symmetry.

To get rid of the parity symmetries, simply add a term,

$$\mathcal{L}_! = -\frac{1}{2}\sum_{i=1}^{5}(\partial_\mu\phi_i\partial^\mu\phi_i + m_i^2\phi_i^2) + g\epsilon^{\mu\nu\sigma\rho}\phi_1\partial_\mu\phi_2\partial_\nu\phi_3\partial_\sigma\phi_4\partial_\rho\phi_5 + g'\phi_1\phi_2\phi_3\phi_4\phi_5\,.$$

The g and g' terms transform exactly oppositely under every $P_{\mathbf{s}}$, so there is no parity symmetry left.

b) This Lagrangian also breaks T, by the same reasoning, but not PT, C (which acts trivially), or CPT.

c) One could try

$$\mathcal{L}_? = -\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4!}\lambda\phi^4 - \frac{1}{4!}\lambda^*\phi^{\dagger 4}\,.$$

For λ complex, $\lambda = e^{i\theta}|\lambda|$, this is not invariant under $C^{-1}\phi(x)C = \phi(x)^\dagger$, but it is invariant under $C^{-1}\phi(x)C = e^{-i\theta/2}\phi(x)^\dagger$. This counts, because it includes reversing ϕ and $\phi^\dagger$. Equivalently, if we define $\phi' = e^{i\theta/4}\phi$, we get

$$\mathcal{L}_? = -\partial_\mu\phi'^\dagger\partial^\mu\phi' - m^2\phi'^\dagger\phi' - \frac{1}{4!}|\lambda|(\phi'^4 + \phi'^{\dagger 4})\,,$$

which is invariant under $C^{-1}\phi'(x)C = \phi'(x)^\dagger$.

As above, we break the symmetry by adding another term that destroys the new symmetry:

$$\mathcal{L}_{!} = -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2(\phi^2 + \phi^{\dagger 2}) - \frac{1}{4!}\lambda\phi^4 - \frac{1}{4!}\lambda^* \phi^{\dagger 4}\,.$$

This now violates all possible charge conjugation symmetries, for complex λ.

This was a rather unconventional problem. I was pleased that a number of people got the whole thing (there were various alternate solutions, all using the same ideas), but in any case I hope that it was instructive for how to think about symmetries. They are rather important: for any $\mathcal{L}$, the first thing to think about is what are its symmetries? Another common question: for a given set of fields and symmetries, what is the most general renormalizable $\mathcal{L}$ that one can write.