Homework 8 solutions

1. Srednicki 28.1. Following the steps in Srednicki, 28.13 to 28.29,

$$\begin{aligned} 0 &= \lambda\mu\partial_\mu|_{\lambda_0,m_0} \ln \lambda_0 \\ &= \lambda\mu\partial_\mu|_{\lambda_0,m_0} \ln(Z_\lambda Z_\phi^{-2}\tilde{\mu}^\epsilon\lambda) \\ &= \epsilon\lambda + (\mu\partial_\mu|_{\lambda_0,m_0}\lambda)\left(1+\lambda\partial_\lambda \ln(Z_\lambda Z_\phi^{-2}\right) \end{aligned}$$

Now, $\mu\partial_\mu|_{\lambda_0,m_0}\lambda = -\epsilon\lambda + \beta(\lambda)$, $Z_\phi = 1 + O(\lambda^2)$, and $Z_\lambda = 1 + 3\lambda/16\pi^2\epsilon + O(\lambda^2)$ (previous problem set), so we get to one-loop order

$$0 = \epsilon\lambda + (-\epsilon\lambda + \beta(\lambda))(1 + 3\lambda/16\pi^2\epsilon + \ldots)\,.$$

The terms of order ϵ cancel, and those of order ϵ^0 give

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$$

Similarly,

$$\gamma_m = \lambda\partial_\lambda M_1$$

where

$$\ln(Z_m^{1/2} Z_\phi^{-1/2}) = \frac{M_1(\lambda)}{\epsilon} + \ldots\,.$$

An easy one-loop calculation gives $Z_m = 1 + \lambda/16\pi^2\epsilon$, and so

$$\gamma_m = \frac{\lambda}{32\pi^2}\,.$$

2. a)

$$S[\phi] = -\int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{\lambda}{24}\phi^4(x)\right)\,.$$

$$\begin{aligned} S[\phi'] &= -\int d^4x \left(s^{2a}\frac{1}{2}\partial_\mu\phi(x')\partial^\mu\phi(x') + s^{4a}\frac{\lambda}{24}\phi^4(x')\right)\,, \quad x' = sx \\ &= -\int d^4x \left(s^{2a+2}\frac{1}{2}\partial'_\mu\phi(x')\partial'^\mu\phi(x') + s^{4a}\frac{\lambda}{24}\phi^4(x')\right) \\ &\overset{\text{if } a=1}{=} -\int d^4x' \left(\frac{1}{2}\partial'_\mu\phi(x')\partial'^\mu\phi(x') + \frac{\lambda}{24}\phi^4(x')\right) \\ &= S[\phi]\,. \end{aligned} \tag{1}$$

b) For $s = 1 + \epsilon$, $\delta\phi = \epsilon(\phi + x^\mu \partial_\mu \phi)$. One finds that $\delta\mathcal{L} = \epsilon\partial_\mu(x^\mu \mathcal{L})$, so 22.27 becomes (Mark seems to leave out some ϵ's)

$$\begin{aligned} j^\mu &= -(\phi + x^\nu \partial_\nu \phi)\partial^\mu \phi - x^\mu \mathcal{L} \\ &= -\phi\partial^\mu\phi - x^\nu\partial_\nu\phi\partial^\mu\phi + x^\mu\left(\frac{1}{2}\partial_\nu\phi\partial^\nu\phi + \frac{\lambda}{24}\phi^4\right) . \end{aligned} \tag{2}$$

c)

$$\partial_\mu(T^{\mu\nu}x_\nu) = (\partial_\mu T^{\mu\nu})x_\nu + T^{\mu\nu}g_{\mu\nu} = T^\mu{}_\mu \,.$$

d) 22.31 gives

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi + \frac{\lambda}{24}\phi^4\right) .$$

Then

$$T^\mu{}_\mu = -\partial_\sigma\phi\partial^\sigma\phi - \frac{\lambda}{6}\phi^4 \neq 0\,.$$

e) Straightforward.

f)

$$I^\mu{}_\mu = -3\partial^2(\phi^2) = -6\partial_\sigma\phi\partial^\sigma\phi - 6\phi\partial^2\phi = -6\partial_\sigma\phi\partial^\sigma\phi - \lambda\phi^4\,,$$

where the last line uses the equation of motion. So $c = -1/6$ cancels the trace.

