Selected Homework 5 solutions

2. Note that these 16 matrices represent all products of 0,1,2,3, or 4 γ matrices, e.g we could write them alternately as

$$\Gamma^{abcd} = (\gamma^0)^a(\gamma^1)^b(\gamma^2)^c(\gamma^3)^d$$

for $a, b, c, d = 0, 1$. Now, $\mathrm{Tr}(\Gamma^{abcd})$ vanishes unless $a = b = c = d = 0$, using known results. (Explicit argument: $\mathrm{Tr}(X)$ vanishes if there is any Y such that $Y^2 = \alpha I$, with α a nonzero constant, and $XY = -YX$, because $\mathrm{Tr}(X) = \alpha^{-1}\mathrm{Tr}(Y^2X) \overset{\text{anticom}}{=} -\alpha^{-1}\,\mathrm{Tr}(YXY) \overset{\text{cyclic}}{=} -\alpha^{-1}\mathrm{Tr}(Y^2X) = -\,\mathrm{Tr}(X)$. For any case except $a = b = c = d = 0$ you can find such a Y: if $a+b+c+d$ is even, take any of the γ^μ that has exponent 1, and if $a+b+c+d$ is odd, take any of the γ^μ that has exponent 0.)

But also

$$\Gamma^{abcd}\Gamma^{a'b'c'd'} = \pm\Gamma^{a+a',b+b',c+c',d+d'}\,,$$

(exponents mod 2)using the anticommutation relations. So $\mathrm{Tr}(\Gamma^{abcd}\Gamma^{a'b'c'd'})$ vanishes unless $(a, b, c, d) = (a', b', c', d')$. If any of these matrices could be written as a linear combination of the others, there would be a discrepancy. So we have $2^4 = 4^2$ linearly indendent matrices. This works in any even number of dimensions: $2^d = (2^{d/2})^2$.

3. a) Labeling the momenta $k \to p_1', p_2'$ for $\varphi \to e^-e^+$, we have $\mathcal{T} = g\bar{u}_{s_1'}(p_1')v_{s_2'}(p_2')$. Then

$$\sum_{\text{spins}} |\mathcal{T}|^2 = g^2\mathrm{Tr}[(m - \not{p}_1')(-m - \not{p}_2')] = -4g^2(p_1' \cdot p_2' + m^2)\,.$$

Srednicki 11.48 and 11.30 give

$$d\Gamma = \frac{1}{2M}\sum_{\text{spins}} |\mathcal{T}|^2 \frac{|\vec{p}_1{}'|}{16\pi^2 M} d\Omega\,.$$

Inserting some kinematics, we have $E_1' = E_2' = M/2$ and $|p_1'| = |p_2'| = \frac{1}{2}(M^2 - 4m^2)^{1/2}$, so

$$\sum_{\text{spins}} |\mathcal{T}|^2 = 4g^2(E_1'E_2' + |p_1'||p_2'| - m^2) = 2g^2(M^2 - 4m^2)\,,$$

and integrating over angles gives

$$\Gamma = g^2\frac{(M^2 - 4m^2)^{3/2}}{8\pi M^2}\,. \tag{1}$$

b) According to Srednicki 38.28,

$$
\begin{aligned}
|\mathcal{T}|^2 &= \frac{g^2}{4}\mathrm{Tr}[(1 - s_1\gamma_5\not{x})(m - \not{p}'_1)(1 - s_2\gamma_5\not{x})(-m - \not{p}'_2)] \\
&= \frac{g^2}{4}\mathrm{Tr}[-m^2 - m^2 s_1 s_2\gamma_5\not{x}\gamma_5\not{x} + \not{p}'_1\not{p}'_2 + s_1 s_2\gamma_5\not{x}\not{p}'_1\gamma_5\not{x}\not{p}'_2] \\
&= -g^2(m^2 + p'_1\cdot p'_2)(1 + s_1 s_2) \\
&= g^2(M^2 - 4m^2)(1 + s_1 s_2)/2\,. \qquad (2)
\end{aligned}
$$

In the second line we have expanded and dropped traces that vanish. Here, I am interpreting Mark's "let the x-axis be the spin-quantization axis" as using γ^1 in place of γ^3 above 38.26. Now, the property $x\cdot p = 0$ means that $\not{x}$ and $\not{p}$ anticommute. Also, $\not{x}\not{x} = -1$.

