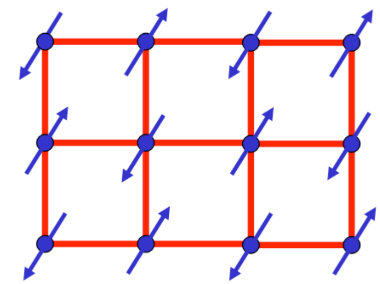
Spin Liquids with Topological Order

2d Mott Insulators with one electron per unit cell

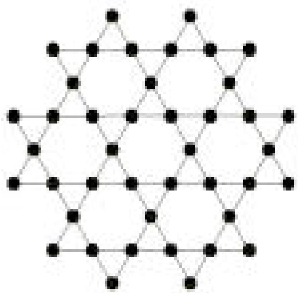
Quantum s=1/2 magnets



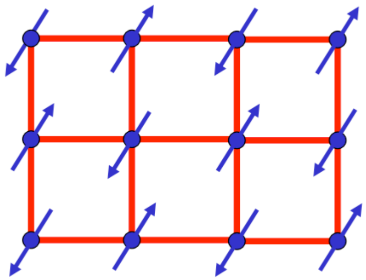
Will show that topologically ordered spin liquids have:

- an emergent “gauge structure”
- Quantum number fractionalization
- Ground state degeneracy on a torus

Focus on Spin liquids with:
- Z_2 Topological order
- Fully gapped with bosonic “spinons” and “visons”
- 2d square lattice and Kagome lattice



Resonating Valence Bond "Picture"

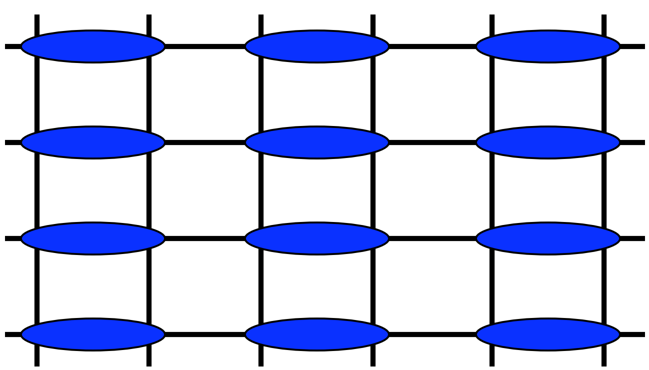


2d square lattice s=1/2 AFM

$$H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j + \ldots$$

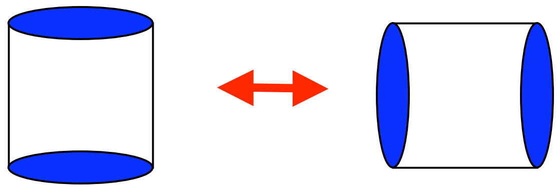
 $= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

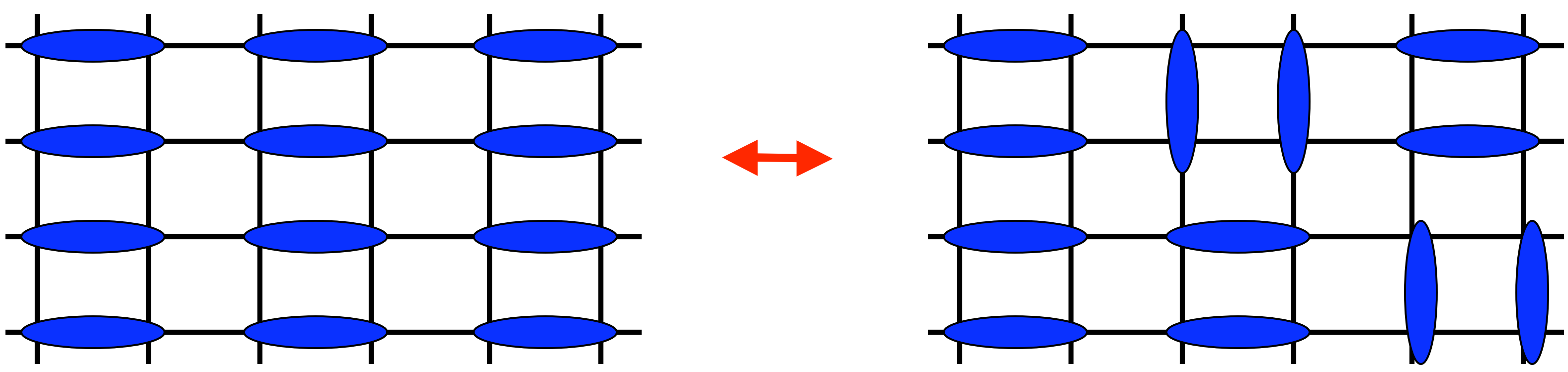
Singlet or a Valence Bond - Gains exchange energy J



