**SOLUTION SET 5 215B-QUANTUM MECHANICS, WINTER 2019**

**Problem 1. PROBLEMS FROM SAKURAI**

**4.1) Calculate the three lowest energy levels, together with their degeneracies for the following systems (assuming equal mass distinguishable particles)**

**(a) Three non-interacting spin $1/2$ particles in a one-dimensional box of length L**

The Hamiltonian for three non-interacting particles of mass $m$ in a one-dimensional box of length $L$ is

$$H = \frac{1}{2m}\sum_{i=1}^{3} p_i^2 \tag{1}$$

Consequently the state completely factorizes and we can independently designate the spin and position eigenstates of each particle. The energy levels are independent of spin and given by

$$E_{\vec{n}} = \frac{\pi^2\hbar^2}{2mL^2}\sum_{i=1} 3n_i^2 \tag{2}$$

The ground state has energy

$$E_{(1,1,1)} = 3\frac{\pi^2\hbar^2}{2mL^2}\,, \tag{3}$$

with no degeneracy in the position wave-function, but a 2-fold degeneracy in equal energy spin states for each of the three particles. Thus the ground state degeneracy is 8.

The first excited states have energy

$$E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = 3\frac{\pi^2\hbar^2}{mL^2}\,, \tag{4}$$

with a 3-fold degeneracy in position wavefunctions and 8-fold degeneracy in spin giving a total degeneracy of 24.

The second exited states have energy

$$E_{(2,2,1)} = E_{(1,2,2)} = E_{(2,1,2)} = 9\frac{\pi^2\hbar^2}{2mL^2}\,, \tag{5}$$

with a 3-fold degeneracy in position wavefunctions and 8-fold degeneracy in spin giving a total degeneracy of 24.

**(b) Four non-interacting spin $1/2$ particles in a one-dimensional box of length L**

Four particles is exactly analogous. The ground state has energy

$$E_0 = 2\frac{\pi^2\hbar^2}{mL^2}\,, \tag{6}$$

with no degeneracy in the position wave-function, but a 2-fold degeneracy in equal energy spin states for each of the four particles. Thus the ground state degeneracy is 16.

The first excited states have energy

$$E_1 = 7\frac{\pi^2\hbar^2}{2mL^2}\,, \tag{7}$$

with a $\binom{4}{1}$-fold degeneracy in position wavefunctions and 16-fold degeneracy in spin giving a total degeneracy of 64.

The second excited states have energy

$$E_2 = 5\frac{\pi^2\hbar^2}{mL^2}\,, \tag{8}$$

with a $\binom{4}{2}$-fold degeneracy in position wavefunctions and 16-fold degeneracy in spin giving a total degeneracy of 96.

**4.2) Which of the following commute?**

**(a)** $\mathcal{T}_d$ and $\mathcal{T}_{d'}$

They commute because the generators of translation commute: $[P_i, P_j] = 0$.

**(b)** $\mathcal{D}(\hat{n}, \phi)$ and $\mathcal{D}(\hat{n}', \phi')$

They do not commute in general because the generators of rotation do not all commute: $[J_i, J_k] \neq 0$

**(c)** $\mathcal{T}_d$ and $\pi$

They do not commute because parity does not commute with the generator of translations: $[\pi, P_i] \neq 0$ (but $\{\pi, \mathbf{P}_i\} = 0$).

**(d)** $\mathcal{D}(\hat{n}, \phi)$ and $\pi$

They commute because the parity operator commutes with the generators of rotations: $[\mathbf{J}_i, \pi] = 0$.

**4.3)**

For $\Psi$ an eigenstate of anticommuting operators $A, B$ with respective eigenvalues $a, b$, we have that

$$\{A, B\} = 0 \Rightarrow (AB + BA)|\Psi\rangle = (2ab)|\Psi\rangle = 0\,. \tag{9}$$

Thus, at least one of the eigenvalues $a, b$ must be zero.

As an example, consider an eigenstate of the parity, $\pi$, an momentum operators, $\mathbf{P}$, with eigenvalues $\pi$ and $\mathbf{p}$ respectively. Then one of the two must be zero. However, in this case we can go further: because $\pi$ is an invertible operators, it can not have a non-trivial nullspace. Thus, we must have that $\mathbf{p} = 0$.

