Homework 6 solutions

1. #1) Let I_s, I_f be the number of internal scalar and fermion propagators, E_s, E_f be the number of external scalar and fermion propagators, and n_{si}, n_{fi}, n_{di} be the number of scalar fields, fermi fields, and derivatives in the i'th vertex. Then the total degree of divergence is

$$D = dL - 2I_s - I_\psi + \sum_i n_{di}\,,$$

counting the total number of momentum integrals, the momenta in the propagators, and the momenta in the vertices. Matching the number of propagator ends,

$$2I_s + E_s = \sum_i n_{si}\,, \quad 2I_\psi + E_\psi = \sum_i n_{\psi i}\,.$$

The total number of loop integrations is

$$L = I_s + I_\psi + 1 - \sum_i 1$$

(one momentum per internal propagator, constrained by one δ-function per vertex, except that one of δ-functions constrains the *external* momenta to sum to zero. We can solve for I_s, I_ψ, L to get

$$D = -\frac{d-2}{2}E_s - \frac{d-1}{2}E_\psi + d + \sum_i \left(\frac{d-2}{2}n_{si} + \frac{d-1}{2}n_{\psi i} + n_{di} - d\right)\,.$$

#2) From $[S]=0$, the kinetic terms give $[\phi] = \frac{d-2}{2}$ and $[\psi] = \frac{d-1}{2}$. So

$$[d^d x][\phi^{n_s}][\psi^{n_\psi}][\partial^{n_d}] = -d + \frac{d-2}{2}n_s + \frac{d-1}{2}n_\psi + n_d$$

and for the corresponding coupling

$$[g] = d - \frac{d-2}{2}n_s - \frac{d-1}{2}n_\psi - n_d\,.$$

So dimensional analysis gives

$$D = [g_E] - \sum_i [g_i]\,,$$

and g must be ≥ 0 in a renormalizable theory.

In $d=4$, the condition for renormalizability is $n_s + \frac{3}{2}n_\psi + n_d \leq 4$, with solutions

$\phi^{n_s}\,,\ n_s \leq 4:$ scalar self-interactions terms,

$\partial\phi\partial\phi:$ scalar kinetic term,

$\psi^2:$ fermion mass, $\quad \phi\psi^2:$ Yukawa coupling, $\quad \psi\partial\psi:$ fermion kinetic term.

I've omitted total derivatives and terms that aren't Lorentz invariant (one derivative or one fermion)

In $d = 3$, the condition for renormalizability is $\frac{1}{2}n_s + n_\psi + n_d \leq 3$, with solutions

$$\begin{aligned} &\phi^{n_s}\,,\ n_s \leq 6 : \text{ scalar self-interactions terms,} \\ &\partial\phi\partial\phi : \text{ scalar kinetic term,} \quad \psi\partial\psi : \text{ fermion kinetic term,} \\ &\psi^2 : \text{ fermion mass,} \quad \phi\psi^2 : \text{ Yukawa coupling,} \quad \phi^2\psi^2 : \text{ higher Yukawa}\,. \end{aligned}$$

In $d = 2$, the condition for renormalizability is $\frac{1}{2}n_\psi + n_d \leq 2$, with solutions

$$\begin{aligned} &\phi^{n_s}\,,\ \text{any } n_s : \text{ scalar self-interactions terms,} \\ &\partial\phi\partial\phi : \text{ scalar kinetic term,} \quad \phi^{n_s-2}\partial\phi\partial\phi\,,\ n_s \geq 3 : \text{ nonlinear scalar kinetic term,} \\ &\psi\partial\psi : \text{ fermion kinetic term,} \quad \phi^{n_s}\psi\partial\psi\,,\ n_s \geq 3 : \text{ scalar-dependent fermion kinetic term,} \\ &\psi^2 : \text{ fermion mass,} \quad \phi\psi^2 : \text{ Yukawa coupling,} \quad \phi^{n_s}\psi^2 : \text{ higher Yukawas,} \\ &\psi^4 : \text{ fermion self-interaction,} \quad \phi^{n_s}\psi^4 : \text{ higher interaction}\ . \end{aligned}$$

In $d = 2$ there is an infinite number of allowed couplings, since we can add scalars for free, but these quantum field theories still make sense.

2. The suggested method: Evaluate

$$\int_{-\infty}^{\infty} d^d x\, e^{-x\cdot x}$$

in two ways. First, it's

$$\left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \pi^{d/2}\,.$$

Second, it's

$$\Omega_d \int_0^\infty dr\, r^{d-1} e^{-r^2} \overset{r^2=y}{=} \frac{\Omega_d}{2}\int_0^\infty dy\, y^{(d/2)-1} e^{-y} = \frac{\Omega_d}{2}\Gamma(d/2)\,.$$

Thus $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$.

Direct route: let

$$\begin{aligned} x_1 &= \cos\theta_1\,, \quad x_2 = \sin\theta_1\cos\theta_2\,, \quad x_3 = \sin\theta_1\sin\theta_2\cos\theta_3\,, \\ x_d &= \sin\theta_1\sin\theta_2\ldots\sin\theta_{d-1}\,. \end{aligned}$$

Each θ_i runs from 0 to π except θ_{d-1} which runs 0 to 2π (good to check the cases $d = 2, 3$ to see how this works). Then

$$\begin{aligned}(dx\cdot dx)_d &= d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \ldots + \sin^2\theta_{d-2}d\theta_{d-1}^2)))) \\ &= d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2 + \ldots + \sin^2\theta_1\ldots\sin^2\theta_{d-2}d\theta_{d-1}^2\,. \end{aligned} \tag{1}$$

Then

$$\begin{aligned}\Omega_d &= 2\pi\int_0^\pi d\theta_1\ldots d\theta_{d-2}\sin\theta_1^{d-2}\sin\theta_2^{d-3}\ldots\sin\theta_{d-2} \\ &= 2\pi\prod_{n=1}^{d-2}\int_0^\pi \sin^n\theta \\ &= 2\pi\int_0^\pi\prod_{n=1}^{d-2}\frac{\Gamma(\frac{1}{2}(n+1))\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(n+2))} \\ &= 2\pi\int_0^\pi\prod_{n=1}^{d-2}\frac{\Gamma(\frac{1}{2}(n+1))\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(n+2))} \\ &= 2\pi\frac{\Gamma(1)}{\Gamma(\frac{d}{2})}\Gamma(\tfrac{1}{2})^{d-2} \\ &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\,.\end{aligned}$$