Valence Bond Solid

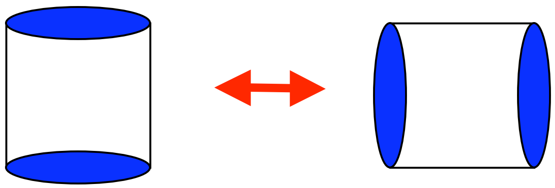
Plaquette Resonance

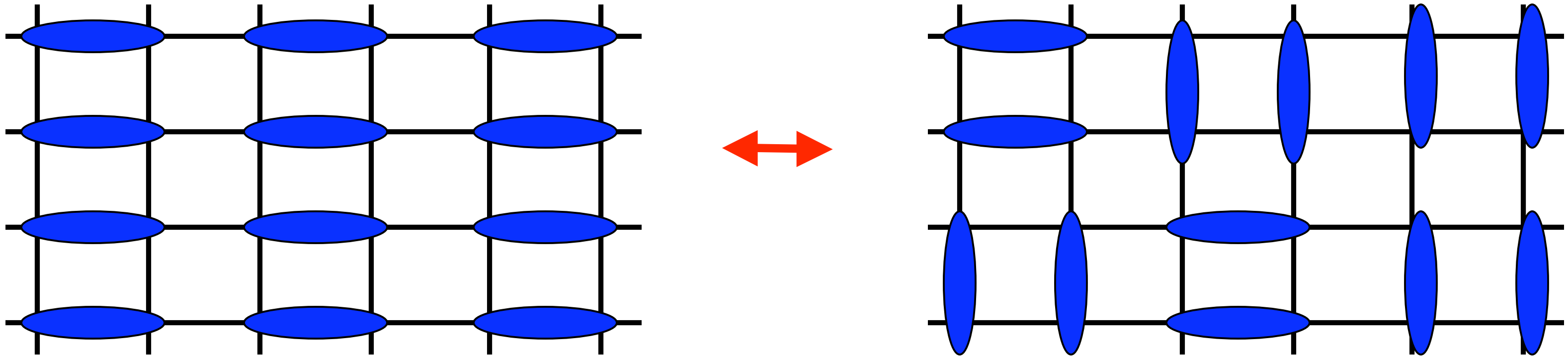




Resonating Valence Bond "Spin liquid"

