Chapter 13

DUALITY IN LOW DIMENSIONAL QUANTUM FIELD THEORIES

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Abstract In some strongly correlated electronic materials Landau's quasiparticle concept appears to break down, suggesting the possibility of new quantum ground states which support particle-like excitations carrying fractional quantum numbers. Theoretical descriptions of such exotic ground states can be greatly aided by the use of duality transformations which exchange the electronic operators for new quantum fields. This chapter gives a brief and self-contained introduction to duality transformations in the simplest possible context - lattice quantum field theories in one and two spatial dimensions with a global Ising or XY symmetry. The duality transformations are expressed as exact operator change of variables performed on simple lattice Hamiltonians. A Hamiltonian version of the Z_2 gauge theory approach to electron fractionalization is also reviewed. Several experimental systems of current interest for which the ideas of duality might be beneficial are briefly discussed.

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1. Introduction

At the heart of quantum mechanics is the wave-particle dualism. Quantum particles such as electrons when detected "are" particles, but exhibit many wavelike characteristics such as diffraction and interference. In condensed matter physics one is often interested in the collective behavior of 10^{23} electrons, which must be treated quantum mechanically

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even at room temperature [1]. Fluids of light atoms such as He-3 and He-4 also exhibit collective quantum phenomena in the Kelvin temperature range [2, 3], and these days heavier atoms can be much further cooled to exhibit Bose condensation. The two low temperature phases of He-4 are a beautiful manifestation of this wave-particle dichotomy in the many-body context - the superfluid at ambient pressures behaving as a single collective "wavefunction" [4] and the crystalline solid at high pressures best thought of in terms of the "particles".

The collective behavior of such many-particle quantum systems is usually discussed in terms of "particles" rather than in terms of "waves", and this preference is mirrored in the theoretical approaches which work with "particle" creation and destruction operators. But in some instances it is exceedingly helpful to have an alternative framework, particularly when one wants to focus attention on some underlying wave-like phenomena. Duality transformations can sometimes serve this purpose, since they exchange the particle creation operators for a new set of "dual" operators which typically create "collective" excitations such as solitons (in 1d) or vortices (in 2d). Moreover, duality is playing an increasingly important role in describing novel electronic ground states which support excitations which carry fractional quantum numbers. The best studied situation is the one-dimensional interacting electron gas, a "quantum wire" [5], which exhibits a novel "Luttinger liquid" phase [6]. The "bosonization" reformulation of 1d interacting electrons, discussed in detail in Chapter 4, is in fact closely related to the 1d duality transformations introduced below.

This chapter provides a brief yet self-contained introduction to duality transformations, focussing on simple quantum Hamiltonians with global Ising or XY symmetry. In Section II the Hamiltonian for the quantum Ising model in transverse field is discussed, and is dualized in one dimension (1d) and two dimensions (2d) in subsections A and B, respectively. Section III considers a model of interacting bosons formulated in terms of quantum "rotors" - nicely exhibiting the phase-number uncertainty and being readily dualized in both 1d and 2d. A Hamiltonian version of the Z_2 gauge theory of electron fractionalization is discussed in Section IV. Finally, Section V is devoted to a brief discussion of some experimental systems of current interest for which the theoretical ideas introduced above might prove helpful.

2. Quantum Ising models

Consider the quantum Ising model in a transverse field with lattice Hamiltonian [7],

$$\mathcal{H}_I = -J \sum_{\langle ij \rangle} S_i^z S_j^z - K \sum_i S_i^x, \tag{13.1}$$

where S_i^x and S_i^z are Pauli matrices defined on the sites of a 1d lattice or a 2d square lattice and the sum in the first term is over near-neighbor sites. Here and throughout the rest of the chapter the 1d and 2d lattices are assumed to be infinite. The Pauli "spins" satisfy $S_i^x S_i^x = 1$ and $S_i^z S_i^z = 1$, commute on different sites and anticommute on the same site,

$$S_i^z S_i^x = -S_i^x S_i^z. \tag{13.2}$$

In the absence of the transverse field, $K = 0$, it is most convenient to work in a basis diagonal in $S^z = \pm 1$. The model then reduces to a *classical* Ising model with ferromagnetic exchange interaction J, and the ground state is the ferromagnetically ordered state with $S^z = 1$ (or $S^z = -1$) on every site. This ground state spontaneously breaks the global spin-flip (or Z_2) symmetry. A small transverse field will cause the spins to flip and the ground state will be more complicated, but provided $K \ll J$ one expects the ferromagnetic order to survive, $\langle 0|S_i^z|0\rangle \neq 0$. In the opposite extreme, $K \gg J$, all the spins will point in the $x-$direction, $S^x = 1$, which corresponds to a quantum paramagnetic ground state with zero magnetization, $\langle S_i^z \rangle = 0$. Based on this reasoning, one expects a quantum phase transition between the ferro- and para-magnetic ground states when J is of order K.