$$\begin{aligned} T'^{\mu\nu}x_\nu &= x_\nu\partial^\mu\phi\partial^\nu\phi - x^\mu\left(\frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi + \frac{\lambda}{24}\phi^4\right) - \frac{1}{6}(x_\nu\partial^\mu\partial^\nu - x^\mu\partial^2)\phi^2 \\ &= \frac{2}{3}x_\nu\partial^\mu\phi\partial^\nu\phi - \frac{1}{3}x_\nu\phi\partial^\mu\partial^\nu\phi - \frac{1}{6}x^\mu\partial_\sigma\phi\partial^\sigma\phi - \frac{\lambda}{24}x^\mu\phi^4 + \frac{1}{3}x^\mu\phi\partial^2\phi \\ &= \frac{2}{3}x_\nu\partial^\mu\phi\partial^\nu\phi - \frac{1}{3}x_\nu\phi\partial^\mu\partial^\nu\phi - \frac{1}{6}x^\mu\partial_\sigma\phi\partial^\sigma\phi + \frac{\lambda}{72}x^\mu\phi^4\,. \end{aligned} \tag{3}$$

This is different from your answer to (b), I forgot that there was one more step. Consider

$$\begin{aligned} I^\mu &= \partial_\nu(x^\nu\phi\partial^\mu\phi - x^\mu\phi\partial^\nu\phi) = 3\phi\partial^\nu\phi + x^\nu\partial_\nu\phi\partial^\mu\phi - x^\mu\partial_\nu\phi\partial^\nu\phi + x^\nu\phi\partial_\nu\partial^\mu\phi - \phi\partial^2\phi \\ &= 3\phi\partial^\mu\phi + x^\nu\partial_\nu\phi\partial^\mu\phi - x^\mu\partial_\nu\phi\partial^\nu\phi + x^\nu\phi\partial_\nu\partial^\mu\phi - \frac{\lambda}{6}\phi^4\,. \end{aligned} \tag{4}$$

Like $I^{\mu\nu}$ this is trivially conserved and can be used to redefine the Noether current. Then j^μ from part (b) plus $\frac{1}{3}I^\mu$ is equal to $-T'^{\mu\nu}x_\nu$. So there is also a sign flip - sorry about the difference of conventions. Whew!

g)

$$I_{00} = -\partial_i\partial_i(\phi^2)\,, \quad I_{0i} = -\partial_i\partial_0(\phi^2)\,,$$

so both are total spatial derivatives.

h)

$$\partial_\mu (T^{\mu\nu} v_\nu) = (\partial_\mu T^{\mu\nu}) v_\nu + T^{\mu\nu} \partial_\mu v_\nu \,.$$

The first term vanishes by conservation of T. The second vanishes if

$$\partial_\mu v_\nu + \partial_\nu v_\mu \propto g_{\mu\nu} \,.$$

The point is that the antisymmetric part of $\partial_\mu v_\nu$ drops out because T is symmetric, and then the second term in the conservation law vanishes because T is traceless. By taking the traces of both sides, one can make the more precise statement that

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{1}{2} g_{\mu\nu} \partial_\sigma v^\sigma \,.$$

Extra: For small deviations $h_{\mu\nu}$ from a flat metric, the extra coupling is

$$\phi^2 (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) h^{\mu\nu} \,.$$

This is the linearized form of $R\phi^2$ (up to normalization), where R is the curvature scalar. The point is that when we couple a flat spacetime field theory to gravity, besides replacing the flat metric in the action with the curved metric, there may also be such 'non-minimal' couplings to the curvature, which we don't see in the flat space theory.

