Problem 1. Correlations for two spin-$1/2$’s

The singlet state of the two spin-$1/2$ particles can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+aA; -aB\rangle - |-aA; +aB\rangle \right)$$

where $|+aA; -aB\rangle$ denotes the state where Alice’s particle has spin up along $\hat{a}$, and Bob’s particle has spin down along $\hat{a}$, and so on.

(a)

The probability that Alice and Bob will measure the spins of their respective particles up along $\hat{a}$ and $\hat{b}$ is

$$\mathcal{P}_{++} = |\langle +aA; +bB | \psi \rangle|^2 = \frac{1}{2} |\langle +bB | -aB \rangle|^2$$

Take $\hat{a}$ in the $z$-direction, and $\hat{b}$ in the $(y,z)$-plane. Let $\theta$ be the angle between the two, so that $\hat{a} \cdot \hat{b} = \cos \theta$. Then

$$|+a\rangle = \exp(-\frac{i}{2} \theta \sigma_x) |+a\rangle = [\cos(\theta/2)\hat{\lambda} - \sin(\theta/2)\sigma_x] |+a\rangle = \cos(\theta/2) |+a\rangle - \sin(\theta/2) |-a\rangle ,$$

up to an overall phase. Therefore, $|\langle +b | -a \rangle|^2 = \sin^2(\theta/2) = \frac{1}{2} (1 - \cos \theta)$. Using this in eq. (1),

$$\mathcal{P}_{++} = \frac{1}{4} (1 - \hat{a} \cdot \hat{b})$$

(b)

Alice measures her particle to have spin down along the $\hat{a}$-axis; this projects the spin singlet to

$$|\psi\rangle \rightarrow |-aA; +aB\rangle$$

Thus, the conditional probability that Bob measures his particle to have spin up along $\hat{b}$, given that Alice measured spin down along $\hat{a}$, is, using eq. (2),

$$|\langle +bB | +aB \rangle|^2 = \cos^2(\theta/2) = \frac{1}{2} (1 + \hat{a} \cdot \hat{b})$$

(c)

There are four possible measurement outcomes, $++, --, +-, -$ and $-+$. The first two are assigned a value $+1$, and the second two $-1$. Thus

$$\mathcal{D}(\hat{a}, \hat{b}) = (\mathcal{P}_{++} + \mathcal{P}_{--}) - (\mathcal{P}_{+-} + \mathcal{P}_{-+})$$
By symmetry, \( P_- = P_{++} \) and \( P_+ = P_{+-} \), so
\[
\mathcal{D}(\hat{a}, \hat{b}) = 2(P_{++} - P_{+-}) \tag{3}
\]

Following part a,
\[
P_{+-} = |\langle +a_A; -b_B | \psi \rangle|^2 = \frac{1}{2}|\langle -b_B | -a_B \rangle|^2 = \frac{1}{2} \cos^2(\theta/2) = \frac{1}{4}(1 + \hat{a} \cdot \hat{b})
\]

Notice that this joint probability is half the conditional probability of part b, as it should be. Using this with the result of part a in eq. (3), we get
\[
\mathcal{D}(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}
\]

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**Problem 2. Three photons: beyond Bell**

In the \(|H\rangle, |V\rangle \) basis, the three-photon state is
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( |H\rangle_1 |H\rangle_2 |H\rangle_3 + |V\rangle_1 |V\rangle_2 |V\rangle_3 \right) \tag{4}
\]

(a)

The 45-degree rotated basis is
\[
|H'\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}, \quad |V'\rangle = \frac{|H\rangle - |V\rangle}{\sqrt{2}}
\]

Inverting this,
\[
|H\rangle = \frac{|H'\rangle + |V'\rangle}{\sqrt{2}}, \quad |V\rangle = \frac{|H'\rangle - |V'\rangle}{\sqrt{2}} \tag{5}
\]

Substituting into eq. (4), and dropping the ket symbols to simplify notation,
\[
|\psi\rangle = \frac{1}{4} \left[ (H_1' + V_1')(H_2' + V_2')(H_3' + V_3') + (H_1' - V_1')(H_2' - V_2')(H_3' - V_3') \right]
\]

Terms with an odd number of \( V' \)’s cancel between the two parts; terms with an even number of \( V' \)’s add:
\[
|\psi\rangle = \frac{1}{2} \left[ |H_1'H_2'H_3'\rangle + |H_1'V_2'V_3'\rangle + |V_1'H_2'V_3'\rangle + |V_1'V_2'H_3'\rangle \right]
\]

Thus, in an \textsc{noa} experiment, the outcomes \( H'H'H', H'V'V', V'H'V' \) and \( V'V'H' \) each occur with probability 1/4, while the remaining outcomes \( H'H'V', H'V'H', V'H'H' \) and \( V'V'V' \) never occur.