In the third line we have evaluated the traces; note that $\not{x}$ anticommutes with $\not{p}$ because the momentum is perpendicular to x.

We see that this vanishes if $s_1 = -s_2$. This is from a combination of P and angular momentum. The interaction is parity invariant for ϕ a scalar, so the parity and angular momentum are both zero. Now, in the final state, we have

$$
P b^\dagger_{s_1}(\vec{p}) d^\dagger_{s_2}(-\vec{p})|0\rangle = -b^\dagger_{s_1}(-\vec{p}) d^\dagger_{s_2}(+\vec{p})|0\rangle\,, \qquad (3)
$$

with the minus sign from 40.17. We can bring this back to to original state with a rotation $e^{i\pi J_x}$, but this introduces also a phase from rotating the spins,

$$
e^{i\pi J_x}(-b^\dagger_{s_1}(-\vec{p}) d^\dagger_{s_2}(+\vec{p}))|0\rangle = -e^{i\pi(s_1+s_2)/2} b^\dagger_{s_1}(\vec{p}) d^\dagger_{s_2}(-\vec{p})|0\rangle\,, \qquad (4)
$$

Since $e^{i\pi J_x}$ and P are both symmetries, the amplitude to produce $b^\dagger_{s_1}(\vec{p}) d^\dagger_{s_2}(-\vec{p})|0\rangle$ must be equal to $-e^{i\pi(s_1+s_2)/2}$ times itself, so $s_1 - s_2 = \pm 2$. Thus, the combination of parity and angular momentum forbid the decay with $s_1 = -s_2$. I think that this is

Also, if $M = 2m$ then $p'_1 = p'_2 = (m, 0, 0, 0)$ and it vanishes, as explained in part (a). This follows from P alone. By 40.17, the state $b^\dagger_{s_1}(0) d^\dagger_{s_2}(0)|0\rangle$ has odd parity, so $\mathcal{T}$ must vanish. Note that in the total rate (1) there are *two* vanishing factors, from $|\vec{p}_1{}'|$ and from $|\mathcal{T}|^2$. The first is because there is vanishing phase space for this decay, and the second is due to parity.

c) Choose coordinates so the e^- momentum is along the $+z$ direction and the e^+ momentum along the $-z$ direction. This problem is stated somewhat carelessly. So z_1 is the $+\eta$ boost of $(0, 1)$ and z_2 is the $-\eta$ boost of $(0, -1)$:

$$
p'_1 = m(\cosh\eta, \sinh\eta)\,,\;\; z_1 = (\sinh\eta, \cosh\eta)\,,\;\; p'_2 = m(\cosh\eta, -\sinh\eta)\,,\;\; z_2 = m(\sinh\eta, -\cosh\eta)\,,
$$

where I've written only the $0, z$ components.

$$|\mathcal{T}|^2 = \frac{g^2}{4}\mathrm{Tr}[(1 - s_1\gamma_5 \not{z}_1)(m - \not{p}_1')(1 - s_2\gamma_5 \not{z}_2)(-m - \not{p}_2')]$$