2.1 Quantum Ising duality in 1d

Further insight into the quantum Ising model follows upon performing a duality transformation [8]. Generally, a duality transformation is simply a change of variables wherein the original fields - in this case the Ising spins - are exchanged for a new set of "dual" fields. For the Ising model in 1d the Pauli spin operators, S^z and S^x are exchanged for a new set of (dual) Pauli matrices, σ^z and σ^x, defined on the *sites* of the dual lattice (which are the links of the original lattice),

$$S_i^z = \prod_{j \leq i} \sigma_j^x, \tag{13.3}$$

$$S_i^x = \sigma_i^z \sigma_{i+1}^z. \tag{13.4}$$

The product runs over a semi-infinite "string" of sites j on the dual lattice which satisfy $j \leq i$. One can readily check that provided the σ_i^μ fields obey the Pauli matrix algebra, then so do the spin fields S_i^μ. These expressions can be inverted,

$$\sigma_i^z = \prod_{j<i} S_j^x, \tag{13.5}$$

$$\sigma_i^x = S_i^z S_{i-1}^z, \tag{13.6}$$

taking essentially the same form.

The Hamiltonian in Eq. (13.1) when re-expressed in terms of the dual operators has precisely the same form as originally,

$$\mathcal{H}_I = -K \sum_i \sigma_i^z \sigma_{i+1}^z - J \sum_i \sigma_i^x, \tag{13.7}$$

except with an interchange of the coupling constants, $J \leftrightarrow K$. Then, $\langle \sigma^z \rangle$ serves as a *disorder* parameter, being non-zero in the paramagnetic ground state, and vanishing in the ferromagnetic state. The Hamiltonian is self-dual when $J = K$, and this point corresponds to the quantum phase transition separating the two phases.

The duality transformation is also useful in identifying the excitations above the ground state. Consider first the ferromagnetically ordered state with $J \gg K$ where $S^z = 1$. The lowest energy *classical* excitation (when $K = 0$) consists of a domain wall separating two domains with $S^z = \pm 1$. Notice that such an excitation can be created by acting with the operator σ_i^z on the classical ground state: $\sigma_i^z|0\rangle$. When K is small but non-zero this "domain-wall" excitation is no longer an eigenstate of the Hamiltonian, since acting with the first term in the dual Hamiltonian Eq. (13.7) can be seen to move the location of the domain-wall. All of these domain-wall excitations have the same energy when $K = 0$, but this low energy manifold of states is split by non-zero K. One can use standard degenerate perturbation theory to calculate the energy splitting of this degenerate manifold to leading (first) order in small K, and obtain the associated eigenstates. One thereby obtains a set of states in which the domain wall is propagating along, and behaves like a particle. Indeed, since a single domain wall is topologically protected, this "particle" will not decay. But *two* domain-wall "particles" can annihilate another and disappear altogether. For this reason one says that such domain walls carry an Ising or Z_2 "charge". The fact that domain walls are point-like objects in one spatial dimension and can propagate like particles is exploited in the bosonization approach to 1d interacting systems and underlies the physics of 1d particle "fractionalization".

The paramagnetic ground state when $J \ll K$ also supports particle-like excitations. These excitations correspond to domain walls in the ordered state of the dual Ising model, i.e. walls separating the two phases with $\langle \sigma_i^z \rangle \approx \pm 1$. The operator which creates this "particle" is simply the Ising spin itself, S_i^z, as is readily apparent from Eq. (13.3). By treating the original Ising spin Hamiltonian perturbatively to first order in small $J \ll K$, one can construct these gapped particle-like excitations in the paramagnetic phase of the Ising model, and obtain their dispersion relation. As one increases J towards K from below, the energy gap for creating these Ising-spin excitations vanishes, and in the ferromagnetic phase these particles "condense", exhibiting long-ranged order,

$$\langle S_i^z S_j^z \rangle \neq 0; \qquad |i-j| \to \infty. \tag{13.8}$$

2.2 Quantum Ising Duality in 2d

We next turn to the transverse field quantum Ising model in two spatial dimensions, which for simplicity we place on a 2d square lattice with nearest-neighbor exchange interaction J. As in 1d, this model is expected to have two quantum ground states as the couplings are varied, a ferromagnetic ground state when $J \gg K$, a paramagnetic state in the opposite limit $J \ll K$, and an intervening quantum phase transition when J is comparable to K. As we shall see, the duality transformation in 2d relates the quantum Ising model (with *global* Z_2 symmetry under $S^z \to -S^z$) to a dual *gauge* theory - specifically a gauge theory with a *local* Z_2 symmetry [8]. At the operator level, the duality transformation is implemented by re-expressing S^x and S^z directly in terms of the dual gauge fields, σ_{ij}^μ - a set of Pauli matrices defined on the *links* of the dual square lattice,

$$S_i^x = \prod_{pl(i)} \sigma_{jl}^z, \tag{13.9}$$

and

$$S_i^z = \overrightarrow{\prod_{jl=i}^{\infty}} \sigma_{jl}^x. \tag{13.10}$$

Here, the first product is taken around an elementary four-sided plaquette on the dual square lattice (which encircles the site i of the original lattice). The second product involves an infinite string which connects sites of the original (direct) lattice, emanating from the site S_i^z and running to spatial infinity. For every bond of the dual lattice which is bisected by this string, a factor of σ_{ij}^x is present in the product. To assure that this definition is independent of the precise path taken by the string requires imposing the constraint that the product of σ_{ij}^x on all bonds connected to each site on the dual square lattice is set equal to unity,

$$\mathcal{G}_i = \prod_{j \in i} \sigma^x_{ij} = 1, \tag{13.11}$$

where j labels the nearest-neighbor sites of i. These local Z_2 gauge constraints must be imposed on the Hilbert space of the dual theory. (Note that because there are two bonds for every site of the 2d square lattice, the unconstrained dual Hilbert space is larger than the original Hilbert space, so it is reasonable that the dual Hilbert space be constrained.) In the resulting dual gauge theory, these constraints are analogous to Coulomb's law ($\nabla \cdot E = 0$) in conventional electromagnetism.