3. a)

$$\partial_{\ln\mu} \rho = \frac{\partial_{\ln\mu} \lambda}{g^2} - 2 \frac{\lambda \partial_{\ln\mu} g}{g^3} = \frac{1}{16\pi^2} \left(\frac{3\lambda^2}{g^2} - 2\lambda - 48 g^2 \right) = \frac{g^2}{16\pi^2} (3\rho^2 - 2\rho - 48) \,.$$

Or we can write this at

$$\frac{d\rho}{d\ln g} = \frac{1}{5}(3\rho^2 - 2\rho - 48) = \frac{3}{5}(\rho - \rho_+^*)(\rho - \rho_-^*)$$

where

$$\rho_\pm^* = \frac{1}{3} \pm \frac{1}{3}\sqrt{145} \approx 4.35, -3.68 \,.$$

This separates the flow into two one-dimensional equations, which give $\rho(g)$ and $g(\mu)$.

b) See above.

c,d,e,g) For $\rho > \rho_+$, ρ increases toward the UV and decreases toward the IR. A flow that starts here, e.g. at $\rho = 5$, will flow to strong coupling in the UV and to ρ_+^*. For $\rho_+^* > \rho > \rho_-^*$, the flow goes to ρ_-^* in the UV and ρ_+^* in the IR; the initial value $\rho = 0$ is in this range. For $\rho_-^* > \rho$, ρ flows toward ρ_-^* in the UV and strong coupling in the IR. So ρ_-^* is a UV fixed point and ρ_+^* in an IR fixed point.

f) Writing

$$\frac{d\rho}{d\ln g} = \frac{3}{5}(\rho - \rho_+^*)(\rho - \rho_-^*)\,,$$

you can integrate the flow: solve for g first and then ρ. You find that

$$\nu^{-1} = \frac{3}{5}(\rho_+^* - \rho_-^*) = \frac{2}{5}\sqrt{145}\,.$$

4. a) The flow of λ_1 is given by the first set of graphs, where solid lines are ϕ and dashed lines are χ. The three on the left are just like $\lambda\phi^4$ with one field (as in problem 1),

