Physics 215c: Problem Set 1

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Please let me know if you encounter any typos in the solutions.

Problem 1 (20)

We have the spin singlet state
\[ |\psi\rangle = \frac{1}{\sqrt{2}} [|+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B] \] (1)

(a)

Since the singlet state is invariant under rotations, we can equally write it in terms of the the \( \hat{a} \) basis:
\[ |\psi\rangle = \frac{1}{\sqrt{2}} [|+, \hat{a}\rangle_A |-, \hat{a}\rangle_B - |-, \hat{a}\rangle_A |+, \hat{a}\rangle_B] \] (2)

In terms of another basis vector \( \hat{b} \), where \( \hat{a} \cdot \hat{b} = \cos(\theta) \), we can write (c.f., for example, Sakurai Problem 1.9)
\[ |+, \hat{a}\rangle = \cos(\theta/2)|+, \hat{b}\rangle + e^{i\phi}\sin(\theta/2)|-, \hat{b}\rangle , |-, \hat{a}\rangle = \sin(\theta/2)|+, \hat{b}\rangle - e^{i\phi}\cos(\theta/2)|-, \hat{b}\rangle \] (3)

We then find that the probability of Alice measuring + in the \( \hat{a} \) basis and Bob measuring + in the \( \hat{b} \) basis is given by
\[ P_{++} = \frac{1}{\sqrt{2}} \sin(\theta/2)^2 = \frac{1}{4} (1 - \cos(\theta)) = \frac{1}{4} \left( 1 - \hat{a} \cdot \hat{b} \right) \] (4)

(b)

Using our work in (a), the probability of Bob measuring + in the \( \hat{b} \) basis given that Alice measured – in the \( \hat{a} \) basis is given by first projecting with \( |-, \hat{a}\rangle_A \langle -, \hat{a}| \) and then re-normalizing the state before measuring the probability of of Bob measuring + in the \( \hat{b} \) basis. This gives the conditional probability
\[ P(+, \hat{b}|-, \hat{a}) = |\cos(\theta/2)|^2 = \frac{1}{2} (1 + \cos(\theta)) = \frac{1}{2} \left( 1 + \hat{a} \cdot \hat{b} \right) \] (5)
(c)

Proceeding as in part (a), one finds the probabilities

\[
P_{++} = P_{--} = \frac{1}{4} \left( 1 - \hat{a} \cdot \hat{b} \right)
\]

\[
P_{+-} = P_{-+} = \frac{1}{4} \left( 1 + \hat{a} \cdot \hat{b} \right)
\]

so that

\[
\mathcal{D}(\hat{a}, \hat{b}) = \frac{1}{4} \left( 2(1 - \hat{a} \cdot \hat{b}) - 2(1 + \hat{a} \cdot \hat{b}) \right) = -\hat{a} \cdot \hat{b}.
\]

Problem 2 (20)

(a)

The probability of selecting your friend at random after any number of tries is just $1/4$. However, if you select your friend on the 4th try, we consider this actually to require only 3 lookups: not having found them on 3rd try necessarily implies they are in the last remaining spot. Thus the expectation value for the number of lookups $N$ is given by

\[
E(N) = \frac{1}{4} (1 + 2 + 3 + 3) = \frac{9}{4}
\]

(b)

We need only consider two cases: If $x \neq x_0$, then

\[
U_{F_0}|x\rangle|y\rangle = \frac{1}{\sqrt{2}} |x\rangle (|0 + 0 \pmod{2}\rangle - |1 + 0 \pmod{2}\rangle) = |x\rangle|y\rangle
\]

and if $x = x_0$, then

\[
U_{F_0}|x_0\rangle|y\rangle = \frac{1}{\sqrt{2}} |x_0\rangle (|0 + 1 \pmod{2}\rangle - |1 + 1 \pmod{2}\rangle) = -|x\rangle|y\rangle
\]

Likewise, if $x \neq x_0$, then

\[
\tilde{U}_{F_0}|x\rangle|y\rangle = (I - 2|x_0\rangle\langle x_0|) \frac{1}{\sqrt{2}} |x\rangle (|0\rangle - |1\rangle) = |x\rangle|y\rangle
\]

and if $x = x_0$, then

\[
\tilde{U}_{F_0}|x_0\rangle|y\rangle = (I - 2|x_0\rangle\langle x_0|) \frac{1}{\sqrt{2}} |x_0\rangle (|0\rangle - |1\rangle) = -|x\rangle|y\rangle
\]

Thus we conclude that $\tilde{U}_{F_0}$ and $U_{F_0}$ act equivalently on these states.
(c)

Let $|s\rangle = \frac{1}{2} \sum_{i=1}^{4} |i\rangle$, and $U_s = 2|s\rangle\langle s| - \mathbb{1}$ (Hereafter we are suppressing the register bit). Then

$$\tilde{U}_{F_0}|s\rangle = |s\rangle - |x_0\rangle$$

so that

$$G_{F_0}|s\rangle = U_s(|s\rangle - |x_0\rangle) = [2|s\rangle\langle s| - \mathbb{1}] (|s\rangle - |x_0\rangle) = 2|s\rangle - |s\rangle - |s\rangle + |x_0\rangle = |x_0\rangle$$

so that we find the correct state $|x_0\rangle$ with probability 1.

(d)

Iterating the algorithm a second time, we find

$$G_{F_0}^2|s\rangle = [2|s\rangle\langle s| - \mathbb{1}] (-|x_0\rangle) = -|s\rangle + |x_0\rangle.$$ 

Then the probability of measuring $|x_0\rangle$ is

$$P(x_0) = |1/2|^2 = \frac{1}{4}.$$