Due: Tuesday, October 10, 2017 by 11pm
Put homework in mailbox labelled 215A on 1st floor of Broida (by elevators).

1.) **Hermitian Operators**

(a) For two Hermitian operators, \( \hat{A}, \hat{B} \), both will all eigenvalues positive, show that \( Tr(\hat{A}\hat{B}) > 0 \).

(b) For an Hermitian operator \( \hat{A} \) with all eigenvalues positive, show that the matrix elements satisfy the inequality:
\[
|\langle w|\hat{A}|v\rangle|^2 \leq \langle w|\hat{A}|w\rangle \langle v|\hat{A}|v\rangle,
\]
for any two kets, \( |v⟩ \) and \( |w⟩ \), in the Hilbert space.

(c) Consider Hermitian operators \( \hat{A}, \hat{B} \) and \( \hat{H} \) which satisfy the commutations relations, \([\hat{A}, \hat{B}] = [\hat{B}, \hat{H}] = 0\), but \([\hat{A}, \hat{B}] \neq 0\). Prove that at least one of the eigenvalues of \( \hat{H} \) must be degenerate.

(d) For an Hermitian operator \( \hat{A} \) show that,
\[
\det(e^{\hat{A}}) = e^{Tr(\hat{A})}.
\]
For an Hermitian operator \( \hat{B} \) with all positive eigenvalues show that,
\[
\ln(\det(\hat{B})) = Tr(\ln(\hat{B})).
\]

2.) **Anti-Hermitian operators**

(a) An operator \( \hat{K} \) is said to be anti-Hermitian if it satisfies \( \hat{K}^\dagger = -\hat{K} \). Show that an anti-Hermitian operator can have at most one real eigenvalue (possibly degenerate).

(b) Show that the commutator \([\hat{A}, \hat{B}]\) of two Hermitian operators, \( \hat{A}, \hat{B} \), is either anti-Hermitian or zero.

(c) Demonstrate that it is not possible for two Hermitian operators, \( \hat{Q}, \hat{P} \) to satisfy \([\hat{Q}, \hat{P}] = i\hbar \hat{I} \), where \( \hat{I} \) denotes the identity operator in the \( n \)-dimensional Hilbert space. How do you reconcile this with with the familiar position-momentum commutation relations?

3.) **Functions of operators**

(a) Consider an operator \( \hat{C} \equiv (\hat{A} - \lambda\hat{B})^{-1} \). Assuming that the operator \( \hat{A} \) is invertible, with inverse denoted \( \hat{A}^{-1} \), derive a formal Taylor series expansion for \( \hat{C} \) as a power series in the parameter \( \lambda \).

(b) Consider an operator \( \hat{O}_\lambda \equiv \exp(\lambda\hat{A} + \hat{B}) \). What conditions must be placed on the operators \( \hat{A} \) and \( \hat{B} \) so that \( \partial_\lambda \hat{O}_\lambda = \hat{A}\hat{O}_\lambda \)? By performing the Taylor series expansion in powers of \( \lambda \), show that \( Tr(\partial_\lambda \hat{O}_\lambda) = Tr(\hat{A}\hat{O}_\lambda) \) for arbitrary operators \( \hat{A} \) and \( \hat{B} \).

(c) Consider two operators \( \hat{A} \) and \( \hat{B} \) whose commutator is a c-number. Show that \( \exp(\lambda\hat{A})\exp(\lambda\hat{B}) = \exp(\lambda\hat{A} + \lambda\hat{B}) \exp(\lambda^2[\hat{A}, \hat{B}]/2) \).

(d) Find the eigenvalues of the three operators \( \hat{A}, \hat{B} \) and \( \hat{C} \) which satisfy \( \hat{A}^2 = p^2, \hat{B}^2 = p\hat{B} \) and \( \hat{C}^3 = p^2\hat{C} \) for real c-number \( p \).

4.) **Unitary Operators**

(a) Show that an operator \( \hat{A} \) that satisfies any two of the following conditions: (i) \( \hat{A} \) is unitary, (ii) \( \hat{A} \) is Hermitian, (iii) \( \hat{A}^2 = \hat{I} \), also satisfies the third.

(b) Show that the product of two unitary operators, \( \hat{U}_1 \hat{U}_2 \), is also unitary.

(c) Demonstrate that the Hermitian and anti-Hermitian parts of any unitary operator commute with one another, so that a unitary operator can always be diagonalized. What are the properties of the eigenvalues of a unitary operator?

(d) Show that an operator of the form \( \hat{U} = \exp(i\hat{H}) \) is unitary, if \( \hat{H} \) is an Hermitian operator.

(e) Consider two operators \( \hat{A} \) and \( \hat{A}' \) which are related by a unitary transformation, \( \hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger \). Show that \( Tr(\hat{A}') = Tr(\hat{A}) \) and \( \det(\hat{A}') = \det(\hat{A}) \).