Selected Homework 4 solutions

Comments added 2/22 marked **update**.

1. a) Let parity flip the 1-direction. Then $D(P)$ must commute with γ^0, γ^2 and anticommute with γ^1. Noting that $\gamma^0\gamma^1\gamma^2 = -iI$, this requires it to anticommute with the identity I. This is impossible. Similarly $D(T)$ must also commute with γ^0, γ^2 and anticommute with γ^1 ($*$ flips γ^2, $\mathcal{T}$ flips γ^0, and $-$ flips all three), again impossible. But $\mathcal{C}$ anticommutes with γ^0, γ^2 and commutes with γ^1. So $\mathcal{C} = \gamma^1$ works.

b) If we consider the combination of P and T, part (a) implies that $D(PT)$ commutes with all the γ^μ, so PT is a symmetry with $D(PT) = 1$. The adjoint of this (using unitarity, $\Theta^\dagger = \Theta^{-1}$) is $D(\Theta) = \mathcal{C} = \gamma^1$, this should work. Let's see this more systematically:

$$\Theta^{-1}\Psi(x)\Theta = D(\Theta)\overline{\Psi}^{\mathrm{T}}(x')\,,$$

where $x' = \mathcal{PT}x$. Then

$$\Theta^{-1}\Psi^\dagger(x)\Theta = \Psi^{\mathrm{T}}(x')\beta D(\Theta)^\dagger\,, \quad \Theta^{-1}\overline{\Psi}(x)\Theta = \Psi^{\mathrm{T}}(x')\beta D(\Theta)^\dagger\beta\,.$$

Then

$$\begin{aligned}
&\Theta^{-1}\left(i\overline{\Psi}(x)\gamma^\mu\partial_\mu\Psi(x) + m\overline{\Psi}(x)\Psi(x)\right)\Theta \\
&= -i\Theta^{-1}\overline{\Psi}(x)\Theta\,\gamma^{\mu *}\partial_\mu\,\Theta^{-1}\Psi(x)\Theta + m\Theta^{-1}\overline{\Psi}(x)\Theta\Theta^{-1}\Psi(x)\Theta \\
&= -i\Psi^{\mathrm{T}}(x')\beta D(\Theta)^\dagger\beta\gamma^{\mu *}D(\Theta)\partial_\mu\overline{\Psi}^{\mathrm{T}}(x') + m\Psi^{\mathrm{T}}(x')\beta D(\Theta)^\dagger\beta D(\Theta)\overline{\Psi}^{\mathrm{T}}(x')
\end{aligned}$$

The second line used antilinearity of Θ and the third uses the previous results. Now transposing,

$$\begin{aligned}
&= i\partial_\mu\overline{\Psi}(x')D(\Theta)^{\mathrm{T}}\gamma^{\mu\dagger}\beta D(\Theta)^*\beta\Psi(x') - m\overline{\Psi}(x')D(\Theta)^{\mathrm{T}}\beta D(\Theta)^*\beta\Psi(x') \\
&= -i\overline{\Psi}(x')D(\Theta)^{\mathrm{T}}\gamma^{\mu\dagger}\beta D(\Theta)^*\beta(\mathcal{PT})_\mu{}^\nu\partial'_\nu\Psi(x') - m\overline{\Psi}(x')D(\Theta)^{\mathrm{T}}\beta D(\Theta)^*\beta\Psi(x')\,.
\end{aligned}$$

In the final line we integrate by parts, and then change from ∂_μ to ∂'_μ. From the mass term we need

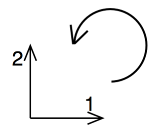
$$D(\Theta)^{\mathrm{T}}\beta D(\Theta)^*\beta = -1$$

so $\beta D(\Theta)^*\beta = -(D(\Theta)^{\mathrm{T}})^{-1}$. From the kinetic term we then need

$$D(\Theta)^{\mathrm{T}}\gamma^{\mu\dagger}(D(\Theta)^{\mathrm{T}})^{-1}(\mathcal{PT})_\mu{}^\nu = \gamma^\nu\,.$$

This works out to a $+$ for $\mu = 1$ and a $-$ for $\mu = 0, 2$. These conditions are indeed satisfied by $D(\Theta)^{\mathrm{T}} = -\gamma^1$, $D(\Theta) = \gamma^1$ (the sign is arbitrary). If you defined parity to flip x^2 you'd get γ^2 instead.

c) The Dirac matrices have two components, half as many as in 4 dimensions, so we get one particle state and one antiparticle state. Let us suppose that the particle state has $S_{12} = +\frac{1}{2}$. From the figure, we see that P (flipping the 1-direction) and T both reverse



the spin, taking the $S_{12} = +\frac{1}{2}$ particle to an $S_{12} = -\frac{1}{2}$ particle. But there is no such state in the spectrum, so these can't be symmetries. PT takes an $S_{12} = +\frac{1}{2}$ particle to an $S_{12} = +\frac{1}{2}$ particle, and is a symmetry. CPT takes an $S_{12} = +\frac{1}{2}$ particle to an $S_{12} = +\frac{1}{2}$ antiparticle. This must be a symmetry, so the antiparticle must have $S_{12} = +\frac{1}{2}$ as in the corrected homework. For a Majorana field, the particle and antiparticle are the same state.

d) Massless: the matrix $D(P) = \gamma^1$ works (γ^2 if your parity flips x^2).