When re-expressed in terms of the dual fields, the Hamiltonian for the 2d quantum Ising model in a transverse field becomes,

$$\mathcal{H}_I = -K \sum_{pl} \prod_{pl} \sigma^z_{ij} - J \sum_{\langle ij \rangle} \sigma^x_{ij}. \tag{13.12}$$

In the first term products are taken around the elementary square plaquettes of the dual square lattice which surround the sites of the original lattice. These products measure "magnetic flux" in the dual gauge fields, that is plaquettes with $\prod_{pl} \sigma^z = -1$.

One can readily verify that the operators which implement a local gauge transformation, $\mathcal{G}_i$ in Eq. (13.11), commute with this dual Hamiltonian. Equivalently, since $\mathcal{G}_i \sigma^z_{ij} \mathcal{G}_i = -\sigma^z_{ij}$, the dual Hamiltonian is invariant under the general Z_2 gauge transformation,

$$\sigma^z_{ij} \rightarrow \epsilon_i \sigma^z_{ij} \epsilon_j, \tag{13.13}$$

with arbitrary $\epsilon_i = \pm 1$. We thus end up with a Z_2 gauge theory.

To gain some intuition for the behavior of this gauge theory we first focus on the limit $J \gg K$, where the global Ising model is ferromagnetically ordered. In this limit, the ground state of the gauge theory is simply $\sigma^x_{ij} = 1$ for all links ij. The low energy excitations about the ferromagnetically ordered state are droplets of $S^z = -1$ in the background of up spins ($S^z = 1$), and the "domain walls" are 1d closed paths (or "strings") which encircle the droplet (in contrast to the point-like domain walls for the 1d Ising model). To create such a droplet excitation from the ground state requires flipping all of the spins inside the droplet, that is,

$$|drop\rangle = \prod_{i \in drop} S^x_i |0\rangle, \tag{13.14}$$

where $|0\rangle$ denotes the ferromagnetically ordered ground state. This can be re-expressed in terms of the dual gauge fields as,

$$|drop\rangle = \prod_{\langle ij\rangle \in \mathcal{C}} \sigma^z_{ij}, \tag{13.15}$$

where $\mathcal{C}$ denotes the closed path that encircles the droplet. The energy of this droplet excited state is roughly JL, where L is the linear dimension (circumference) of the droplet. This is called the "confining" phase of the gauge theory, since two "test Z_2-charges" placed on sites i and j of the dual lattice (with $\prod_{\ell\in i} \sigma^x_{i\ell} = \prod_{\ell\in j} \sigma^x_{j\ell} = -1$), will cost an energy linear in their separation - the two particles are "confined" together in much the same way that the quarks are confined inside the mesons and hadrons in the standard model of the strong interaction (QCD).

Consider next the paramagnetic phase of the Ising model with $K \gg J$. In this limit, the gauge theory ground state corresponds to a state with $\prod_{pl} \sigma^z_{ij} = 1$ for all plaquettes. Excited states correspond to making a single plaquette with $\prod_{pl} \sigma^z_{ij} = -1$, a plaquette with a penetrating Z_2 "magnetic flux". This point-like excitation is reminiscent of a vortex in a 2d superconductor, and has been christened a "vison" due to it's Ising-like character (see below). To study the dynamics of the vison, it is simplest to return to the original global Ising model, where the paramagnetic ground state corresponds to all sites having $S^x_i = 1$. As is clear from the definition in Eq. (13.10), the vison excitation can be created by acting on the ground state with S^z_i, where i is the site of the original lattice which is in the center of the corresponding dual plaquette. Thus, in terms of the original Ising spins, a vison simply consists of a site with $S^x = -1$.

When $J = 0$ there is a large manifold of degenerate single vison states (with energy $2K$), since the vison can occupy any site of the original lattice. This degeneracy will be split by a small non-zero J, and these single vison states will broaden into a dispersing band.

The paramagnetic phase of the global Ising model corresponds to the "deconfined" phase of the gauge theory. In this phase, "test Z_2-charges" introduced into the theory (with $\prod_{j\in i} \sigma^x_{ij} = -1$ at the sites i of the "test charges") cost a finite energy to create. In particular, the energy to separate two such particles does not grow linearly with separation, but saturates at some finite value even as the separation is taken to infinity. One of the key signatures of such a deconfined phase of the Z_2 gauge theory is the presence of the vison as a finite energy excitation. (In the context of high-temperature superconductivity, an experiment was recently proposed to detect whether or not the vison was present in the underdoped region of the phase diagram. Detection of the vison would establish the existence of electron fractionalization (or spin-charge separation). Upon increasing J and approaching the transition into the ferromagnetic phase of the Ising model the energy cost of the vison is reduced. In the ferromagnetically ordered phase the vison has condensed, with $\langle S^z \rangle \neq 0$, since S^z is the vison creation operator.