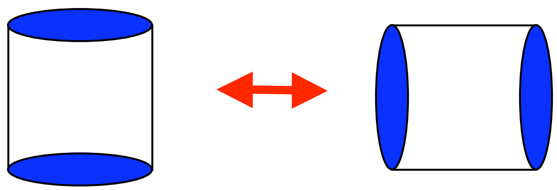
Plaquette Resonance

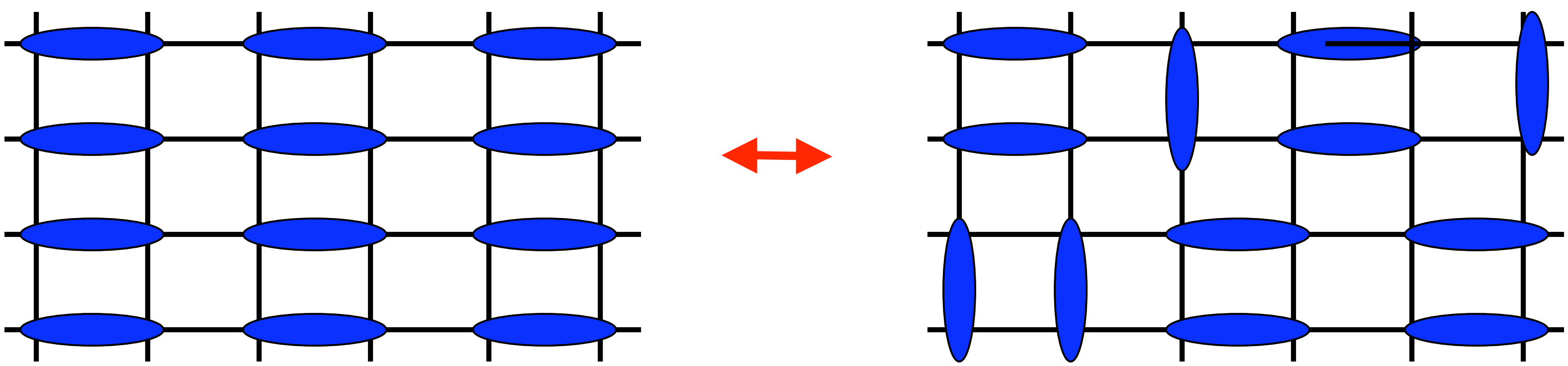




Resonating Valence Bond "Spin liquid"

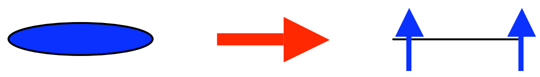
Plaquette Resonance





Resonating Valence Bond “Spin liquid”

Gapped Spin Excitations

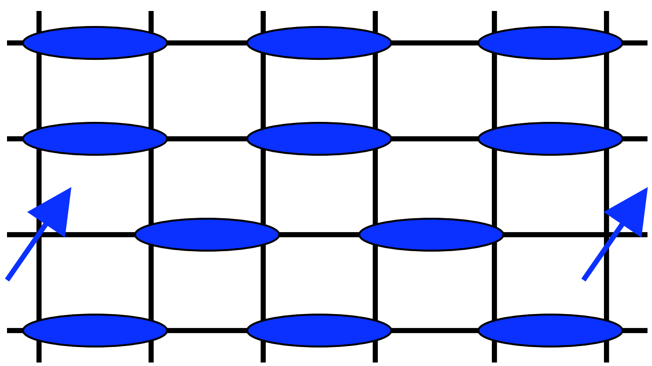


Create s=1 excitation

“Break” a Valence Bond - costs energy of order J

Try to separate two s=1/2 “spinons”

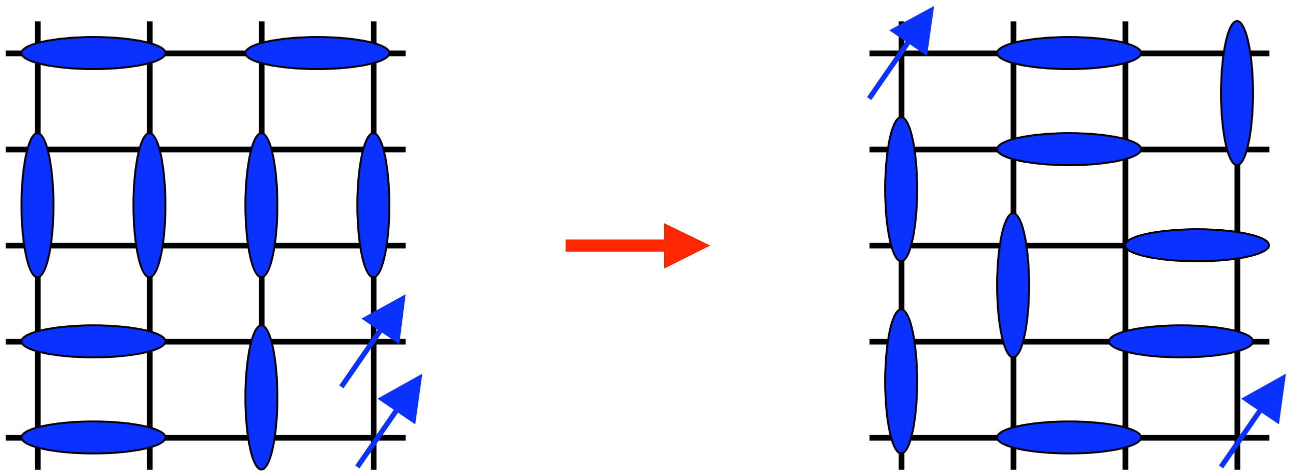
Valence Bond Solid



Energy cost is linear in separation

Spinons are “Confined” in VBS

RVB State: Exhibits Fractionalization!