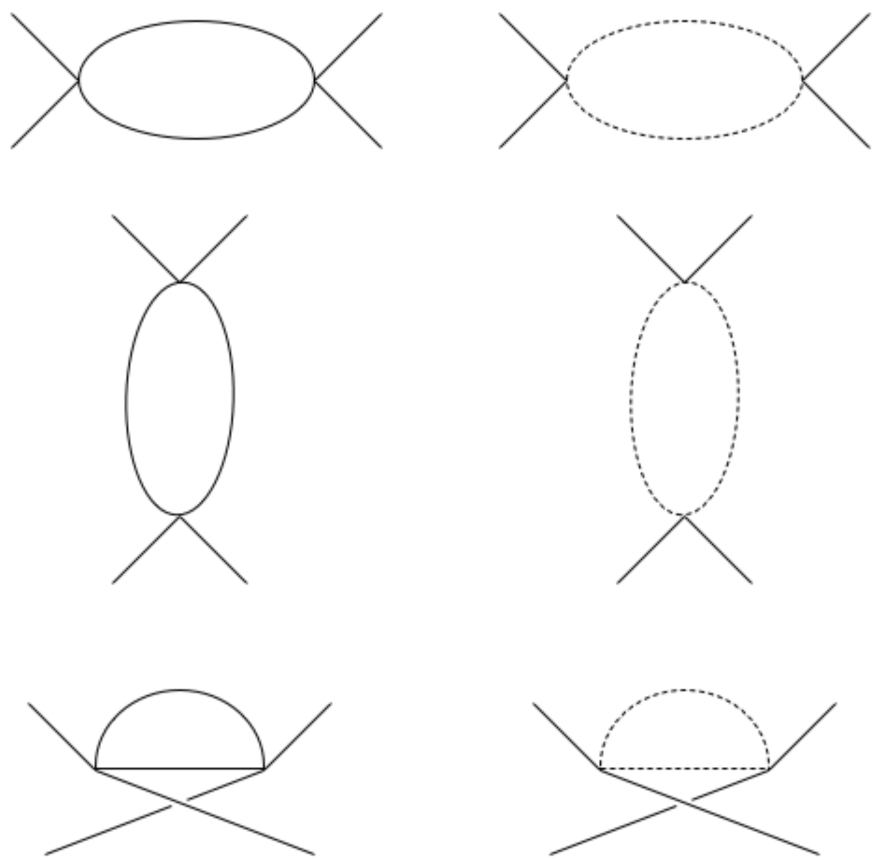

Figure 1: Renormalization of λ_1

and the three on the right are the same with λ_2^2 in place of λ_1^2. So

$$\beta_1 = \frac{3(\lambda_1^2 + \lambda_2^2)}{16\pi^2}\,.$$

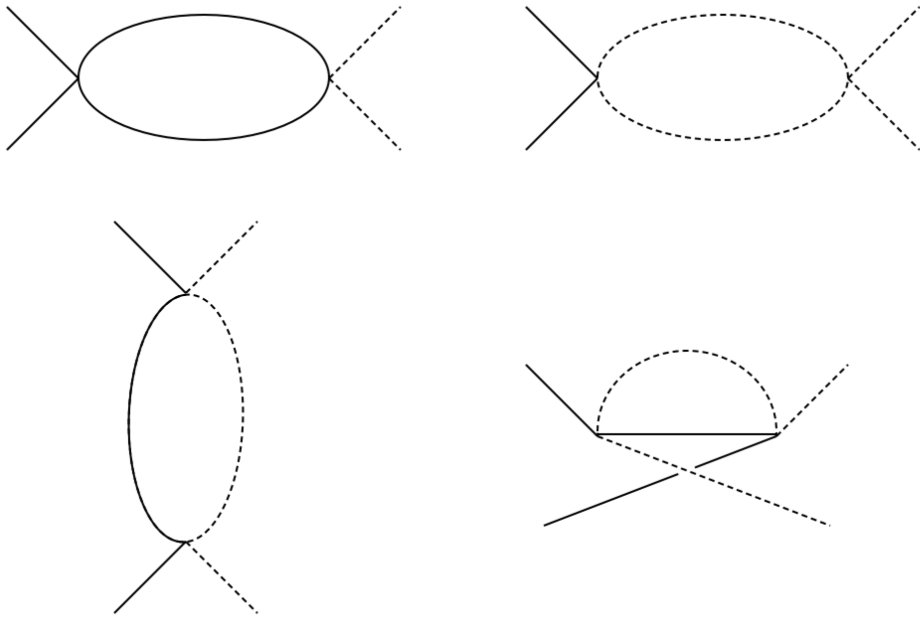

Figure 2: Renormalization of λ_2

The flow of λ_2 is given by the second set of graphs, Noting that the two on the second line have symmetry factor 1 instead of 1/2, we get

$$\beta_2 = \frac{\lambda_1\lambda_2 + 2\lambda_2^2}{8\pi^2}\,.$$

In case that went by too fast, let me give a longer derivation.

$$\begin{aligned} 0 &= \mu\partial_\mu|_{\lambda_0} \ln\lambda_{i0} \\ &= -\epsilon + (\mu\partial_\mu|_{\lambda_0}\lambda_j)\partial_j \ln(Z_{\lambda_i}\lambda_i) \\ &= -\epsilon + \hat\beta_i/\lambda_i + \hat\beta_j\partial_j \ln(Z_{\lambda_i}) \\ &\approx -\epsilon + \hat\beta_i/\lambda_i + \hat\beta_j\partial_j C_{i1}/\epsilon\,. \end{aligned} \tag{5}$$

(I've used $Z_\phi = Z_\chi = 1$.) The ϵ^0 terms come from the β_i piece of the second term and the $-\epsilon\lambda^j\partial_j$ piece of the second term, just as for $\lambda\phi^4$. We find

$$C_{11} = \frac{3(\lambda_1^2+\lambda_2^2)}{16\pi^2\lambda_1}\,,\quad C_{21} = \frac{\lambda_1 + 2\lambda_2}{8\pi^2}\,.$$

Noting that $\lambda^j\partial_j C_{i1} = C_{i1}$, we get $\beta_i = \lambda_i C_{i1}$.

A simple check is that if λ_2 starts out 0 we have two decoupled theories and it stays zero. A fancy check is that if $\lambda_1 = 3\lambda_2$ then there is an $O(2)$ symmetry mixing ϕ and χ. We find then that $\beta_1 = 3\beta_2$, so this symmetry is preserved by the flow.

