Problem Set 1 215A-Quantum Mechanics, Fall 2017

Due: Tuesday, October 10, 2017 by 11pm

Put homework in mailbox labelled 215A on 1st floor of Broida (by elevators).

1.) **Hermitian Operators**

(a) For two Hermitian operators, $\hat{A}, \hat{B}$, both will all eigenvalues positive, show that $Tr(\hat{A}\hat{B}) > 0$.

(b) For an Hermitian operator $\hat{A}$ with all eigenvalues positive, show that the matrix elements satisfy the inequality:

$$|\langle v|\hat{A}|w\rangle|^2 \leq \langle v|\hat{A}|v\rangle\langle w|\hat{A}|w\rangle, \tag{1}$$

for any two kets, $|v\rangle$ and $|w\rangle$, in the Hilbert space.

(c) Consider Hermitian operators $\hat{A}, \hat{B}$ and $\hat{H}$ which satisfy the commutations relations, $[\hat{A}, \hat{H}] = [\hat{B}, \hat{H}] = 0$, but $[\hat{A}, \hat{B}] \neq 0$. Prove that at least one of the eigenvalues of $\hat{H}$ must be degenerate.

(d) For an Hermitian operator $\hat{A}$ show that,

$$\det(e^{\hat{A}}) = e^{Tr(\hat{A})}. \tag{2}$$

For an Hermitian operator $\hat{B}$ with all positive eigenvalues show that,

$$\ln(\det(\hat{B})) = Tr(\ln(\hat{B})). \tag{3}$$

2.) **Anti-Hermitian operators**

(a) An operator $\hat{K}$ is said to be anti-Hermitian if it satisfies $\hat{K}^\dagger = -\hat{K}$. Show that an anti-Hermitian operator can have at most one real eigenvalue (possibly degenerate).

(b) Show that the commutator $[\hat{A}, \hat{B}]$ of two Hermitian operators, $\hat{A}, \hat{B}$, is either anti-Hermitian or zero.

(c) Demonstrate that it is not possible for two Hermitian operators, $\hat{Q}, \hat{P}$ to satisfy $[\hat{Q}, \hat{P}] = i\hbar\hat{1}$, where $\hat{1}$ denotes the identity operator in the $n-$dimensional Hilbert space. How do you reconcile this with with the familiar position-momentum commutation relations?

3.) **Functions of operators**

(a) Consider an operator $\hat{C} \equiv (\hat{A} - \lambda\hat{B})^{-1}$. Assuming that the operator $\hat{A}$ is invertible, with inverse denoted $\hat{A}^{-1}$, derive a formal Taylor series expansion for $\hat{C}$ as a power series in the parameter λ.

(b) Consider an operator $\hat{\mathcal{O}}_\lambda \equiv \exp(\lambda\hat{A} + \hat{B})$. What conditions must be placed on the operators $\hat{A}$ and $\hat{B}$ so that $\partial_\lambda\hat{\mathcal{O}}_\lambda = \hat{A}\hat{\mathcal{O}}_\lambda$? By performing the Taylor series expansion in powers of λ, show that $Tr(\partial_\lambda\hat{\mathcal{O}}_\lambda) = Tr(\hat{A}\hat{\mathcal{O}}_\lambda)$ for arbitrary operators $\hat{A}$ and $\hat{B}$.

(c) Consider two operators $\hat{A}$ and $\hat{B}$ whose commutator is a c-number. Show that $\exp(\lambda\hat{A})\exp(\lambda\hat{B}) = \exp(\lambda\hat{A} + \lambda\hat{B})\exp(\lambda^2[\hat{A}, \hat{B}]/2)$.

(d) Find the eigenvalues of the three operators $\hat{A}, \hat{B}$ and $\hat{C}$ which satisfy $\hat{A}^2 = p^2$, $\hat{B}^2 = p\hat{B}$ and $\hat{C}^3 = p^2\hat{C}$ for real c-number p.

4.) **Unitary Operators**

(a) Show that an operator $\hat{A}$ that satisfies any two of the following conditions: (i) $\hat{A}$ is unitary, (ii) $\hat{A}$ is Hermitian, (iii) $\hat{A}^2 = \hat{I}$, also satisfies the third.

(b) Show that the product of two unitary operators, $\hat{U}_1\hat{U}_2$, is also unitary.

(c) Demonstrate that the Hermitian and anti-Hermitian parts of any unitary operator commute with one another, so that a unitary operator can always be diagonalized. What are the properties of the eigenvalues of a unitary operator?

(d) Show that an operator of the form $\hat{U} = \exp(i\hat{H})$ is unitary, if $\hat{H}$ is an Hermitian operator.

(e) Consider two operators $\hat{A}$ and $\hat{A}'$ which are related by a unitary transformation, $\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$. Show that $Tr(\hat{A}') = Tr(\hat{A})$ and $\det(\hat{A}') = \det(\hat{A})$.