Energy cost stays finite when spinons are separated

Spinons are “deconfined” in the RVB state

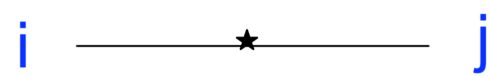
Spinon carries the electrons spin, but not its charge !

The electron is “fractionalized”.

Gauge Theory Formulation of RVB Spin liquid

Focus on the Valence bonds and their quantum dynamics - “A Quantum Dimer model”

Place a “spin” on each link of the lattice



$$\vec{\sigma}_{ij}$$

vector of Pauli matrices

$$\sigma^x_{ij} = 1$$

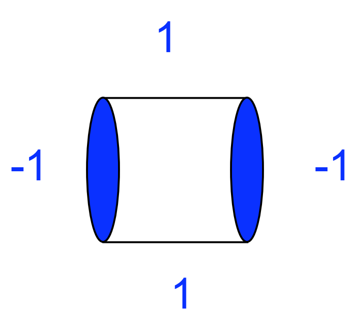
no bond on link ij

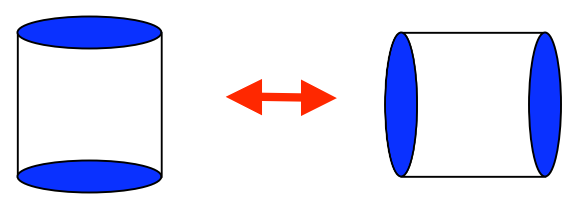
$$\sigma^x_{ij} = -1$$

bond on link ij

Define: Plaquette “Flux”

$$\mathcal{F}_p = \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li}$$



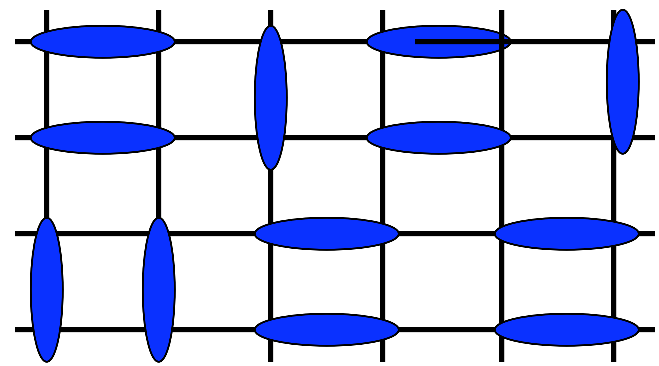


$$p(ijkl)$$

$$(\sigma^x\sigma^z = -\sigma^z\sigma^x)$$

Hamiltonian:

$$\mathcal{H} = -K \sum_{p(ijkl)} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li}$$



Need a Constraint: One valence bond coming out of each site

Z_2 Gauge Theory:

$$\mathcal{H} = -K \sum_{p(ijkl)} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} - J \sum_{ij} \sigma^x_{ij}$$

Constraint on "gauge" charges Q_i :

$$Q_i = \prod_{j=1}^{4} \sigma^x_{ij} = -1$$

Q_i = -1 implies one or three bonds out of each site

Large J energetically selects one valence bond only

Since $[Q_i, \mathcal{H}] = 0$ can diagonalize Q_i and H

Given an eigenstate: $\mathcal{H}|E\rangle = E|E\rangle$
Can construct: $|\tilde{E}\rangle \equiv \mathcal{P}|E\rangle$ with $\mathcal{P} = \prod_i [1 - Q_i]/2$
where $\mathcal{H}|\tilde{E}\rangle = E|\tilde{E}\rangle$

Cf: Maxwell Electrodynamics

$$H = \mathbf{B^2} + \mathbf{E^2}; \qquad \nabla \cdot \mathbf{E} = \mathbf{0}$$

Gauge Redundancy (not symmetry!!)

$$\sigma^z_{ij} \rightarrow \epsilon_i \sigma^z_{ij} \epsilon_j$$

Leaves Hamiltonian invariant for arbitrary $\epsilon_i = \pm 1$

(Gauge transformation)