2. a) Copying the 4d case

$$\Psi(x) = u(p)e^{ipx} + v(p)e^{-ipx}\,,$$

$$0 = (\not{p} + m)u = \begin{bmatrix} m & p^1 - p^0 \\ -p^1 - p^0 & m \end{bmatrix} u\,, \quad u = \begin{bmatrix} (p^0 - p^1)^{1/2} \\ (p^0 + p^1)^{1/2} \end{bmatrix}\,,$$

$$0 = (\not{p} - m)v = \begin{bmatrix} -m & p^1 - p^0 \\ -p^1 - p^0 & -m \end{bmatrix} v\,, \quad v = \begin{bmatrix} (p^0 - p^1)^{1/2} \\ -(p^0 + p^1)^{1/2} \end{bmatrix}\,.$$

I have normalized these so that $\bar{u}u = 2m$, $\bar{v}v = -2m$ by analogy to $4d$. You can also see that

$$u\bar{u} = m - \not{p}\,, \quad v\bar{v} = -m - \not{p}\,.$$

b,c) The full expansion would be

$$\Psi(x) = \int \frac{dp^1}{4\pi\omega} \left(b(p)u(p)e^{ipx} + d^\dagger(p)v(p)e^{-ipx} \right)\,.$$

Following exactly as in 4d,

$$\{\Psi_\alpha(\vec{x},t), \overline{\Psi}_\beta(\vec{y},t)\} = \gamma^0_{\alpha\beta}\delta(\vec{x}-\vec{y})\,, \quad \{b(\vec{p}), b^\dagger(\vec{p}\,')\} = 4\pi\omega\delta(\vec{p}-\vec{p}\,') = \{d(\vec{p}), d^\dagger(\vec{p}\,')\}\,,$$

with other anticommutators vanishing.

d) Just as in 4d, the matrix element is

$$-i\int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{m - \not{p}}{p^2 + m^2 - i\epsilon} = \theta(x^0) \int_{-\infty}^{\infty} \frac{dp^1}{4\pi\omega} e^{ip^1x^1 - i\omega x^0} (m + \gamma^0\omega - \gamma^1 p^1)\Big|_{\omega=\sqrt{(p^1)^2+m^2}}$$
$$- \theta(-x^0) \int_{-\infty}^{\infty} \frac{dp^1}{4\pi\omega} e^{-ip^1x^1 + i\omega x^0} (-m + \gamma^0\omega - \gamma^1 p^1)\Big|_{\omega=\sqrt{(p^1)^2+m^2}}\,.$$

I've set $y = 0$ to simplify, you can restore it by $x \to x - y$. In the massless case, $\omega = |p^1|$, so we get

$$\theta(x^0)\int_{-\infty}^{\infty}\frac{dp^1}{4\pi}e^{ip^1x^1-i|p^1|x^0}(\gamma^0-\gamma^1 s^1) - \theta(-x^0)\int_{-\infty}^{\infty}\frac{dp^1}{4\pi}e^{-ip^1x^1+i|p^1|x^0}(\gamma^0-\gamma^1 s^1)\,.$$

Here $s^1 \equiv \mathrm{sign}(p^1)$. Separate each into two integrals, $p_1 < 0$ and $p_1 > 0$, and use

$$\int_0^\infty dk\, e^{ikx} = \lim_{\epsilon\to 0}\frac{i}{x+i\epsilon}\,,\qquad \int_{-\infty}^0 dk\, e^{ikx} = \lim_{\epsilon\to 0}\frac{-i}{x-i\epsilon}\,.$$

to get the final result

$$\frac{i(\gamma^0-\gamma^1)}{4\pi(x^1-x^0+i\epsilon s^0)} + \frac{-i(\gamma^0+\gamma^1)}{4\pi(x^1+x^0-i\epsilon s^0)} = \frac{i}{2\pi}\frac{\not x}{x^2-i\epsilon}\,, \tag{1}$$

where $s^0 \equiv \mathrm{sign}(x^0)$. You can check that the final ϵ prescription follows as stated after combining denominators. (The $\epsilon \to 0$ limit is implicit.)

The integral was a bit tricky to get straight. For example, if you used $\omega = p^1$ instead of $|p^1|$ you'd get delta functions. Note that the final form is nicely covariant. As it typical in massless field theories, the time-ordered correlator is essentially a power of the separation.

Update: Only a few people got close to (1), but no one got the ϵ's. Note that they are important. e.g.

$$\frac{1}{x\pm i\epsilon} = \frac{P}{x} \mp i\pi\delta(x)\,.$$

So there is a delta-function piece on the light-cone, which you only get correctly by keeping the ϵ's.

3. $e^+e^+ \to e^+e^+$ is $e^-e^- \to e^-e^-$ (eq. 45.24) with $u_i \to v_i'$ and $\bar u \to \bar v_i'$.

$\varphi\varphi \to e^+e^-$ is $e^+e^- \to \varphi\varphi$ (eq. 45.23) with $u \to v'$ and $\bar v \to \bar u'$, and $p_i \to -p_i'$, $k_i' \to -k_i$.

4. Labeling the interaction vertices $r_{1,2,3,4}$ clockwise from the upper right, we have

$$(-i)^8(ig)^4\int d^{4\times 4}r\,\Delta(r_1-r_2)\Delta(r_3-r_4)[S(x-r_1)S(r_1-r_4)S(r_4-y)]_{\alpha\beta}[S(z-r_2)S(r_2-r_3)S(r_4-w)]_{\gamma\delta}$$
$$-(x,\alpha\leftrightarrow z,\gamma)\,. \tag{2}$$

Update: Note that there is no symmetry factor: in a Green function the external lines are distinguishable, since the positions x, y, z, w are generically distinct.

There is also a crossed-ladder graph, which is distinct and I hadn't meant to include it, but it would be part of the full amplitude and some people did. You can draw it by crossing the two scalar lines to make an X, or keep the scalars uncrossed but reverse the direction of the lower line.

Some people wrote it in momentum space (sorry, I didn't specify).