(b)

We have
\[
|L\rangle = \frac{|H\rangle - i|V\rangle}{\sqrt{2}}, \quad |R\rangle = \frac{|H\rangle + i|V\rangle}{\sqrt{2}}
\]

so
\[
|H\rangle = \frac{|R\rangle + |L\rangle}{\sqrt{2}}, \quad |V\rangle = \frac{|R\rangle - |L\rangle}{\sqrt{2}i} \tag{6}
\]

Using eqs. (5) and (6) in eq. (4),
\[
|\psi\rangle = \frac{1}{4} \left[ (R_1 + L_1)(R_2 + L_2)(H_3' + V_3') - (R_1 - L_1)(R_2 - L_2)(H_3' - V_3') \right]
\]

Expanding as before,
\[
|\psi\rangle = \frac{1}{2} \left[ |R_1'R_2'V_3'\rangle + |L_1'L_2'V_3'\rangle + |R_1'L_2'H_3'\rangle + |L_1'R_2'H_3'\rangle \right] \tag{7}
\]
(c) From eq. (7), in a $\beta\beta\alpha$ experiment, the outcomes $RR'V'$, $LL'V'$, $LR'H'$ and $RL'H'$ each occur with probability $1/4$, while the remaining outcomes ($RR'H'$, $LL'H'$, $LR'V'$ and $RL'V'$) never occur.

Photons are indistinguishable, so $\beta\alpha\beta$ and $\alpha\beta\beta$ experiments must give the same results (with the labels appropriately permuted). That is, if $\beta', \beta = R/L$ and $\alpha = H', V'$, then the probability of observing $\beta\alpha\beta'$ in a $\beta\alpha\beta$ experiment is the same as that of observing $\beta'\alpha$ in the $\beta\beta\alpha$ experiment, and so on. Mathematically, this is ensured by permutation symmetry of $|\psi\rangle$.

(d) The functions $A_i(\lambda)$ take the values $\pm 1$ for measurement outcomes $H'$ and $V'$ of the $i$th photon, and $B_i(\lambda)$ take the values $\pm 1$ for measurement outcomes $L$ and $R$. The four outcomes of the $\beta\beta\alpha$ experiment that occur with nonzero probability are $RR'V'$, $LL'V'$, $LR'H'$ and $RL'H'$, and $B_1B_2A_3 = -1$ for each of these. Thus,

$$B_1(\lambda)B_2(\lambda)A_3(\lambda) = -1$$

Similarly,

$$B_1A_2B_3 = A_1B_2B_3 = -1$$

(e) Since $[B_i(\lambda)]^2 = 1$, the above results imply that

$$A_1A_2A_3 = (A_1B_2B_3)(B_1A_2B_3)(A_1B_2A_3) = (-1)^3 = -1$$

The results of an $\alpha\alpha\alpha$ experiment that would be consistent with local hidden variables are therefore $H'H'V'$, $H'V'H'$, $V'H'H'$ and $V'V'V'$.

(f) According to part a, quantum mechanics predicts that in an $\alpha\alpha\alpha$ experiment, the outcomes $H'H'H'$, $H'V'V'$, $V'H'V'$ and $V'V'H'$ each occur with probability $1/4$. Each of these has

$$A_1A_2A_3 = +1$$

According to part e, any local hidden variables theory predicts that these four outcomes will never occur.

(g) The conflict in Bell’s inequality is for statistical predictions, while in this problem the conflict arises even in definite predictions.
Problem 3. **Searching a small quantum phonebook**

(a)

Consider the general case where there are \(N\) entries in the phonebook. The probability that you find your friend’s name on the *first* lookup is \(p_1 = \frac{1}{N}\). The probability of finding it on the *second* lookup is

\[
p_2 = \left( \frac{N-1}{N} \right) \left( \frac{1}{N-1} \right) = \frac{1}{N}
\]

since this involves picking any of the \(N - 1\) incorrect entries the first time, and then picking the correct one among the \(N - 1\) remaining entries the second time. Similarly,

\[
p_3 = \left( \frac{N-1}{N} \right) \left( \frac{N-2}{N-1} \right) \left( \frac{1}{N-2} \right) = \frac{1}{N}
\]

and in the same way,

\[
p_4 = \cdots = p_{N-2} = \frac{1}{N}
\]

Of course, you never need more than \(N - 1\) lookups. Thus,

\[
p_{N-1} = 1 - \sum_{n=1}^{N-2} p_n = 1 - \frac{N-2}{N} = \frac{2}{N}
\]