**4.7)**

**(a)** A plane wave in three dimensions has the wavefunction

$$\psi_{\mathbf{p}}(x, t) = \frac{1}{(2\pi)^{3/2}} \exp\{(i\mathbf{p} \cdot \mathbf{x} - i\omega t)\}\,. \tag{10}$$

Thus we have that

$$\psi^*_{\mathbf{p}}(x, -t) = \frac{1}{(2\pi)^{3/2}} \exp\{(-i\mathbf{p} \cdot \mathbf{x} - i\omega t)\} = \psi_{-\mathbf{p}}(x, t)\,. \tag{11}$$

**(b)** Recall (or consult Sakurai to find) that the two component eigenspinor of $\sigma \cdot \hat{\mathbf{n}}$ with eigenvalue $+1$ can be written explicitly as

$$\chi_+(\hat{\mathbf{n}}) = \begin{pmatrix} \cos(\beta/2)e^{-i\gamma/2} \\ \sin(\beta/2)e^{+i\gamma/2} \end{pmatrix}\,. \tag{12}$$

Then we have that

$$-i\sigma_2\chi^*_+(\hat{\mathbf{n}}) = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\beta/2)e^{+i\gamma/2} \\ \sin(\beta/2)e^{-i\gamma/2} \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} -\sin(\beta/2)e^{-i\gamma/2} \\ \cos(\beta/2)e^{+i\gamma/2} \end{pmatrix} \tag{14}$$

Now we could explicitly check that this is the eigenspinor with the spin direction reversed. However, note that the two eigenspinors with reversed spin directions are orthogonal and span the Hilbert space. Thus it suffices to check that this new spinor is orthogonal to $\chi_+$:

$$\begin{aligned} -i\sigma_2\chi^*_+(\hat{\mathbf{n}}) \cdot \chi^*_+(\hat{\mathbf{n}}) &= \begin{pmatrix} -\sin(\beta/2)e^{-i\gamma/2} \\ \cos(\beta/2)e^{+i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2)e^{+i\gamma/2} \\ \sin(\beta/2)e^{-i\gamma/2} \end{pmatrix} \\ &= -\sin(\beta/2)\cos(\beta/2) + \sin(\beta/2)\cos(\beta/2) \\ &= 0\,. \end{aligned} \tag{15}$$

**4.8)**

**(a)** The question as written is false. The statement is in fact only true for an energy eigenfunction. This is proved in Sakurai already as Theorem 4.2.

**(b)** Plane wave eigenfunctions of the free particle Hamiltonian are not a violation of part (a) because the energy spectrum is degenerate (there are many states with the same energy $\mathbf{p^2}/2m$).

**4.9)**

Let

$$|\alpha\rangle = \int d^3p\phi(p)|p\rangle\,. \tag{16}$$

Then we have that

$$\Theta|\alpha\rangle = \int d^3p\phi^*(p)\Theta|p\rangle \tag{17}$$

and thus that

$$\Theta|\alpha\rangle = \int d^3p\,\phi^*(p)|-p\rangle = \int d^3p\,\phi^*(-p)|p\rangle\,. \tag{18}$$

We conclude $\Theta\phi(p) = \phi^*(-p)$.

**4.11)**

First recall that a non-degenerate spectrum implies that an energy eigenstate $|E\rangle$ is invariant under time reversal up to a phase:

$$|\tilde{E}\rangle = \Theta|E\rangle = e^{i\delta}|E\rangle \tag{19}$$

and that angular momentum is odd under time-reversal:

$$\Theta\mathbf{L}\Theta^{-1} = -\mathbf{L}\,. \tag{20}$$

From these two facts, we find (cf. (4.4.44)

$$\langle E|\mathbf{L}|E\rangle = -\langle\tilde{E}|\mathbf{L}|\tilde{E}\rangle = -\langle E|e^{-i\delta}\mathbf{L}e^{-i\delta}|E\rangle = -\langle E|\mathbf{L}|E\rangle\,. \tag{21}$$

This can only be true if

$$\langle E|\mathbf{L}|E\rangle = 0\,. \tag{22}$$

Recall from problem 4.8 that, because we have a nondegenerate spectrum, $\psi_E(x)e^{i\delta} = \psi^*(x)$. If we expand the wavefunction as

$$\psi(x) = \sum_l\sum_m F_{l,m}(r)Y_l^m(\theta,\phi) \tag{23}$$

then this equation becomes

$$e^{i\delta}\sum_l\sum_m F_{l,m}(r)Y_l^m(\theta,\phi) = \sum_l\sum_m F^*_{l,m}(r)Y_l^{m*}*(\theta,\phi) = \sum_l\sum_m F^*_{l,m}(r)(-1)^mY_l^{-m}(\theta,\phi) \tag{24}$$