3. Quantum XY or Rotor models

We next turn attention to quantum Hamiltonians in 1d and 2d which have a conserved $U(1)$ symmetry [9, 10]. In particular, we focus on bosons hopping on a 1d or a 2d lattice with boson creation operators, $b_i^\dagger$ satisfying the usual Bose commutation relations, $[b_i, b_j^\dagger] = \delta_{ij}$. A simple Hamiltonian which conserves the total number of bosons is,

$$\mathcal{H}_{boson} = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + h.c. + U \sum_i (b_i^\dagger b_i - \bar{n})^2. \tag{13.16}$$

The first term describes the hopping of bosons between nearest-neighbor sites, and the second term is an on-site repulsive interaction. Here, $\bar{n}$ plays the role of a chemical potential in setting the mean number of bosons, $\langle n_i \rangle$. This Hamiltonian is invariant under the global $U(1)$ symmetry: $b_i \to e^{i\Phi} b_i$, with a site-independent phase Φ. This global symmetry reflects the conservation of the total boson number.

Often it is convenient to consider a slight modification of this model, working with "rotor" variables rather than boson operators. In particular, we replace the boson creation operator by the exponential of a phase $\varphi_i \in [0, 2\pi]$: $b_i^\dagger \to e^{i\varphi_i}$, and the boson density by a number operator, n_i, which has integer eigenvalues, $b_i^\dagger b_i \to n_i$. The phase of the "rotor", φ_i, and the number operator are taken to satisfy the commutation relations,

$$[n_i, e^{i\varphi_j}] = \delta_{ij} e^{i\varphi_j}, \tag{13.17}$$

so that n_i can be thought of an "angular momentum" which is conjugate to the rotor phase. This commutation relation is directly analogous to,

$$[b_i^\dagger b_i, b_j^\dagger] = \delta_{ij} b_i^\dagger, \tag{13.18}$$

and indeed the operator $e^{i\varphi_i}$ increases the (boson) number n_i by one. In contrast to the operator $b_i^\dagger b_i$, which has non-negative eigenvalues, the eigenvalues of n_i span all the integers.

The "rotor" or XY Hamiltonian analogous to $\mathcal{H}_{boson}$ is,

$$\mathcal{H}_{XY} = -t \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j) + U \sum_i (n_i - \bar{n})^2. \tag{13.19}$$

Notice that for large U the states with negative number $n_i < 0$ are up at high energy and can generally be neglected. Let us consider briefly the ground state phases of this quantum rotor or quantum XY model. When $U = 0$, the model reduces to a classical XY model, and the ground state is an ordered state with spatially constant rotor phases, $\varphi_i = \phi$ for all sites i. There is an associated non-vanishing order parameter, $\langle e^{i\varphi_i}\rangle \neq 0$. This ground state corresponds to the superfluid phase of the bosons, and exhibits off-diagonal long-ranged order,

$$G_{ij} = \langle e^{i\varphi_i} e^{-i\varphi_j}\rangle = |\langle e^{i\varphi_i}\rangle|^2 \neq 0 \qquad |r_i - r_j| \to \infty. \tag{13.20}$$

For small but non-zero $U \ll t$ the ground state will be more complicated since the interaction term will induce some quantum fluctuations in the phases, but the off-diagonal long-range order and superfluidity should survive. (Actually, in 1d there will only be off-diagonal quasi-long-ranged order, and the correlator G_{ij} will vanish algebraically in the spatial separation.) The low-energy excitations above this ground state are the gapless Goldstone modes associated with the spontaneous breaking of the continuous $U(1)$ symmetry. (In 1d these should perhaps be called "quasi-Goldstone" modes, since the symmetry is not truly broken.) An effective Hamiltonian for these modes is obtained by expanding the cosine for small phase gradients,

$$\mathcal{H}_{Gold} = \frac{t}{2}\sum_{\langle ij\rangle}(\varphi_i - \varphi_j)^2 + U\sum_i (n_i - \bar{n})^2. \tag{13.21}$$

This Hamiltonian is quadratic in the conjugate variables (φ and n) and can be readily diagonalized to obtain the Goldstone modes. One can then evaluate the off-diagonal correlator G_{ij} in the ground state, and show that it decays algebraically in 1d but is infinitely long-ranged in 2d.

The behavior of the ground state in the opposite strong-interaction limit with $U \gg t$ depends sensitively on the average boson occupancy, $\langle n_i\rangle$ (which is only equal to $\bar{n}$ in the opposite $U \ll t$ limit). To understand this, consider the extreme limit with $t = 0$. For integer filling, such as $\langle n_i\rangle = 1$ say, the ground state will be unique with one boson on each site, and excited states with zero or two bosons on a site will cost a large energy of order U. This is a "Mott insulating" state with a large gap to charged excitations, and will be robust at integer filling, provided that $t \ll U$. Away from integer filling the ground states at $t = 0$ will be strongly degenerate, since the bosons can be arranged in many different ways on the lattice. In this case, non-zero hopping t will lift this degeneracy, leading to superfluidity. Henceforth, we focus primarily on the more interesting situation with integer boson filling.