Physical observables are gauge invariant - such as the "electric field" σ^x_{ij}

and the "magnetic flux" $\mathcal{F}_p = \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li}$

but NOT the "gauge field" σ^z_{ij} ➡ description using σ^z_{ij} is redundant

Hilbert Space

Square lattice with N sites and 2N links $\sigma^z_{ij} = +1, -1$ ➡ 2^{2N} total states

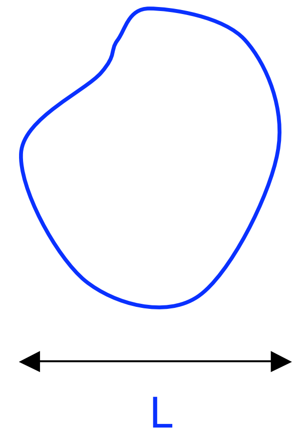
States related by a gauge transformation are physically equivalent, ie. each gauge inequivalent class has a redundancy of 2^N

Number of physically distinct states is $2^{2N}/2^N = 2^N$ corresponding to fluxes $\mathcal{F}_p = \boxed{\pm 1}$

Phase Diagram of Z_2 Gauge Theory

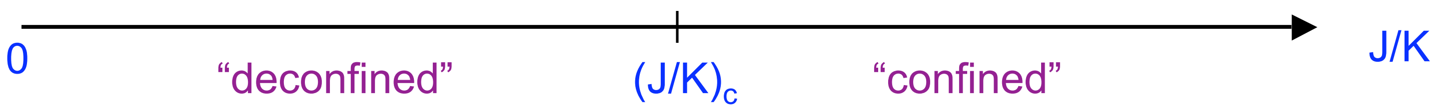
$$\mathcal{H} = -K \sum_{p(ijkl)} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} - J \sum_{ij} \sigma^x_{ij}$$

Characterize phases by gauge invariant Wilson loop operator $W_L = \prod_{loop}^{L} \sigma^z$



Perimeter Law: $\langle W_L \rangle = \exp(-cL)$ in "deconfined" phase,

Area law: $\langle W_L \rangle = \exp(-cL^2)$ in "confined" phase



Deconfined phase: "Magnetic flux" fixed, $\mathcal{F}_p = \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \approx 1$
"Electric field" fluctuating

← RVB Spin liquid

Confined phase: "Electric field" fixed, $\sigma^x_{ij} \approx \pm 1$
"Magnetic flux" fluctuating

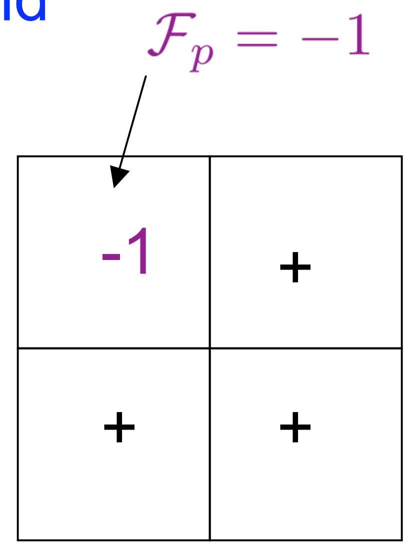
← Valence bond solid

“Vison” Excitations in the Deconfined Spin liquid

Assume “magnetic” flux is +1 thru all plaquettes in the ground state

Excited state: Put flux -1 thru a single plaquette - “vison”

Energy cost of vison is roughly K -
visons are gapped in RVB phase

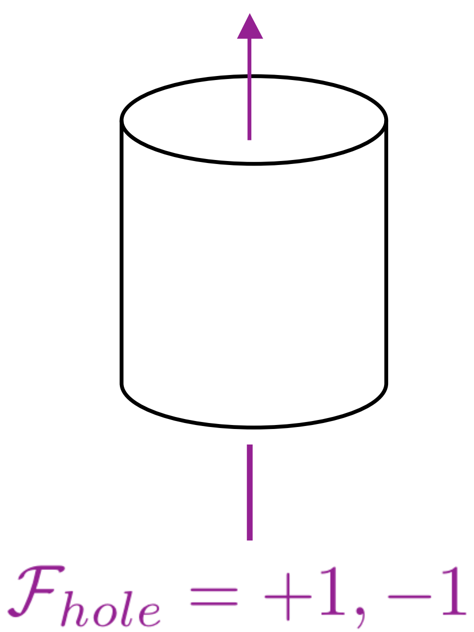


Topological Order - Ground State Degeneracies

Put the 2d system on a cylinder, and in the deconfined spin liquid phase with $\mathcal{F}_p \approx 1$