We can then project onto one mode

$$e^{i\delta}\int\sum_l\sum_m F_{l,m}(r)Y_l^m(\theta,\phi)Y_L^M(\theta,\phi)d\Omega = \int\sum_l\sum_m F^*_{l,m}(r)(-1)^mY_l^{-m}(\theta,\phi)Y_L^M(\theta,\phi)d\Omega$$
$$\Rightarrow e^{i\delta}F_{L,M}(r) = (-1)^MF^*_{L,-M}(r) \tag{25}$$

**4.12)**

The Hamiltonian

$$H = AS_z^2 + B(S_x^2 - S_y^2) \tag{26}$$

can be written explicitly as

$$H = \hbar^2\begin{pmatrix} A & 0 & B\\ 0 & 0 & 0\\ B & 0 & A\end{pmatrix}\,. \tag{27}$$

This matrix has eigenvectors

$$|1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}\,,\ |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\,,\ |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\,, \tag{28}$$

with respective eigenvalues $0, \hbar^2(A+B), \hbar^2(A-B)$.

Now $\Theta S_i \Theta^{-1} = -S_i$ implies $\Theta S_i^2 \Theta^{-1} = S_i^2$ so $\Theta H \Theta^{-1} = H$ and the Hamiltonian is invariant.

We can rewrite the eigenstates in terms of spin eigentstates

$$\begin{aligned}
|1\rangle &= |1,0\rangle \\
|2\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \\
|3\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) \,.
\end{aligned} \tag{29}$$

Then recall the convention in (4.4.77)

$$\Theta|l,m\rangle = (-1)^m |l,-m\rangle \tag{30}$$

so that we find

$$\begin{aligned}
\Theta|1\rangle &= |1,0\rangle = |1\rangle \\
\Theta|2\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) = \frac{1}{\sqrt{2}}(-|1,-1\rangle - |1,1\rangle) = -|2\rangle \\
\Theta|3\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) = \frac{1}{\sqrt{2}}(-|1,-1\rangle + |1,1\rangle) = |3\rangle \,.
\end{aligned} \tag{31}$$

**Problem 2. SHOR'S ERROR CORRECTING CODE**

**(a.1)** Noticing that

$$\mathcal{O}_i\mathcal{O}'_j = \mathcal{O}'_j\mathcal{O}_i, \text{ where } i \neq j, \text{ and } \mathcal{O}, \mathcal{O}' = X, Z; \tag{32}$$

$$Z_iX_i = -X_iZ_i, \tag{33}$$

we can check that $g_ig_j = g_jg_i$ for any pair of $i, j$, by verifying that the number of sites where $g_i$ has $Z$ ($X$) and $g_j$ has $X$ ($Z$) is even.

Noticing that

$$Z_1Z_2\left(|000\rangle \pm |111\rangle\right) = Z_2Z_3\left(|000\rangle \pm |111\rangle\right) = |000\rangle \pm |111\rangle\,, \tag{34}$$

$$X_1X_2X_3\left(|000\rangle \pm |111\rangle\right) = \pm\left(|000\rangle \pm |111\rangle\right), \tag{35}$$

we can see that $g_i\,|\overline{z}\rangle = |\overline{z}\rangle$ for $z = 0, 1$.

**(a.2)** The independence is somewhat obvious. One can explicitly show that by computing the rank of some corresponding matrix on $\mathbb{Z}_2$.

We can use the same observation in **(a.1)** to show that the operators commute.

Suppose $|\psi\rangle$ is a codeword, that is, $g_i\,|\psi\rangle = |\psi\rangle\,\forall i$. We see that

$$g_i\overline{Z}\,|\psi\rangle = \overline{Z}g_i\,|\psi\rangle = \overline{Z}\,|\psi\rangle\,, \tag{36}$$

$$g_i\overline{X}\,|\psi\rangle = \overline{X}g_i\,|\psi\rangle = \overline{X}\,|\psi\rangle\,. \tag{37}$$

Thus, $\overline{Z}\,|\psi\rangle$ and $\overline{X}\,|\psi\rangle$ are still codewords.