3.1 Quantum XY Duality in 1d

Here we focus first on duality for the 1d rotor model. In close analogy with the Ising duality in 1d (Eq. (13.3) and 13.4), consider the change of variables,

$$e^{i\varphi_i} = \prod_{j\le i} e^{iE_j}, \tag{13.22}$$

$$n_i = \theta_{i+1} - \theta_i, \tag{13.23}$$

where the dual "phase" field $E_i \in [0, 2\pi]$ and the integer-eigenvalue operator θ_i occupy the sites of the dual lattice. The dual operators are taken to satisfy

$$[e^{iE_i}, \theta_j] = \delta_{ij} e^{iE_i}, \tag{13.24}$$

which enables one to establish the desired commutator between n_i and $e^{i\varphi_i}$. In terms of these new fields the rotor Hamiltonian becomes,

$$\mathcal{H}_{XY} = -t\sum_i \cos(E_i) + U\sum_i(\theta_{i+1} - \theta_i - \bar{n})^2. \tag{13.25}$$

While formally exact, this dual Hamiltonian is often rather difficult to work with due to the integer constraints on the field θ_i. For this reason, it is both convenient and illuminating to modify the model by "softening" this integer constraint, allowing θ_i to take all real values and then adding a "potential term" acting on θ which favors integer values: $V(\theta) = -t_v \cos(2\pi\theta)$. Once this dual "angular momentum" θ_i is no longer quantized, it is legitimate to extend the "phase" field E_i to all real values, and to expand the cosine potential. In this way we arrive at an approximation to the dual Hamiltonian of the rotor model which should describe the same physics,

$$\tilde{\mathcal{H}}_{XY} = \sum_i \{\frac{t}{2}E_i^2 + U(\theta_{i+1} - \theta_i - 2\pi\bar{n})^2 - t_v\cos(2\pi\theta_i)\}, \tag{13.26}$$

where θ_i and E_i are now generalized coordinates and momenta which satisfy the canonical commutation relations,

$$[\theta_i, E_j] = i\delta_{ij}. \tag{13.27}$$

For integer boson occupancy, $\bar{n}$ can be eliminated from the theory by shifting the fields $\theta_j \to \theta_j + j(2\pi\bar{n})$, and $\tilde{\mathcal{H}}_{XY}$ reduces to a lattice sine-Gordon Hamiltonian. The associated Euclidian Lagrangian follows from,

$$\tilde{S}_{XY} = i\int d\tau \sum_i E_i \partial_\tau \theta_i + \int d\tau \tilde{\mathcal{H}}_{XY}, \tag{13.28}$$

and after integrating over the conjugate momenta becomes,

$$\tilde{S}_{XY} = \int d\tau \sum_i \{ \frac{1}{2t}(\partial_\tau \theta_i)^2 + U(\theta_{i+1} - \theta_i - 2\pi\bar{n})^2 \quad (13.29)$$

$$- \quad t_v \cos(2\pi\theta_i) \}. \quad (13.30)$$

When $U \ll t$ the field θ is very soft and strongly fluctuating, and the cosine term becomes ineffective - this is the superfluid phase. After discarding the cosine term the Euclidian action (or Hamiltonian) is quadratic, and can be diagonalized to obtain the gapless "quasi-Goldstone" modes of the superfluid phase.

In the Mott insulating phase with $U \gg t$ (for integer boson density $\bar{n}$), the θ fluctuations are very "stiff" and become "pinned" in the minima of the cosine potential. In this limit one expects that the modes will become gapped. This can be verified by expanding the cosine potential to quadratic order for small θ and diagonalizing the resulting Hamiltonian to show that the normal-mode dispersion is gapped.

In addition to the gapped sound waves, the sine-Gordon theory will support "soliton"-like excitations separating regions in which the θ field is trapped in neighboring minima of the cosine potential. These correspond to single-boson excitations above the Mott ground state.

3.2 Quantum XY Duality in 2d

Finally, we consider dualizing the 2d quantum rotor model [10]. As for the Ising duality in $2+1$d, the duality transformation for the 2d rotor model with global $U(1)$ (or XY) symmetry will take one to a gauge theory - but now a gauge theory with a (local) $U(1)$ gauge symmetry. Specifically, we re-express φ_i and n_i in terms of gauge fields defined on the links of the dual square lattice,

$$n_i = \Delta_x a_i^y - \Delta_y a_i^x \equiv \vec{\Delta} \times \vec{a}, \quad (13.31)$$

$$e^{i\varphi_i} = \prod_{j,\alpha=i}^{\infty} \!\!\rightarrow e^{iE_j^\alpha}, \quad (13.32)$$

where a_i^α and E_i^α with $\alpha = x, y$ are vector fields defined on the links of the dual square lattice (a_i^x lives on the link running from site i to the site $i + \hat{x}$, and similarly for a_i^y). As above, E_i^α is a "phase" field defined on the interval $[0, 2\pi]$ and the operators a_i^α have integer eigenvalues. Here Δ_α denotes a discrete difference, $\Delta_x f_i = f_{i+\hat{x}} - f_i$. As for the Ising duality in $2+1$d, the product above is along an infinite string - the string links sites of the original lattice starting at site i and running to spatial infinity, and for every link of the dual lattice bisected by the string a factor of $e^{iE_i^\alpha}$ is present in the product. The dual "vector potential" and "electric fields" are canonically conjugate variables, as in ordinary quantum electromagnetism,

$$[a_i^\alpha, e^{iE_j^\beta}] = \delta_{ij}\delta_{\alpha\beta}e^{iE_i^\alpha}. \tag{13.33}$$

To assure path-independence we must impose a constraint on the dual Hilbert space,

$$\mathcal{G}(\Lambda) = \prod_i e^{i\Lambda_i \vec{\Delta}\cdot\vec{E}_i} = 1, \tag{13.34}$$

for arbitrary integers Λ_i. Equivalently, the divergence of the "electric field" $\vec{\Delta}\cdot\vec{E}_i$ must equal $2\pi N_i$ for some integer N_i at each site of the dual lattice.

