Homework 7 solutions

1. Srednicki 14.3. For all three integrals, change variables \( q'^{\mu} = \Lambda^{\mu}_{\nu} q^{\nu} \). Using the lemma that the only invariants are products of metrics and the Levi-Civita tensor, we get that

\[
I^{\mu} = 0, \quad I^{\mu\nu} = cg^{\mu\nu}, \quad I^{\mu\nu\rho\sigma} = c'(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}).
\]

Then

\[
I^{\mu\mu} = cd, \quad I^{\mu\nu\mu\nu} = c'(2d + d^2).
\]

The constants \( c \) and \( c' \) are thus expressed in terms of integrals over functions of \( q^2 \) only.

2. Graph similar to propagator in \( g\phi^3 \), but now \( d = 4 - \epsilon \):

\[
\frac{V_4}{\lambda \bar{\mu}^\epsilon} = - Z_\lambda - i \frac{\lambda^2 \bar{\mu}^\epsilon}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{1 - (l^2 + m^2)((l + k_1 + k_2)^2 + m^2)} + 2 \text{ perms.} + O(\lambda^2)
\]

\[
= -(1 + C_1 \lambda) + \frac{\lambda \bar{\mu}^\epsilon}{2} \int \frac{d^d \bar{q}}{(2\pi)^d} \int_0^1 dx \frac{1}{(\bar{q}^2 + D_{12})^\epsilon} + 2 \text{ perms.} + O(\lambda^2)
\]

\[
= -(1 + C_1 \lambda) + \frac{\lambda}{2(4\pi)^2} \mu^\epsilon \Gamma(\epsilon/2)(4\pi)^{\epsilon/2} \int_0^1 dx D_{12}^{-\epsilon/2} + 2 \text{ perms.} + O(\lambda^2)
\]

\[
\epsilon \to 0, -(1 + C_1 \lambda) + \frac{\lambda}{2(4\pi)^2} \left( \frac{2}{\epsilon} - \int_0^1 dx \ln(D_{12}/\mu^2) \right) + 2 \text{ perms.} + O(\lambda^3)
\]

\[
= -1 - \frac{\lambda}{2(4\pi)^2} \int_0^1 dx \ln(D_{12}/\mu^2) + 2 \text{ perms.} + O(\lambda^2).
\]

Note that I divided through by \( \bar{\mu}^\epsilon \) to make both sides dimensionless. Here \( D_{12} = x(1-x)(k_1 + k_2)^2 + m^2 \), and in the last line we canceled the pole in \( \epsilon \) (\( \overline{\text{MS}} \) scheme),

\[
C_1 = \frac{3}{16\pi^2 \epsilon}.
\]

Finally, for \( m^2 = 0 \) we can do the integral, with the result

\[
\frac{V_4}{\lambda} = -1 - \frac{3\lambda}{16\pi^2} - \frac{\lambda}{32\pi^2} \left( \ln \left( \frac{(k_1 + k_2)^2}{\mu^2} \right) + \ln \left( \frac{(k_1 + k_3)^2}{\mu^2} \right) + \ln \left( \frac{(k_1 + k_4)^2}{\mu^2} \right) \right) + O(\lambda^2).
\]

Note again the appearance of logarithms, and that the magnitude grows at large \( k \) as expected from the RG for this theory.

Note that the \( \lambda\phi^4 \) vertex is \( -i\lambda Z_\lambda \), the sign because the potential is defined to be positive for positive \( \lambda \).

3. Each propagator now has an index denoting the field. The vertex \( -i\lambda Z_\lambda \) is now multiplied by \( \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \), in terms of the indices of the attached propagators.
For the first graph below, the s-channel, the integral from problem 2, for external lines 
\((k_1, i; k_2, j; k_3, k; k_4, l)\) is now multiplied by
\[
(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{in} + \delta_{kn}\delta_{lm}) = (N + 4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk}.
\]
The indices \(m, n\) on the internal lines have been summed.

Summing the other two channels then gives
\[
\frac{V_{ijkl}}{\lambda} = -(1 + C_1\lambda)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\]
\[
+ \frac{\lambda}{2(4\pi)^2} \left\{ (N + 4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} \right\} \left( \frac{2}{\epsilon} - \int_0^1 dx \ln(D_{12}/\mu^2) \right) + 2 \text{ perms.} + O(\lambda^3)
\]
\[
= -(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda}{2(4\pi)^2} \left\{ (N + 4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} \right\} \int_0^1 dx \ln(D_{12}/\mu^2)
\]
\[
+ ((N + 4)\delta_{ik}\delta_{jl} + 2\delta_{ij}\delta_{kl} + 2\delta_{kl}\delta_{jk} \right\} \int_0^1 dx \ln(D_{13}/\mu^2)
\]
\[
+ ((N + 4)\delta_{il}\delta_{kj} + 2\delta_{ik}\delta_{jl} + 2\delta_{ij}\delta_{lk} \right\} \int_0^1 dx \ln(D_{14}/\mu^2)
\]
where we have used \(\overline{\text{MS}}\) to set
\[
C_1 = \frac{(N + 8)}{16\pi^2 \epsilon}.
\]
Note that the indices are permuted along with the momenta, e.g. \((k_2, j) \leftrightarrow (k_4, l)\) in the last line, and that the tensor structure doesn’t just add up to a multiple of \(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\).

A check: for \(N = 1\) and \(\lambda \to \lambda/3\) it reduces to the previous problem.