Two fold degenerate ground state - flux/no-flux thru hole in cylinder

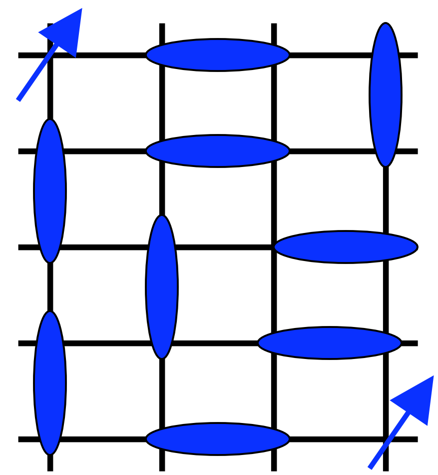
Ground state degeneracy depends on the topology (ie. 4-fold for torus) !



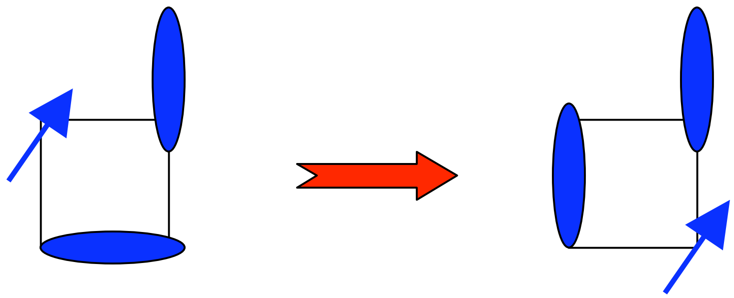
Put back in the “Spin(ons)”

Site with spinon has no connecting valence bond

Spinon carries “electric” gauge charge Q_i = -1



Introduce spinon creation operator at each site with spin up/down: $b^\dagger_{i\alpha}$

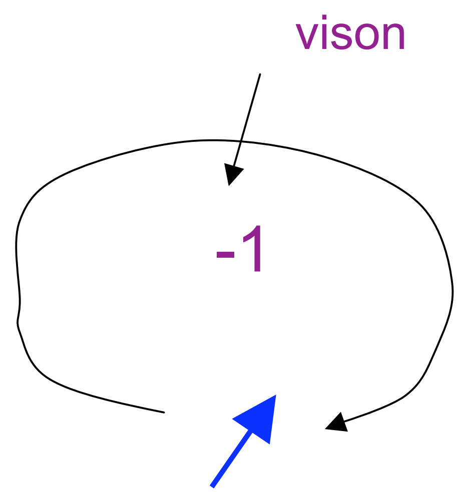


Spinon “Hopping” Hamiltonian:

$$\mathcal{H}_s = -t_s \sum_{ij} \sigma^z_{ij} b^\dagger_{i\alpha} b_{j\alpha} + h.c.$$

Spinons are “minimally” coupled to the Z_2 gauge field (cf. Maxwell)

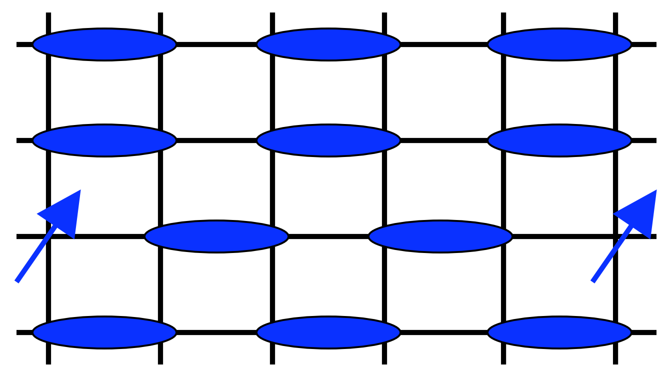
“Statistical” Interaction between spinon and vison
(in Z_2 spin liquid)



Taking a spinon (Z_2 “electric” charge) around a vison (Z_2 “magnetic flux”) gives a sign change to the spinon wavefunction

$$\psi_s \rightarrow -\psi_s$$

Confinement at large J/K - appropriate for quantum dimer model
(the Valence Bond Solid phase)