These integer "charges" actually correspond to vortices - point-like singularities around which the phase field φ_i winds by $2\pi N$. To see this, note that we can relate spatial gradients in the phase φ to the electric field

$$e^{i\Delta_\alpha \varphi_i} = e^{i\epsilon_{\alpha\beta}E_i^\beta}. \tag{13.35}$$

This is the discrete lattice version of $\nabla\varphi = \hat{z}\times\vec{E}$, and implies that $\nabla\times\nabla\varphi = \nabla\cdot\vec{E}$. As for the Ising case, the gauge constraints are generators of the local gauge transformations,

$$\mathcal{G}^\dagger a_i^\alpha \mathcal{G} = a_i^\alpha + \Delta_\alpha\Lambda_i. \tag{13.36}$$

In terms of the dual variables the 2d quantum XY model takes the form,

$$\mathcal{H}_{XY} = -t\sum_{i\alpha}\cos(E_i^\alpha) + U\sum_i(\vec{\Delta}\times\vec{a}_i - \bar{n})^2. \tag{13.37}$$

As in $1+1$d we now soften up the integer constraint on a_i^α, defining E_i^α in the range $[-\infty,\infty]$. Upon expanding the cosine term one obtains,

$$\begin{aligned} \tilde{\mathcal{H}}_{XY} &= \sum_i\{\frac{t}{2}\vec{E}_i^2 + U(\vec{\Delta}x\vec{a}_i - \bar{n})^2\} && (13.38)\\ &- t_v\sum_{i\alpha}\cos(\Delta_\alpha\theta_i - 2\pi a_i^\alpha), && (13.39)\end{aligned}$$

where we have explicitly displayed the longitudinal part of the gauge field, $2\pi\vec{a}_i^\ell \equiv \vec{\Delta}\theta_i$. After softening this constraint the appropriate local gauge symmetry becomes,

$$\tilde{\mathcal{G}}(\Lambda) = \prod_i e^{i\Lambda_i(\vec{\Delta}\cdot\vec{E}_i - 2\pi N_i)} = 1, \tag{13.40}$$

where N_i is a vortex number *operator* with integer eigenvalues which satisfies,

$$[N_i, e^{i\theta_j}] = \delta_{ij} e^{i\theta_i}, \tag{13.41}$$

so that

$$\tilde{\mathcal{G}}^\dagger a_i^\alpha \tilde{\mathcal{G}} = a_i^\alpha + \Delta_\alpha \Lambda_i \tag{13.42}$$
$$\tilde{\mathcal{G}}^\dagger \theta_i \tilde{\mathcal{G}} = \theta_i + 2\pi \Lambda_i. \tag{13.43}$$

We can now interpret the physics of the final dual Hamiltonian, $\tilde{\mathcal{H}}_{XY}$. The field, $e^{i\theta_i}$, is a vortex creation operator since it's action raises the vortex number, N_i, by one. The last term in the dual Hamiltonian thus describes the vortex kinetic energy, and the vortices are seen to be minimally coupled to the dual "electromagnetic" field. Thus the dual field mediates a logarithmic interaction between vortices. The dual $U(1)$ gauge symmetry can be interpreted as the conservation of vorticity.

When the vortices are absent from the ground state, with $t_v = 0$, the remaining terms in the Hamiltonian are quadratic and can be diagonalized to obtain the Goldstone modes of the 2d superfluid phase. In terms of the dual "electromagnetic field", this is nothing other than the massless "photon". Since the original boson density is equal to the curl of the dual "vector potential", this Goldstone mode is a longitudinal density (or sound) wave.

To describe the Mott insulating state we have to allow the proliferation of vortices and anti-vortices. Since the vortices are bosons, when present at zero temperature they will condense so that the ground state can be considered as a condensation of vortices, $\langle e^{i\theta} \rangle \neq 0$. Since the vortices are minimally coupled to the dual "electromagnetic" field, their condensation will lead to an expulsion of this dual "flux". For integer boson densities (i.e. integer $\bar{n}$) this phase will be the dual analog of the Meissner state in a superconductor. The gapless Goldstone mode of the superfluid (the dual "photon") will become gapped. To see this explicitly, one can expand the cosine to quadratic order in the dual "vector potential", and after choosing a convenient gauge ($\vec{\Delta} \cdot \vec{a_i} = 0$) diagonalize the resulting quadratic Hamiltonian.

After condensing the vortex, an externally applied dual "magnetic field" will be quantized into dual "flux quanta", analogous to the Abrikosov vortices in type-II superconductors. However, in this dual representation, a single quantized flux actually correspond to a boson excitation in the Mott insulating state. The dual "Abrikosov flux lattice" would then be a crystal of bosons.

This illustrates an appealing feature of dualizing to a vortex description when considering systems of 2d bosons: vortex-condensation gives one an order parameter for insulating (non-superfluid) phases of the bosons. It is very interesting to consider the possibility of *pairing* vortices, and condensing the pair, leaving single vortices uncondensed. As recently argued, this procedure leads to an exotic insulating state of bosons which supports fractionalized excitations - a gapped “half-boson” excitation (the dual quantized flux in the vortex pair condensate) and a gapped vison excitation (essentially an unpaired vortex).