The average number of lookups needed is then

\[
\langle n \rangle_N = \frac{1}{N} \sum_{n=1}^{N-1} n p_n = \frac{1}{N} \left( \sum_{n=1}^{N-2} n + 2(N-1) \right) = \frac{1}{N} \left( \frac{(N-1)(N-2)}{2} + 2(N-1) \right) = \frac{(N-1)(N+2)}{2N}
\]

Setting \(N = 4\),

\[
\langle n \rangle_4 = \frac{9}{4}
\]

(b)

When \(x \neq x_0\) (that is, when \(\langle x | x_0 \rangle = 0\),

\[U_{F_0} |x \rangle \langle 0| \rightarrow |x \rangle |0 \oplus 0\rangle = |x \rangle |0\rangle \quad \text{and} \quad U_{F_0} |x \rangle \langle 1| \rightarrow |x \rangle |1 \oplus 0\rangle = |x \rangle |1\rangle \]

When \(x = x_0\),

\[U_{F_0} |x_0 \rangle \langle 0| \rightarrow |x_0 \rangle |0 \oplus 1\rangle = |x_0 \rangle |1\rangle \quad \text{and} \quad U_{F_0} |x_0 \rangle \langle 1| \rightarrow |x_0 \rangle |1 \oplus 1\rangle = |x_0 \rangle |0\rangle \]

Therefore, \(U_{F_0}\) acts on the state \(|x\rangle (|0\rangle - |1\rangle)\sqrt{2}\) as \(\mathbb{1}\) when \(x \neq x_0\), but as \(-\mathbb{1}\) when \(x = x_0\). In symbols,

\[U_{F_0} |x \rangle (|0\rangle - |1\rangle)\sqrt{2} \rightarrow (\tilde{U}_{F_0} |x \rangle) (|0\rangle - |1\rangle)\sqrt{2}, \]

where \(\tilde{U}_{F_0} = \mathbb{1} - 2|x_0 \rangle \langle x_0|\) acts only in the computational subspace.

(c)

In general, the initial state is

\[|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \]
and the Grover operation is
\[ \mathcal{G}_F = U_s \tilde{U}_F = (2 |s \rangle \langle s| - 1)(1 - 2 |x_0 \rangle \langle x_0|) \]

The state after a single iteration is therefore
\[ |\psi_1 \rangle = \mathcal{G}_F |s \rangle = (2 |s \rangle \langle s| - 1)(1 - 2 |x_0 \rangle \langle x_0|) |s \rangle \]
\[ = |s \rangle - 4 |s \rangle |x_0 \rangle^2 + 2 |x_0 \rangle \langle x_0| s \rangle \]
\[ = \left(1 - \frac{4}{N}\right) |s \rangle + \frac{2}{\sqrt{N}} |x_0 \rangle \]

For the special case \(N = 4\), we see that
\[ |\psi_1 \rangle = |x_0 \rangle \]
so in this case the Grover algorithm locates your friend’s name with unit probability after a single iteration.

(d)

For general \(N\),
\[ \mathcal{G}_F |x_0 \rangle = U_s (- |x_0 \rangle) = (2 |s \rangle \langle s| - 1)(- |x_0 \rangle) \]
\[ = |x_0 \rangle - 2 |s \rangle \langle s|x_0| \]
\[ = |x_0 \rangle - \frac{2}{\sqrt{N}} |s \rangle \]

Together with the result of part c and linearity, this yields the state after \(T = 2\) iterations:
\[ |\psi_2 \rangle = \mathcal{G}_F |\psi_1 \rangle = \left(1 - \frac{4}{N}\right) \mathcal{G}_F |s \rangle + \frac{2}{\sqrt{N}} \mathcal{G}_F |x_0 \rangle \]
\[ = \left(1 - \frac{4}{N}\right) \left(\left(1 - \frac{4}{N}\right) |s \rangle + \frac{2}{\sqrt{N}} |x_0 \rangle\right) + \frac{2}{\sqrt{N}} \left(|x_0 \rangle - \frac{2}{\sqrt{N}} |s \rangle\right) \]

For the special case \(N = 4\), the first term above vanishes, leaving
\[ |\psi_2 \rangle = |x_0 \rangle - |s \rangle \]

The probability of finding your friend’s name after two iterations of Grover’s algorithm is therefore
\[ P_2 = |\langle x_0 |\psi_2 \rangle|^2 = \left|1 - \frac{1}{2}\right|^2 = \frac{1}{4} \]

which is as bad as guessing blindly.

If we keep going, we find that \(P_3 = 1/4\), \(P_4 = 1\), and so on, with two 1/4’s following each 1. This type of cyclic behavior will always occur for any \(N\), because we are iterating a unitary transformation on a finite-dimensional Hilbert space.