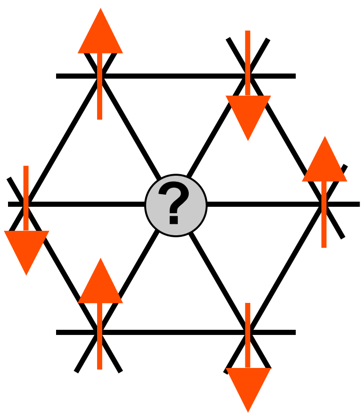
Confined phase: “Electric field” fixed $\sigma^x_{ij} \approx \pm 1$
“Magnetic flux” fluctuating

- Visons have proliferated - ie, they are “condensed”
- The spinons cannot propogate thru the fluctuating “magnetic” flux - they are “confined” and no longer present as finite energy excitations in the VBS phase

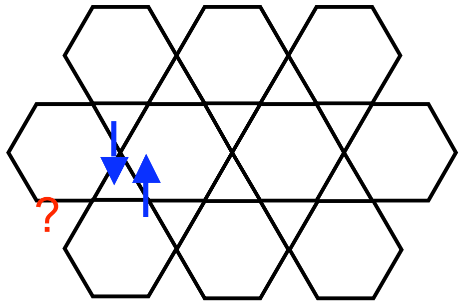
Desperately seeking topologically ordered spin liquids

2d square lattice near-neighbor s=1/2 Heisenburg model orders antiferromagnetically, and even with frustrating further neighbor interactions a Z_2 spin liquid seems unlikely

Try other lattices - with “geometric frustration”

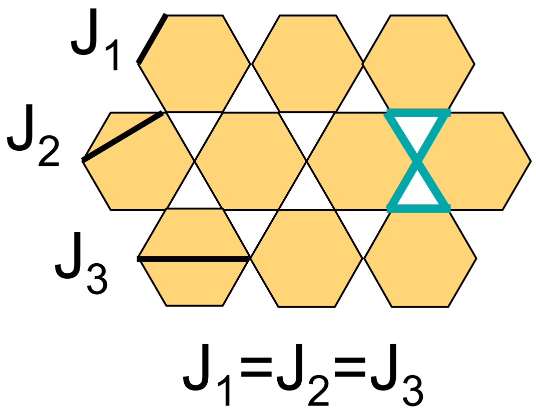


eg. triangular lattice (Lecture 4)

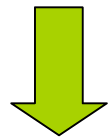


Kagome lattice (Japanese for basket weave)
-lattice of corner sharing triangles, perhaps the “most frustrated” lattice

Example: Generalized Kagome Ising Antiferromagnet



$$H_s = \frac{1}{2}\sum_{ij} J^z_{ij} S^z_i S^z_j + J^{xy}_{ij}(S^x_i S^x_j + S^y_i S^y_j)$$

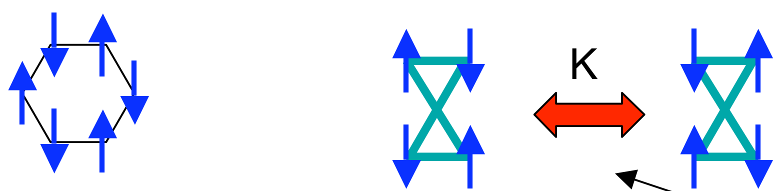


$$J^z \gg J^{xy}$$

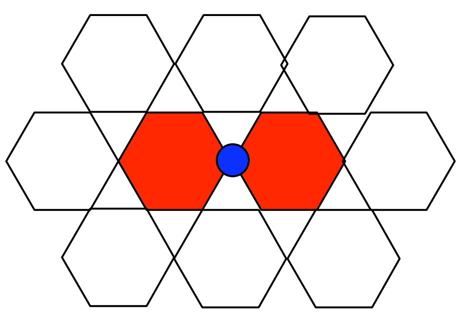
$$H_{eff} = J^z \sum_{\hexagon} (S^z_{\hexagon})^2 - K \sum_{\bowtie} (S_1^+ S_2^- S_3^+ S_4^- + \text{h.c.}) + v \sum_{\bowtie} \hat{P}_{1234}$$

added by hand

J^z enforces 3 up and 3 down spins on every hexagon

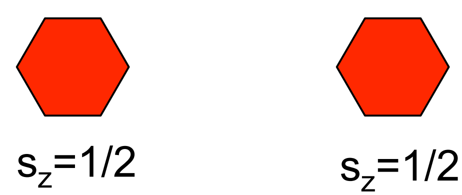


"ring term"