4. Chargons, spinons and the Z_2 gauge theory of 2d electron fractionalization

The quantum Ising and XY Hamiltonians studied in Sections 2 and 3 are the simplest examples of quantum Hamiltonians which can be fruitfully analyzed by “duality” - re-expressing them in terms of a new set of “dual” operators. But many important models relevant to the quantum behavior of solids involve the fermionic electron creation and destruction operators, rather than the commuting bosonic operators entering in the quantum Ising and XY models. The classic example is the Hubbard model, which describes electrons hopping on the sites of a lattice interacting via a short-ranged (on-site) screened Coulomb repulsion. For the 1d Hubbard model and other 1d interacting electron models, a reformulation in terms of new operators - the so-called “bosonization” [6] technique - is possible and well understood. But “dualizing” models of 2d and 3d interacting electrons appears to be much more challenging. Nevertheless, some progress has been made in 2d, usually involving a “spin-charge” decomposition of the electron creation operator into a product of an operator which creates the spin of the electron - a “spinon” - and another which creates the charge of the electron - a “holon” or “chargon”. These reformulations invariably involve a gauge field, which strongly couples together the spinons and chargons, and effectively “glues” them back together [11]. But in some situations the effects of the “gauge glue” can be weak, and exotic quantum ground states emerge within which the spinons and chargons can propagate as “deconfined” particle excitations. In effect, the electron is splintered into two fragments. A theory of such 2d “electron fractionalization” has recently been developed which involves a Z_2 gauge field [11]. The fractionalized state corresponds to the deconfined phase of the Z_2 gauge theory, and therefore supports a vison excitation precisely as discussed in Section IIB.

Here, a simple Hamiltonian version of the Z_2 gauge theory of 2d electrons is briefly presented. In the usual formulation, the $s = 1/2$ spinons carry the Fermi statistics of the electron, and the chargons are bosonic.

The full gauge theory Hamiltonian is [11],

$$
\begin{aligned}
H &= H_c + H_\sigma + H_s, && (13.44)\\
H_c &= -t\sum_{\langle ij\rangle}\sigma^z_{ij}\left(b^\dagger_i b_j + h.c\right) + U\sum_i\left(b^\dagger_i b_i - 1\right)^2, && (13.45)\\
H_\sigma &= -K\sum_{pl}\prod_{pl}\sigma^z_{ij} - J\sum_{\langle ij\rangle}\sigma^x_{ij}, && (13.46)\\
H_s &= -\sum_{\langle ij\rangle}\sigma^z_{ij}\left[t_s\left(f^\dagger_{i\alpha}f_{j\alpha} + h.c\right)\right.\\
&+ \left.\Delta_{ij}\left(f_{i\uparrow}f_{j\downarrow} - f_{i\downarrow}f_{j\uparrow} + h.c.\right)\right]. && (13.47)
\end{aligned}
$$

Here $b^\dagger_i$ creates a chargon at site i while $f^\dagger_{i\alpha}$ creates a spinon with spin $\alpha = \uparrow, \downarrow$ at site i. The operator $b^\dagger_i b_i$ measures the number of bosonic chargons at site i. For simplicity, we have specialized to half-filling, i.e. to an average of one boson per site. The constant Δ_{ij} contains the information about the pairing symmetry of the spinons. The $\sigma^z_{ij}, \sigma^x_{ij}$ are Pauli spin matrices which are defined on the links of the lattice, and H_σ is in fact identical to the Z_2 gauge theory Hamiltonian discussed in Section IIB.

The full Hamiltonian is invariant under the Z_2 gauge transformation $b_i \to -b_i, f_{i\alpha} \to -f_{i\alpha}$ at any site i of the lattice accompanied by $\sigma^z_{ij} \to -\sigma^z_{ij}$ on all the links connected to that site. This Hamiltonian must be supplemented with the constraint equation

$$\mathcal{G}_i = \prod_{j\in i}\sigma^x_{ij}e^{i\pi\left(f^\dagger_{i\alpha}f_{i\alpha} + b^\dagger_i b_i\right)} = 1. \qquad (13.48)$$

Here the product over σ^x_{ij} is over all links that emanate from site i. The operator $\mathcal{G}_i$, which commutes with the full Hamiltonian, is the generator of the local Z_2 gauge symmetry. Thus the constraint $\mathcal{G}_i = 1$ simply expresses the condition that the physical states in the Hilbert space are those that are gauge invariant.

When $J \gg K$ the gauge theory is deep within it's confining phase, and the chargon and spinon are confined back together to form the electron, with destruction operator $c_{i\alpha} = b_i f_{i\alpha}$. On the other hand, the fractionalized insulating phase is described as the deconfined phase of this gauge theory. This is obtained when $K \gg J, U \gg t$. A conventional superconducting state follows when the chargons condense, which occurs when $t \gg U$, or alternatively by doping away from half-filling. Note that the "pairing" symmetry of the superconductor is determined by Δ_{ij}.

5. Physics and Duality

This section provides a brief discussion of several strongly correlated electronic materials which exhibit unusual and in some cases poorly understood behavior, and considers how the theoretical ideas introduced above might provide a framework for gaining further insight into their properties.

5.1 One-dimensional systems

A number of complex molecular crystals exhibit highly anisotropic electrical properties. For crystals comprised of long (often organic), chain-like molecules, the conductivity along the chains can be many orders of magnitude larger than the transverse conductivity. In such cases, progress can be made by focusing on the properties of a single conducting chain. Modern lithographic techniques honed in the semiconductor industry provide another means to access one dimensional conductors, by controlling gates which further restrict the motion of electrons confined at the interface between two semiconductor materials [5]. However, carbon nanotubes - tube-shaped single molecules of carbon a nanometer in diameter and many microns long - provide the cleanest and most accessible example of a one-dimensional conductor [12].