Note also that $z_1 \neq z_2$ because we boost in opposite directions, so there is + sign in the second spin projector. This did not come up in part (b) because x doesn't change under a z boost. So

$$p_1' = m(\cosh\eta, \sinh\eta)\,,\ \ z_1 = (\sinh\eta, \cosh\eta)\,,\ \ p_2' = m(\cosh\eta, -\sinh\eta)\,,\ \ z_2 = m(-\sinh\eta, \cosh\eta)\,,$$

where I've written only the $0, z$ components. Then

$$\begin{aligned}
|\mathcal{T}|^2 &= \frac{g^2}{4}\mathrm{Tr}[(1 - s_1\gamma_5 \not{z}_1)(m - \not{p}_1')(1 - s_2\gamma_5 \not{z}_2)(-m - \not{p}_2')] \\
&= \frac{g^2}{4}\mathrm{Tr}[-m^2 - m^2 s_1 s_2 \gamma_5 \not{z}_1 \gamma_5 \not{z}_2 + \not{p}_1' \not{p}_2' + s_1 s_2 \gamma_5 \not{z}_1 \not{p}_1' \gamma_5 \not{z}_2 \not{p}_2'] \\
&= g^2(-m^2 - m^2 s_1 s_2 p_1' \cdot p_2' - p_1' \cdot p_2' - m^2 s_1 s_2) \\
&= -g^2(m^2 + p_1' \cdot p_2')(1 + s_1 s_2) \\
&= g^2(M^2 - 4m^2)(1 + s_1 s_2)/2\,. \qquad (5)
\end{aligned}$$

This vanishes if $s_1 = -s_2$. Now, if $s_1 = -s_2$, then the z-components of the spins are in the same direction, so the total s_z is ± 1. But an orbital rotation leaves the momenta invariant, so the total z angular momentum is ± 1. This is impossible, because the initial boson was spinless. (Note that parity was not used).

d) Now

$$\sum_{\text{spins}} |\mathcal{T}|^2 = g^2 \mathrm{Tr}[(m - \not{p}_1') i\gamma_5 (-m - \not{p}_2') i\gamma_5] = 4g^2(-p_1' \cdot p_2' + m^2) = 2g^2 M^2\,.$$

For this interaction parity invariance requires ϕ to be a pseudoscalar, $P = -$, so the amplitude can be nonvanishing as $M \to 2m$. There are just two terms, m^2 and $p_1' \cdot p_2'$, and we see that the relative sign must be negative for the first interaction (in order that they cancel at $M = 2m$) and positive for the γ_5 interaction.

e) Repeating (b) ,

$$\begin{aligned}
|\mathcal{T}|^2 &= \frac{g^2}{4}\mathrm{Tr}[(1 - s_1\gamma_5 \not{x})(m - \not{p}_1') i\gamma_5 (1 - s_2\gamma_5 \not{x})(-m - \not{p}_2') i\gamma_5] \\
&= \frac{g^2}{4}\mathrm{Tr}[m^2 - m^2 s_1 s_2 \gamma_5 \not{x} \gamma_5 \not{x} + \not{p}_1' \not{p}_2' - s_1 s_2 \gamma_5 \not{x} \not{p}_1' \gamma_5 \not{x} \not{p}_2'] \\
&= g^2(-p_1' \cdot p_2' + m^2)(1 - s_1 s_2) \\
&= g^2 M^2 (1 - s_1 s_2)/2\,.
\end{aligned}$$

Now is vanishes for $s_1 = +s_2$. In this case the interaction requires that ϕ be pseudoscalar, so the signs are flipped in (3,4) and we need $e^{i\pi(s_1+s_2)} = +1$, opposite to the previous case, by the combination of P and angular momentum.

Repeating (c),

$$\begin{aligned}
|\mathcal{T}|^2 &= \frac{g^2}{4}\mathrm{Tr}[(1 - s_1\gamma_5\not{z}_1)(m - \not{p}'_1)i\gamma_5(1 - s_2\gamma_5\not{z}_2)(-m - \not{p}'_2)i\gamma_5] \\
&= \frac{g^2}{4}\mathrm{Tr}[m^2 - m^2 s_1 s_2\gamma_5\not{z}_1\gamma_5\not{z}_2 + \not{p}'_1\not{p}'_2 - s_1 s_2\gamma_5\not{z}_1\not{p}'_1\gamma_5\not{z}_2\not{p}'_2] \\
&= g^2(m^2 - m^2 s_1 s_2 p'_1\cdot p'_2 - p'_1\cdot p'_2 + 4m^2 h_1 h_2) \\
&= -g^2(-m^2 + p'_1\cdot p'_2)(1 + s_1 s_2) \\
&= g^2 M^2(1 + s_1 s_2)/2\,.
\end{aligned}$$

This still vanishes for $s_1 = -s_2$, as that depended only on angular momentum conservation and not parity.

