Homework 6 solutions

1. #1) Let $I_s, I_f$ be the number of internal scalar and fermion propagators, $E_s, E_f$ be the number of external scalar and fermion propagators, and $n_{s_i}, n_{f_i}, n_{d_i}$ be the number of scalar fields, fermi fields, and derivatives in the $i$'th vertex. Then the total degree of divergence is

$$D = dL - 2I_s - I_\psi + \sum_i n_{d_i},$$

counting the total number of momentum integrals, the momenta in the propagators, and the momenta in the vertices. Matching the number of propagator ends,

$$2I_s + E_s = \sum_i n_{s_i}, \quad 2I_\psi + E_\psi = \sum_i n_{\psi_i}.$$

The total number of loop integrations is

$$L = I_s + I_\psi + 1 - \sum_i 1$$

(one momentum per internal propagator, constrained by one $\delta$-function per vertex, except that one of $\delta$-functions constrains the external momenta to sum to zero. We can solve for $I_s, I_\psi, L$ to get

$$D = -\frac{d-2}{2}E_s - \frac{d-1}{2}E_\psi + d + \sum_i \left( \frac{d-2}{2}n_{s_i} + \frac{d-1}{2}n_{\psi_i} + n_{d_i} - d \right).$$

#2) From $[S] = 0$, the kinetic terms give $[\phi] = \frac{d-2}{2}$ and $[\psi] = \frac{d-1}{2}$. So

$$[d^d x][\phi^{n_s}][\psi^{n_\psi}][\partial^{n_d}] = -d + \frac{d-2}{2}n_s + \frac{d-1}{2}n_\psi + n_d$$

and for the corresponding coupling

$$[g] = d - \frac{d-2}{2}n_s - \frac{d-1}{2}n_\psi - n_d.$$

So dimensional analysis gives

$$D = [g_E] - \sum_i [g_i],$$

and $g$ must be $\geq 0$ in a renormalizable theory.

In $d = 4$, the condition for renormalizability is $n_s + \frac{3}{2}n_\psi + n_d \leq 4$, with solutions

$\phi^{n_s}, n_s \leq 4$: scalar self-interactions terms,
$\partial\phi\partial\phi$: scalar kinetic term,
$\psi^2$: fermion mass, $\phi\psi^2$: Yukawa coupling, $\psi\partial\psi$: fermion kinetic term.
I’ve omitted total derivatives and terms that aren’t Lorentz invariant (one derivative or one fermion)

In $d = 3$, the condition for renormalizability is $\frac{1}{2} n_s + n_\psi + n_d \leq 3$, with solutions

\[ \phi^{n_s}, \ n_s \leq 6 : \text{ scalar self-interactions terms, } \]
\[ \partial \phi \partial \phi : \text{ scalar kinetic term, } \psi \partial \psi : \text{ fermion kinetic term, } \]
\[ \psi^2 : \text{ fermion mass, } \phi \psi^2 : \text{ Yukawa coupling, } \phi^2 \psi^2 : \text{ higher Yukawa. } \]

In $d = 2$, the condition for renormalizability is $\frac{1}{2} n_\psi + n_d \leq 2$, with solutions

\[ \phi^{n_s}, \ \text{any } n_s : \text{ scalar self-interactions terms, } \]
\[ \partial \phi \partial \phi : \text{ scalar kinetic term, } \phi^{n_s-2} \partial \phi \partial \phi, \ n_s \geq 3 : \text{ nonlinear scalar kinetic term, } \]
\[ \psi \partial \psi : \text{ fermion kinetic term, } \phi^{n_s} \psi \partial \psi, \ n_s \geq 3 : \text{ scalar-dependent fermion kinetic term, } \]
\[ \psi^2 : \text{ fermion mass, } \phi \psi^2 : \text{ Yukawa coupling, } \phi^{n_s} \psi^2 : \text{ higher Yukawas, } \]
\[ \psi^4 : \text{ fermion self-interaction, } \phi^{n_s} \psi^4 : \text{ higher interaction. } \]

In $d = 2$ there is an infinite number of allowed couplings, since we can add scalars for free, but these quantum field theories still make sense.

2. The suggested method: Evaluate

\[ \int_{-\infty}^{\infty} d^d x \ e^{-x^2} \]

in two ways. First, it’s

\[ \left( \int_{-\infty}^{\infty} dx \ e^{-x^2} \right)^d = \pi^{d/2}. \]

Second, it’s

\[ \Omega_d \int_0^{\infty} dr \ r^{d-1} \ e^{-r^2} \ r^2 = y \frac{\Omega_d}{2} \int_0^{\infty} dy \ y^{(d/2)-1} \ e^{-y} = \frac{\Omega_d}{2} \Gamma(d/2). \]

Thus \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \).

Direct route: let

\[ x_1 = \cos \theta_1, \quad x_2 = \sin \theta_1 \cos \theta_2, \quad x_3 = \sin \theta_1 \sin \theta_2 \cos \theta_3, \]
\[ x_d = \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{d-1}. \]
Each $\theta_i$ runs from 0 to $\pi$ except $\theta_{d-1}$ which runs 0 to $2\pi$ (good to check the cases $d = 2, 3$ to see how this works). Then

$$(dx \cdot dx)_d = d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 (d\theta_3^2 + \ldots + \sin^2 \theta_{d-2} d\theta_{d-1}^2)))$$

$$= d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 + \ldots + \sin^2 \theta_1 \ldots \sin^2 \theta_{d-2} d\theta_{d-1}^2. \quad (1)$$

Then

$$\Omega_d = 2\pi \int_0^\pi d\theta_1 \ldots d\theta_{d-2} \sin \theta_1^{d-2} \sin \theta_2^{d-3} \ldots \sin \theta_{d-2}$$

$$= 2\pi \prod_{n=1}^{d-2} \int_0^\pi \sin^n \theta$$

$$= 2\pi \int_0^\pi \prod_{n=1}^{d-2} \frac{\Gamma(\frac{1}{2}(n+1)) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(n+2))}$$

$$= 2\pi \int_0^\pi \prod_{n=1}^{d-2} \frac{\Gamma(\frac{1}{2}(n+1)) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(n+2))}$$

$$= 2\pi \frac{\Gamma(1)}{\Gamma\left(\frac{d}{2}\right) \Gamma(\frac{1}{2})^{d-2}}$$

$$= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$