Now plot the vector field

$$-\left(\frac{3\lambda_1^2+3\lambda_2^2}{16\pi^2},\frac{\lambda_1\lambda_2+2\lambda_2^2}{8\pi^2}\right).$$

The minus sign is because the arrows go toward the IR.

b) Now plot

$$\left(\epsilon\lambda_1-\frac{3\lambda_1^2+3\lambda_2^2}{16\pi^2},\epsilon\lambda_2-\frac{\lambda_1\lambda_2+2\lambda_2^2}{8\pi^2}\right).$$

The condition $\hat{\beta}_2 = 0$ factors into $\lambda_2 = 0$ or $\lambda_2 = 4\pi^2\epsilon - \lambda_1/2$. Then solve $\hat{\beta}_1 = 0$. There are four fixed points:

1. $(0,0)$, free UV fixed point

2. $(16\pi^2\epsilon/3, 0)$, decoupled Wilson-Fisher IR fixed points. In the 2-d flow this is a saddle point, because a λ_2 perturbation will grow.

3. $(24\pi^2\epsilon/5, 8\pi^2\epsilon/5)$, $O(2)$ symmetric Wilson-Fisher IR fixed point

4. $(8\pi^2\epsilon/3, 8\pi^2\epsilon/3)$, another saddle point. In fact this is again two decoupled WF theories,

$$V \propto (\phi+\chi)^4 + (\phi-\chi)^4.$$

I couldn't get a nice plot out of my old version of Mathematica, so the figure shows the fixed points for nonzero ϵ and the flow near them, you can piece the rest together by interpolating. (For $\epsilon = 0$ the flow is basically just toward the origin.) Some people continued the plot to negative couplings, but unless $\lambda_1 > 0$ and $\lambda_1 + 3\lambda_2 > 0$ the potential is unbounded below and there is no vacuum. A large region of parameter space ($\lambda_1 > \lambda_2 > 0$, to be precise) ends up at the $O(2)$ symmetric WF point. This means that there is an emergent scale invariance and an emergent $O(2)$ symmetry in the IR.

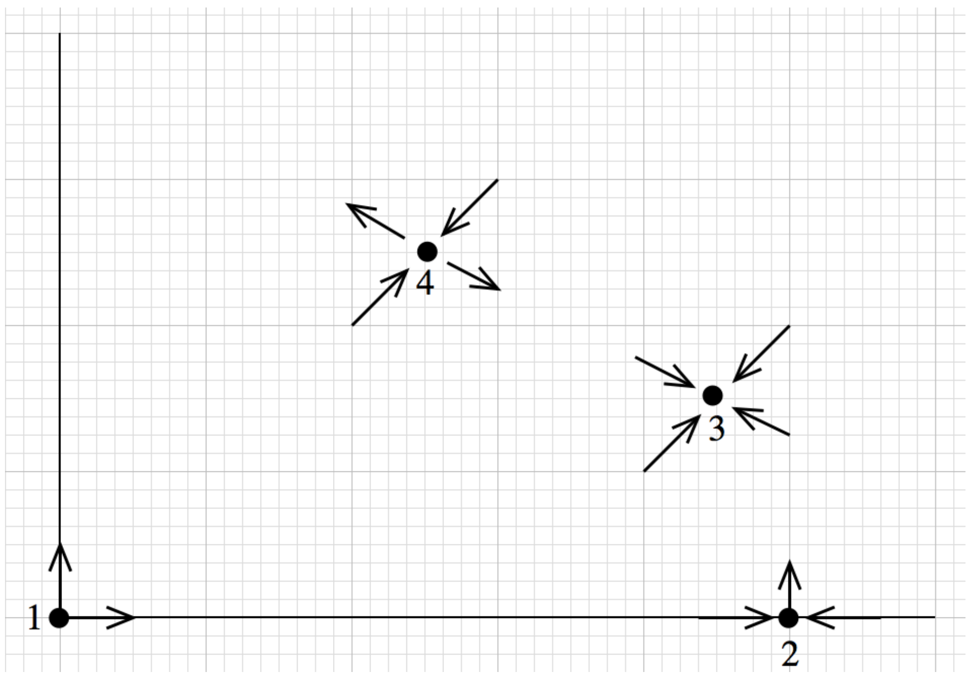

Figure 3: Flow in (λ_1, λ_2) plane, arrows toward the IR! 1. Free fixed point. 2. Decoupled WF saddle. 3. $O(N)$ WF IR fixed point. 4. Decoupled WF saddle.