Whew! That was longer than I expected.

4. We get

$$\mathcal{T} = -ig\bar{u}(p'_1)\not{k}(1 - \gamma_5)v(p'_2)\,.$$

Here I've abbreviated $g = c_1 G_F f_\pi$. Also, ∂_μ gives $-ik_m u$ when the momentum arrow points toward the derivative as here, and ik_μ when it points away. (The sign won't matter here because we square it.) Using

$$(\bar{u}(p'_1)\not{k}(1 - \gamma_5)v(p'_2)) * = \bar{v}(p'_2)(1 - \bar{\gamma}_5)\bar{\not{k}}u(p'_1) = \bar{v}(p'_2)(1 + \gamma_5)\not{k}u(p'_1)\,.$$

this becomes

$$\sum_{\text{spins}}|\mathcal{T}|^2 = g^2\mathrm{Tr}[(m_\mu - \not{p}'_1)\not{k}(1 - \gamma_5)(-m_\nu - \not{p}'_2)(1 + \gamma_5)\not{k}]\,.$$

You can check that $(1 - \gamma_5)(-m_\nu - \not{p}'_2)(1 + \gamma_5) = -2\not{p}'_2(1 + \gamma_5)$. There is now only a single γ_5, and any traces with it will vanish. Using also $k = p'_1 + p'_2$ we get

$$\begin{aligned}
\sum_{\text{spins}}|\mathcal{T}|^2 &= -2g^2\mathrm{Tr}[(m_\mu - \not{p}'_1)(\not{p}'_1 + \not{p}'_2)\not{p}'_2(\not{p}'_1 + \not{p}'_2)] \\
&= 2g^2\mathrm{Tr}[\not{p}'_1(\not{p}'_1 + \not{p}'_2)\not{p}'_2(\not{p}'_1 + \not{p}'_2)] \quad \text{(dropped term with odd \# γ's)} \\
&= -8(m_\mu + m_\nu)^2 g^2 p'_1\cdot p'_2\,.
\end{aligned} \tag{6}$$

The neutrino mass is negligible and we drop it from here. Now some kinematics. $E_1' + E_2' = m_\pi$ and $|\vec{p}_1{}'| = |\vec{p}_2{}'| = E_2'$ (massless ν). So $(m_\mu^2 + |\vec{p}_1{}'|^2)^{1/2} + |\vec{p}_1{}'| = m_\pi$, giving

$$|\vec{p}_1{}'| = (m_\pi^2 - m_\mu^2)/2m_\pi\,, \quad E_1' = (m_\pi^2 + m_\mu^2)/2m_\pi\,, \quad p_1' \cdot p_2' = -(m_\pi^2 - m_\mu^2)/2\,.$$

Using Srednicki 11.48 and 11.30 as above,

$$\Gamma = \frac{1}{8\pi m_\pi^2}|\vec{p}_1{}'| \sum_{\text{spins}} |\mathcal{T}|^2 = \frac{g^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3}\,.$$

So

$$f_\pi = \frac{2\sqrt{\pi} m_\pi^{3/2} \Gamma^{1/2}}{c_1 G_F m_\mu (m_\pi^2 - m_\mu^2)} = 93.15 \text{ MeV}\,.$$

from $\Gamma = \hbar/(2.603 \times 10^{-8}\text{ s}) = 2.529 \times 10^{-14}$ MeV. The measured value (taking into account a $\sqrt{2}$ convention) is 92.21 ± 0.15 MeV,
http://pdg.lbl.gov/2012/reviews/rpp2012-rev-pseudoscalar-meson-decay-cons.pdf. As Mark notes, most of the discrepancy can be understood from a QED correction.