Homework 7 solutions

1. Srednicki 14.3. For all three integrals, change variables $q^{\mu'} = \Lambda^\mu{}_\nu q^\nu$. Using the lemma that the only invariants are products of metrics and the Levi-Civita tensor, we get that

$$I^\mu = 0\,, \quad I^{\mu\nu} = cg^{\mu\nu}\,, \quad I^{\mu\nu\rho\sigma} = c'(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\sigma\nu})\,.$$

Then

$$I^\mu{}_\mu = cd\,, \quad I^{\mu\nu}{}_{\mu\nu} = c'(2d + d^2)\,.$$

The constants c and c' are thus expressed in terms of integrals over functions of q^2 only.

2. Graph similar to propagator in $g\phi^3$, but now $d = 4 - \epsilon$:

$$\begin{aligned}
\frac{\mathbf{V}_4}{\lambda\tilde{\mu}^\epsilon} &= -Z_\lambda - i\frac{\lambda^2\tilde{\mu}^\epsilon}{2}\int\frac{d^dl}{(2\pi)^4}\frac{1}{(l^2+m^2)((l+k_1+k_2)^2+m^2)} + 2\text{ perms.} + O(\lambda^2) \\
&= -(1 + C_1\lambda) + \frac{\lambda\tilde{\mu}^\epsilon}{2}\int\frac{d^d\bar{q}}{(2\pi)^d}\int_0^1 dx\frac{1}{(\bar{q}^2 + D_{12})^2} + 2\text{ perms.} + O(\lambda^2) \\
&= -(1 + C_1\lambda) + \frac{\lambda}{2(4\pi)^2}\tilde{\mu}^\epsilon\Gamma(\epsilon/2)(4\pi)^{\epsilon/2}\int_0^1 dx\, D_{12}^{-\epsilon/2} + 2\text{ perms.} + O(\lambda^2) \\
&\overset{\epsilon\to 0}{=} -(1 + C_1\lambda) + \frac{\lambda}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \int_0^1 dx\,\ln(D_{12}/\mu^2)\right) + 2\text{ perms.} + O(\lambda^3) \\
&= -1 - \frac{\lambda}{2(4\pi)^2}\int_0^1 dx\,\ln(D_{12}/\mu^2) + 2\text{ perms.} + O(\lambda^2)\,.
\end{aligned}$$

Note that I divided through by $\tilde{\mu}^\epsilon$ to make both sides dimensionless. Here $D_{12} = x(1 - x)(k_1 + k_2)^2 + m^2$, and in the last line we canceled the pole in ϵ ($\overline{\text{MS}}$ scheme),

$$C_1 = \frac{3}{16\pi^2\epsilon}\,.$$

Finally, for $m^2 = 0$ we can do the integral, with the result

$$\frac{\mathbf{V}_4}{\lambda} = -1 + \frac{3\lambda}{16\pi^2} - \frac{\lambda}{32\pi^2}\left(\ln\frac{(k_1+k_2)^2}{\mu^2} + \ln\frac{(k_1+k_3)^2}{\mu^2} + \ln\frac{(k_1+k_4)^2}{\mu^2}\right) + O(\lambda^2)\,.$$

Note again the appearance of logarithms, and that the magnitude grows at large k as expected from the RG for this theory.

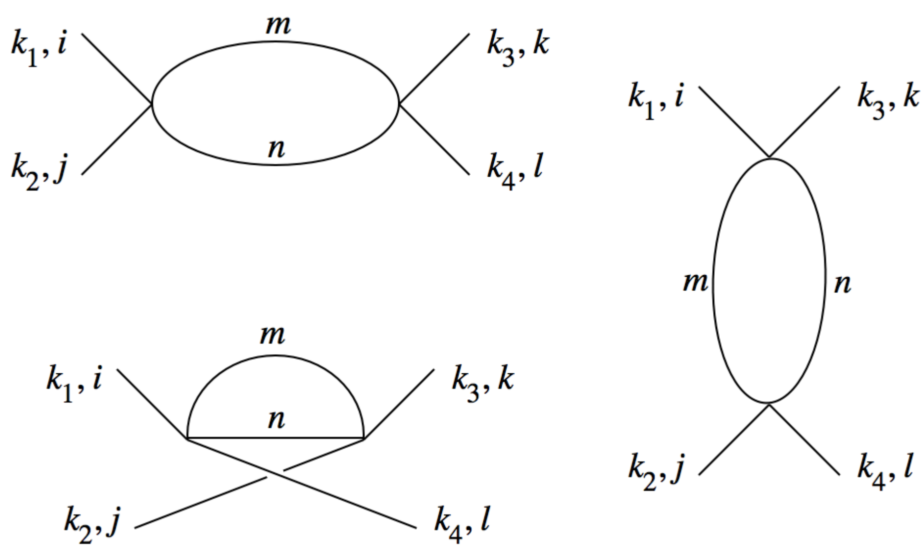
Note that the $\lambda\phi^4$ vertex is $-i\lambda Z_\lambda$, the sign because the potential is defined to be positive for positive λ.

3. Each propagator now has an index denoting the field. The vertex $-i\lambda Z_\lambda$ is now multiplied by $\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, in terms of the indices of the attached propagators.

For the first graph below, the s-channel, the integral from problem 2, for external lines $(k_1, i; k_2, j; k_3, k; k_4, l)$ is now multiplied by

$$(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) = (N+4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} \,.$$

The indices m, n on the internal lines have been summed.



Summing the other two channels then gives

$$\begin{aligned}
\frac{\mathbf{V}_{ijkl}}{\lambda} &\overset{\epsilon\to 0}{=} -(1 + C_1\lambda)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&\quad + \frac{\lambda}{2(4\pi)^2}\left((N+4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk}\right)\left(\frac{2}{\epsilon} - \int_0^1 dx\, \ln(D_{12}/\mu^2)\right) + 2 \text{ perms.} + O(\lambda^3) \\
&= -(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda}{2(4\pi)^2}\Big\{\left((N+4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk}\right)\int_0^1 dx\, \ln(D_{12}/\mu^2) \\
&\quad + \left((N+4)\delta_{ik}\delta_{jl} + 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk}\right)\int_0^1 dx\, \ln(D_{13}/\mu^2) \\
&\quad + \left((N+4)\delta_{il}\delta_{kj} + 2\delta_{ik}\delta_{jl} + 2\delta_{ij}\delta_{lk}\right)\int_0^1 dx\, \ln(D_{14}/\mu^2)\Big\} \,,
\end{aligned}$$

where we have used $\overline{\text{MS}}$ to set

$$C_1 = \frac{(N+8)}{16\pi^2\epsilon} \,.$$

Note that the indices are permuted along with the momenta, e.g. $(k_2, j) \leftrightarrow (k_4, l)$ in the last line, and that the tensor structure doesn't just add up to a multiple of $\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$.

A check: for $N = 1$ and $\lambda \to \lambda/3$ it reduces to the previous problem.