It turns out that the strong effects of the interactions between the electrons moving up and down such nanotubes leads to exotic new behavior which is qualitatively different from the behavior of electrons in an ordinary conductor such as a copper wire [6, 13]. In particular, an electron added to a nanotube, for example by tunnelling from a metallic electrode, effectively splinters into fragments as it is propagates along the tube [14]. More precisely, the added spin and charge of the electron propagate in several "packets", one carrying the spin only and the others some fraction of the electronic charge. These exotic new "particles" are correctly considered as "solitons" in the background 1d fluid of electrons. They are quite similar to the "solitons" discussed in Section IIA in the context of the 1d quantum Ising model, which were domain walls between ferromagnetic domains which propagate as 1d "particles". The fractionally charged carriers in the nanotubes are even more closely related to the solitons mentioned in the context of the 1d quantum XY duality in Section IIIA (the solitons connecting different minima of the sine-Gordon cosine potential).

The central theoretical approach used to describe the physics of 1d interacting electron systems such as those occurring in carbon nanotubes is known as "bosonization" [15, 6]. In the bosonization approach the electron creation operator is exchanged for *two* bosonic fields, often denoted θ and φ. These two fields are essentially the same as the two fields employed in the discussion of 1d XY duality in Section IIIA, and provide two complementary (dual) descriptions of the same physics.

5.2 Two-dimensional systems

Quasi-two-dimensional layered materials occur both naturally (as with mica or graphite) and can also be grown, either out of the melt or layer-by-layer (in the case of semiconductors) using molecular beam epitaxy. Layered materials which exhibit strongly correlated electronic behavior typically have partially filled conduction band states, which can either lead to conduction or, when the band is very narrow, to self-localization. In this latter case, the residual electron spin degrees of freedom comprise a very interesting and challenging many-body system [16, 7]. The canonical examples are provided by the transition-metal oxides, where the 3d or 4d electrons form the localized interacting spin moments. The dynamics of such two-dimensional quantum spin systems can often be captured by (deceptively) simple lattice spin-Hamiltonians [17, 16]. For spin one-half moments, the spin Hamiltonians are in fact quite similar to the Hamiltonians in Eq. (13.1) and (13.16), the main difference being that the physical spin-systems have approximate spin-rotational symmetry rather than the extreme Ising-like "easy-axis" or XY-like "easy-plane" models considered here. Nevertheless, considerable insight can often be gained by appropriately dualizing the spin Hamiltonians [18]. Of interest are the myriad of possible quantum ground states that such many-body systems can possess, ranging from states with spontaneously broken spin-rotational symmetry (i.e. magnetic order) or broken translational symmetry ("spin-Peierls" order) [17] to exotic ground states with hidden "topological order" and fractionalized excitations [19].

Electrically conducting 2d layered materials offer an even more challenging arena of complicated many-body behavior. The high temperature cuprate superconductors [20, 1] offer the classic example. After more than 15 years of intensive effort (and at least tens of thousands of experimental publications), the underlying physics of these fascinating materials remains poorly understood and shrouded with theoretical controversy. The 2d electron system formed near the surface of an oxidized and gated silicon crystal (metal-oxide-semiconductor field-effect transistors or MOSFETS for short) provides another example of a well characterized material which exhibits strange behavior - an apparent 2d "metal-insulator" transition - which continues to defy theoretical consensus [21]. While the *fermionic* character of the conducting electrons is surely central to gaining an understanding of these materials, the 2d duality transformations discussed in this paper (which involve bosonic fields, commuting on different sites) might nevertheless be rather useful. One concrete approach was mentioned in Section IV, where a theory of 2d interacting electrons was reformulated in terms of spin-charge separated variables and a Z_2 gauge field - the same gauge theory shown to be dual to the 2d quantum Ising model in Section IIB. The 2d quantum XY duality transformation of Section IIIB has also been employed to access a new approach to 2d strongly correlated electrons [22, 23]. In this work, the vortices which appear in the dualized model of Section IIIB, are identified with the familiar vortices of a 2d superconductor. Very recent work [24] has exploited such a dual representation to obtain the first example of a genuine 2d "non-Fermi liquid phase" - a quantum ground state of 2d interacting electrons with no broken symmetries which has gapless charge and spin excitations but is *not* connected adiabatically to the free Fermi gas - in contrast to the familiar Fermi liquid phase. This novel quantum phase can apparently be accessed only by looking through a pair of "dual glasses". Determining whether such exotic states actually underlie the mysterious behavior of the cuprates or other 2d strongly correlated materials remains as one of the central challenges in contemporary theoretical physics.

Over the past 20 years my knowledge and appreciation of the wave-particle dualism of quantum mechanics in general and duality transformations of field theories in particular have been greatly aided by intensive and beneficial interactions and collaborations with (among others), Leon Balents, Daniel Fisher, Steve Girvin, Geoff Grinstein, Charlie Kane, Dung-Hai Lee, Chetan Nayak, T. Senthil and A. Peter Young - and I am deeply grateful and indebted to them all. This work has been generously supported by the National Science Foundation under grants DMR-0210790 and PHY-9907947.

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