Physics 221B **Quantum Field Theory** **Winter 2015**

Selected Homework 3 solutions

1. a)

$$(\overline{\Psi}X\Psi)^* = (\Psi^\dagger\beta X\Psi)^* = \Psi^\dagger X^\dagger\beta^\dagger\Psi = \Psi^\dagger\beta^2 X^\dagger\beta\Psi = \overline{\Psi}\,\overline{X}\Psi\,,$$

and use 38.15. Also $U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$, so

$$\begin{aligned} U(\Lambda)^{-1}\overline{\Psi}(x)U(\Lambda) &= U(\Lambda)^{-1}\Psi(x)^\dagger U(\Lambda)\beta\,, \quad \text{mult. by spinor matrix commutes with mult. by operator} \\ &= (U(\Lambda)^\dagger\Psi(x)U(\Lambda)^{-1\dagger})^\dagger\beta\,, \quad ABC = (C^\dagger B^\dagger A^\dagger)^\dagger \\ &= (U(\Lambda)^{-1}\Psi(x)U(\Lambda))^\dagger\beta\,, \quad U(\Lambda) \text{ is unitary} \\ &= \Psi(\Lambda^{-1}x)^\dagger D(\Lambda)^\dagger\beta = \overline{\Psi}(\Lambda^{-1}x)\overline{D(\Lambda)} \\ &= \overline{\Psi}(\Lambda^{-1}x)D(\Lambda)^{-1}\,, \quad \overline{iS^{\mu\nu}} = -i\overline{S^{\mu\nu}} = -iS^{\mu\nu}\,. \end{aligned}$$

(This should be in Srednicki but I could not find it.) Then

$$U(\Lambda)^{-1}\overline{\Psi}(x)X\Psi(x)U(\Lambda) = \overline{\Psi}(\Lambda^{-1}x)D(\Lambda)^{-1}XD(\Lambda)\Psi(\Lambda^{-1}x).$$

Finally, γ_5 commutes with $S^{\mu\nu}$ and so with $D(\Lambda)$.

b)

$$P^{-1}\overline{\Psi}\gamma_5\Psi P = \overline{\Psi}\gamma^0\gamma_5\gamma^0\Psi = -\overline{\Psi}\gamma_5\gamma^0\gamma^0\Psi = -\overline{\Psi}\gamma_5\Psi.$$

c) Write

$$m_1 + im_2\gamma_5 = (m_1 + im_2)P_L + (m_1 - im_2)P_R\,,$$

since $P_{L,R}$ project onto eigenspaces of γ_5. Defining $m_1 + im_2 = e^{i\phi}m$ for m real, we have

$$\mathcal{L} = i\overline{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + i\overline{\Psi}_R\gamma^\mu\partial_\mu\Psi_R - me^{-i\phi}\overline{\Psi}_R\Psi_L - me^{i\phi}\overline{\Psi}_L\Psi_R\,,$$

where $\Psi_{L,R} = P_{L,R}\Psi$. You should check that the last equation is correct.

Then a field redefinition $\Psi'_L = e^{-i\phi}\Psi_L$ leaves the kinetic term invariant and changes the mass term to $m(\overline{\Psi}'_R\Psi'_L + e^{i\phi}\overline{\Psi}'_L\Psi'_R) = m\overline{\Psi}'\Psi'$, which is invariant under the ordinary parity transformation on Ψ'. The redefinition $\Psi'_L = e^{-i\phi/2}\Psi_L, \Psi'_R = e^{+i\phi/2}\Psi_R$ does the same thing. It can be conveniently written as

$$\Psi' = e^{-i\phi\gamma_5/2}\Psi = (\cos\phi/2 - i\gamma_5\sin\phi/2)\Psi.$$

Or, instead of redefining the field, we could redefine the transformation. P takes $m_2 \to -m_2$ and so $\phi \to -\phi$. So combine P with a symmetry rotation $Z^{-1}\Psi_L Z = e^{-2i\phi}\Psi_L$, $Z^{-1}\Psi_R Z = \Psi_R$, and PZ leaves $\mathcal{L}$ invariant.

Note that the field redefinition is part of the $U(1) \times U(1)$ symmetry described in the solution to 36.5, HW #1, which was in turn part of the full $U(2)$ symmetry of the massless theory. Since your field redefinition must leave the kinetic term invariant, it must be part of this $U(2)$. Some people wrote down transformations that did the desired thing to the mass term, but did not leave the kinetic term invariant.

d) We need that $(m_1 + im_2)(g_1 - ig_2)$ is real in order to make both terms invariant at once. That is $m_1 g_2 = m_2 g_1$.

e) Let's redefine the field so that $m_2 = 0$ but $g_2 \neq 0$. According to Srednicki 40.40 and 40.47, the g_2 term will be invariant under C but not T. P and T each flip its sign, so PT and CPT are symmetries.

2. a)

$$\{b_i^+, b_j^-\} = -\delta^{ij}\,, \quad \{b_i^+, b_j^+\} = \{b_i^-, b_j^-\} = 0\,.$$

I should have included an extra i in the definition to get rid of the $-$ sign.

b) If for any $|\psi\rangle$, $b_i^-(b_1^- b_2^- |\psi\rangle) = 0$, by (a). So $b_1^- b_2^- |\psi\rangle$ has the desired property, unless it vanishes. If it does vanish, then one of $b_1^-|\psi\rangle$, $b_2^-|\psi\rangle$, or $|\psi\rangle$ itself is nonvanishing with that property.

c) By the anticommutation relations,

$$b_1^+(u_0, b_1^+ u_0, b_2^+ u_0, b_1^+ b_2^+ u_0) = (b_1^+ u_0, 0, b_1^+ b_2^+ u_0, 0)\,,$$

so

$$b_1^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly,

$$b_1^- = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad b_2^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad b_2^- = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then solve for γ^μ and γ_5 from the definitions. By the way, this can be used to construct the γ matrices in any number of dimensions, it is essentially what is done in vol. 2, appendix B.1 of my book.