**(a.3)** Using the observation in **(a.1)** (that $X$ and $Z$ on the same site anticommute), we can show that $\overline{ZX} = -\overline{XZ}$ (**notice** the typo in the problem).

We can also show that

$$\overline{Z}\,|\overline{0}\rangle = |\overline{0}\rangle, \overline{Z}\,|\overline{1}\rangle = -\,|\overline{1}\rangle\,, \tag{38}$$

$$\overline{X}\,|\overline{0}\rangle = |\overline{1}\rangle, \overline{X}\,|\overline{1}\rangle = |\overline{0}\rangle\,. \tag{39}$$

Thus the matrix representation.

**(a.4)** Since the logical operators commute with the stabilizers, we can assume, without loss of generality,

$$\overline{Z}' = \overline{Z}g_1^{p_1}g_2^{p_2}\ldots g_8^{p_8}, \tag{40}$$

$$\overline{X}' = \overline{X}g_1^{q_1}g_2^{q_2}\ldots g_8^{q_8}, \tag{41}$$

where $p_i, q_i \in \{0, 1\}$. We still have

$$\overline{Z}'\,|\overline{0}\rangle = |\overline{0}\rangle, \overline{Z}'\,|\overline{1}\rangle = -\,|\overline{1}\rangle\,, \tag{42}$$

$$\overline{X}'\,|\overline{0}\rangle = |\overline{1}\rangle, \overline{X}'\,|\overline{1}\rangle = |\overline{0}\rangle\,, \tag{43}$$

because $g_i\,|\overline{z}\rangle = |\overline{z}\rangle$. Thus, $\overline{Z}'$ and $\overline{X}'$ are also good logical operators.

**(b.1)** Noticing that

$$g_1X_1 = -X_1g_1, g_{i>1}X_1 = X_1g_{i>1}, \tag{44}$$

we have

$$g_1X_1\,|\psi\rangle = -X_1g_1\,|\psi\rangle = -X_1\,|\psi\rangle\,, \tag{45}$$

$$g_{i>1}X_1\,|\psi\rangle = X_1g_{i>1}\,|\psi\rangle = X_1\,|\psi\rangle\,. \tag{46}$$

Measuring the stabilizers won't change the wavefunction since it is an eigenstate; we have the error syndrome $(-, +, +, +, +, +, +, +)$.

We correct the error by applying $X_1$ on $|\psi_{X_1}\rangle$.

**(b.2)** Noticing that

$$g_7Z_1 = -Z_1g_7, g_{i\neq7}Z_1 = Z_1g_{i\neq7}, \tag{47}$$

we have the error syndrome $(+,+,+,+,+,+,-,+)$.

We correct the error by applying $Z_1$ on $|\psi_{Z_1}\rangle$.

**(b.3)** Noticing that

$$g_1 Z_1 X_1 = -Z_1 X_1 g_1, g_7 Z_1 X_1 = -Z_1 X_1 g_7, g_{i\neq 1,7} Z_1 X_1 = Z_1 X_1 g_{i\neq 1,7}, \tag{48}$$

we have the error syndrome $(-,+,+,+,+,+,-,+)$.

We correct the error by applying $X_1 Z_1$ on $|\psi_{Y_1}\rangle$.

**(b.4)** We see that $|\psi_E\rangle$ is a superposition of $|\psi\rangle, |\psi_{X_1}\rangle, |\psi_{Z_1}\rangle, |\psi_{Y_1}\rangle$, which are eigenstates of the stabilizers with different eigenvalues (i.e. different error syndrome). They are orthogonal to each other.

Recall that measuring some hermitian operator always results in an eigenstate of that operator; in our case, measuring the stabilizers will result in simultaneous eigenstates of the stabilizers. This should be enough information to figure out the possible error syndromes.