Flipping one spin (s^z=1) makes two "magnetized" hexagons

If they can be separated, one has two(!) s=1/2 spinons



Kagome Phase Diagram

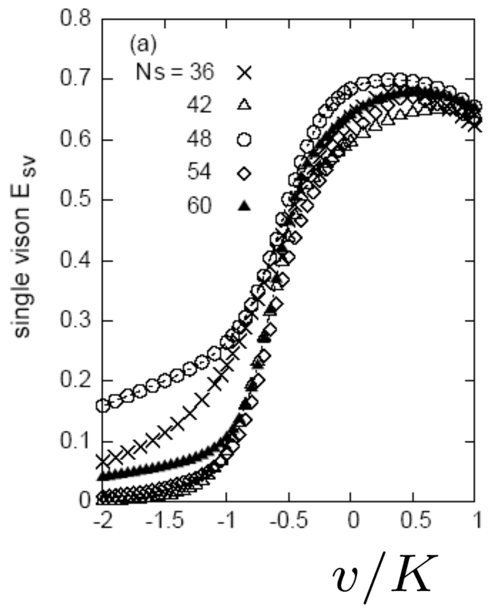
Can also define "vison" excitations, which "live" on the triangles

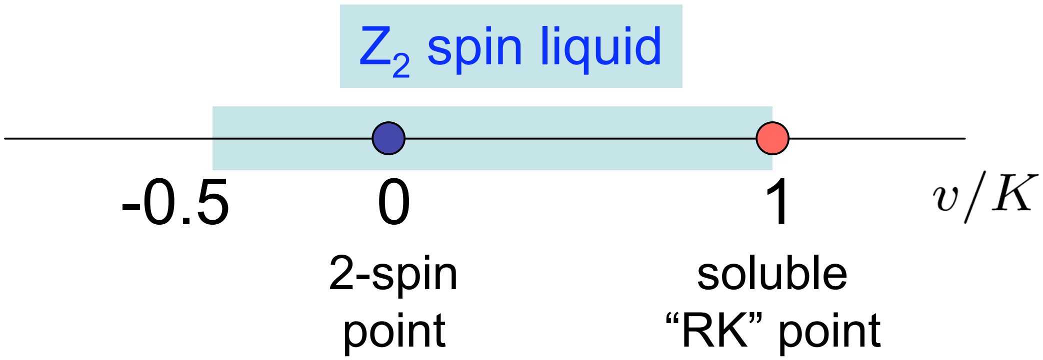
$$\hat{v}_i \sim \prod_{j=i}^{\infty} 2s_j^z$$

(a "string" operator)

Z_2 spin liquid is stable if the visons are gapped

- Exact diagonalization





- Spin liquid state is stable in the two-spin limit!

- c.f. this model is equivalent to a **3**-dimer model on the triangular lattice (Moessner-Sondhi). Appears to have much more stable spin liquid phase than **1**-dimer model.

Other models with topologically ordered spin liquid phases (a partial list)

- Quantum dimer models — Moessner, Sondhi; Misguich *et al*
- Rotor boson models — Motrunich, Senthil
- Honeycomb "Kitaev" model — Kitaev; Freedman, Nayak, Shtengel
- 3d Pyrochlore antiferromagnet — Hermele, Balents, M.P.A.F

■ Models are not crazy but contrived. It remains a huge challenge to find these phases in the lab – and develop theoretical techniques to look for them in realistic models.

Summary & Conclusions

- Quantum spin models can exhibit exotic paramagnet phases - "spin liquids" with topological order and quantum number fractionalization

- Gauge theory offers a simple way to characterize such topologically ordered phases, and to encode the statistical interactions

- The Z_2 spin liquid is the "tip of the iceberg". There are many, many much more intricate topologically ordered phases possible, some with excitations carrying fractional and even non-Abelian statistics - the latter would provide the "hardware" for a decoherence free "topological quantum computer"

- Much future work:
  - Find topological order in experiment??
  - Engineer simple Hamiltonians exhibiting topological order?
  - Experimentally engineer a real topologically ordered phase (eg. in a Josephson junction array)?