We spell out the process of the measurements. Suppose we measure the stabilizers in the order $g_1, \ldots, g_8$. For $g_1$, we have probability $p^1_\pm = \langle\psi_E| \frac{1\pm g_1}{2} |\psi_E\rangle$ of getting the measurement outcome $\pm 1$. Now

$$\frac{1+g_1}{2}|\psi_E\rangle = e_0|\psi\rangle + e_1|\psi_{Z_1}\rangle, \tag{49}$$

$$\frac{1-g_1}{2}|\psi_E\rangle = e_2|\psi_{X_1}\rangle + e_3|\psi_{Y_1}\rangle, \tag{50}$$

so $p^1_+ = |e_0|^2 + |e_1|^2$, and $p^1_- = |e_2|^2 + |e_3|^2$. The corresponding post-measurement wavefunctions, normalized, are $|\psi_+\rangle = \frac{1}{\sqrt{|e_0|^2+|e_1|^2}}(e_0|\psi\rangle + e_1|\psi_{Z_1}\rangle)$ and $|\psi_-\rangle = \frac{1}{\sqrt{|e_2|^2+|e_3|^2}}(e_2|\psi_{X_1}\rangle + e_3|\psi_{Y_1}\rangle)$.

One can easily verify that measuring the stabilizers $g_2$ through $g_6$ always gives result $+1$, and does not change the wavefunction. For $g_7$, we have probability $p^7_{\pm|g_1=\pm} = \langle\psi_\pm| \frac{1\pm g_7}{2} |\psi_\pm\rangle$ of getting the measurement outcome $\pm 1$. Now

$$\frac{1+g_7}{2}|\psi_+\rangle = \frac{e_0}{\sqrt{|e_0|^2+|e_1|^2}}|\psi\rangle, \tag{51}$$

$$\frac{1-g_7}{2}|\psi_+\rangle = \frac{e_1}{\sqrt{|e_0|^2+|e_1|^2}}|\psi_{Z_1}\rangle, \tag{52}$$

$$\frac{1+g_7}{2}|\psi_-\rangle = \frac{e_2}{\sqrt{|e_2|^2+|e_3|^2}}|\psi_{X_1}\rangle, \tag{53}$$

$$\frac{1-g_7}{2}|\psi_-\rangle = \frac{e_3}{\sqrt{|e_2|^2+|e_3|^2}}|\psi_{Y_1}\rangle, \tag{54}$$

$$\tag{55}$$

so $p^7_{+|+} = \frac{|e_0|^2}{|e_0|^2+|e_1|^2}$, $p^7_{-|+} = \frac{|e_1|^2}{|e_0|^2+|e_1|^2}$, $p^7_{+|-} = \frac{|e_2|^2}{|e_2|^2+|e_3|^2}$, and $p^7_{-|-} = \frac{|e_3|^2}{|e_2|^2+|e_3|^2}$. The corresponding post-measurement wavefunctions, normalized, are $|\psi\rangle$, $|\psi_{Z_1}\rangle$, $|\psi_{X_1}\rangle$, and $|\psi_{Y_1}\rangle$.

Similarly, measuring $g_8$ always gives $+1$, and does not affect the wavefunction.

You should be convinced that the order in which we choose to measure the stabilizers is not essential.

The possible error syndromes, their corresponding probabilies, and the error correction unitaries are summarized in the table below.

| Error syndrome | Probability | Resulting state | Correcting unitary |
|---|---|---|---|
| $(++++++++)$ | $\|e_0\|^2 = p^1_+ \times p^7_{+\|+}$ | $\|\psi\rangle$ | Identity |
| $(-+++++++)$ | $\|e_1\|^2 = p^1_+ \times p^7_{-\|+}$ | $\|\psi_{X_1}\rangle$ | $X_1$ |
| $(++++++-+)$ | $\|e_2\|^2 = p^1_- \times p^7_{+\|-}$ | $\|\psi_{Z_1}\rangle$ | $Z_1$ |
| $(-+++++-+)$ | $\|e_3\|^2 = p^1_- \times p^7_{-\|-}$ | $\|\psi_{Y_1}\rangle$ | $X_1 Z_1$ |

Therefore, we can correct an arbitrary single qubit error.

**(b.5)** We will have $g_2 = -1$, but $g_i = 1$ for $i \neq 2$. Therefore the state is not a codeword, and an error has happened – one can detect the error.

However, this error syndrome is the same as the $X_3$ error; instead of flipping $X_1 X_2$, we will try to flip the third physical qubit by $X_3$, effectively applying a logical-$Z$ operator on the state. We have failed correcting this error.

**(b.6)** Using the trick in **(a.4)**, we can have $\overline{X}' = \overline{X} g_2 g_4 g_6 = Z_1 Z_4 Z_7$; this is the minimal support we have. Thus $d = 3$.