Problem Set 2 223B Condensed Matter Physics, Winter 2017

Due: Thursday, February 16, 2017 by 11pm
Put homework in mailbox labelled 223B on 1st floor of Broida (by elevators).

1.) ANISOTROPIC FERROMAGNET

A simple model Hamiltonian for an anisotropic Heisenberg ferromagnet is

$$\hat{\mathcal{H}} = -\sum_{\langle \mathbf{r}\mathbf{r}'\rangle} [J(\hat{S}^x_{\mathbf{r}}\hat{S}^x_{\mathbf{r}'} + \hat{S}^y_{\mathbf{r}}\hat{S}^y_{\mathbf{r}'}) + J_z \hat{S}^z_{\mathbf{r}}\hat{S}^z_{\mathbf{r}'}], \tag{1}$$

where $\mathbf{r}$ and $\mathbf{r}'$ label sites of a d-dimensional (hyper-) cubic lattice, and the sum is over near neighbor sites. In this problem we assume unequal ferromagnetic exchange couplings with, $J_z > J > 0$.

a.) What is a possible physical origin of the anisotropy in the exchange interactions?

b.) For arbitrary spin-S, show that the classical ferromagnetic ground state,

$$|G\rangle = \prod_{\mathbf{r}} |S^z_{\mathbf{r}} = S\rangle, \tag{2}$$

is an exact ground state for the above quantum Hamiltonian.

c.) For large spin S use the Holstein-Primakoff transformation to obtain a quadratic magnon Hamiltonian that describes the low energy spin waves,

$$\hat{H}_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}. \tag{3}$$

Sketch the resulting magnon dispersion relation, $\omega_{\mathbf{k}}$, as a function of k_x (for $k_y = k_z = 0$) and show that it has a gap in contrast to the gapless excitations in the isotropic ferromagnet. Describe the symmetry which has been spontaneously broken, and discuss why the symmetry breaking does *not* lead to gapless excitations in this case.

d.) For low temperatures, $k_B T << J_z - J, J$, calculate the deviation of the spontaneous magnetization from saturation, $\Delta M_z(T) = M_z(T=0) - M_z(T)$. Contrast your result with Bloch's $T^{3/2}$ law. Why the difference?

2.) NON-MAGNETIC GROUND STATES and VALENCE BOND SOLIDS

Quantum spin models in low dimensions often exhibit ground states which are not magnetic, due to quantum fluctuations. A classic example is the $S = 1$ Heisenberg antiferromagnetic chain, which would be expected to exhibit a magnetically ordered ground state if the spin operators were replaced by classical fixed length vectors. The non-magnetic ground state of this model can be understood by employing a Schwinger boson representation of the spin operators. A single spin-S can be represented in terms of two Schwinger bosons, denoted $\hat{a}$ and $\hat{b}$, as:

$$\hat{S}^x + i\hat{S}^y = \hat{a}^\dagger \hat{b}, \quad \hat{S}^x - i\hat{S}^y = \hat{b}^\dagger \hat{a}, \quad \hat{S}^z = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}), \tag{4}$$

where the number of bosons is constrained to satisfy, $\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} = 2S$.

a.) Using the boson commutation relations, (ie. $[\hat{a}, \hat{a}^\dagger] = 1$), show that the spin components $\hat{S}^\alpha$ satisfy the standard canonical spin-commutation relations.

Consider a (generalized) one-dimensional $S = 1$ Heisenberg antiferromagnet with Hamiltonian

$$\hat{\mathcal{H}} = J\sum_{\langle ij\rangle} \hat{\mathcal{P}}_2(ij) \quad ; \qquad \hat{\mathcal{P}}_2(ij) \equiv \vec{S}_i \cdot \vec{S}_j + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{2}{3}, \tag{5}$$

where the sum is over nearest neighbors on a 1d lattice. Here, we have added a bi-quadratic term to the standard Heisenberg Hamiltonian.

b.) By introducing a *bond* spin operator, $\vec{J}_{ij} = \vec{S}_i + \vec{S}_j$, show that the operator $\hat{\mathcal{P}}_2(ij)$ projects $\vec{J}_{ij}$ onto the subspace with magnitude $J = 2$ (that is, it anihilates states with $J = 0, 1$). Using this, show that the Hamiltonian is positive definite (ie. that the ground state energy $E_G \geq 0$).

Consider the state constructed in terms of Schwinger bosons:

$$|\Psi\rangle = \prod_{\langle ij \rangle} (\hat{a}_i^\dagger \hat{b}_j^\dagger - \hat{b}_i^\dagger \hat{a}_j^\dagger)|0\rangle, \tag{6}$$

where $|0\rangle$ denotes the vacuum state for bosons (ie. $\hat{a}_i|0\rangle = \hat{b}|0\rangle = 0$). This state is constructed from "valence bonds" connecting neighboring sites. One can show that the state $|\Psi\rangle$ is invariant under spin rotations. Thus, this state is non-magnetic. It is usually referred to as a "valence bond solid".

c.) In the state $|\Psi\rangle$, how many total bosons $(\hat{a}_i^\dagger \hat{a}_i + \hat{b}_i^\dagger \hat{b}_i)$ are there on each site?

d.) In the state $|\Psi\rangle$ what is the largest value, J_{max}, taken by $\hat{J}_{ij}^z$? (Hint: Use $\hat{S}_i^z = (\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i)/2$).

e.) Since the state $|\Psi\rangle$ is spin-rotationally invariant, J_{max} is also the largest value possible for the total bond spin J. Using this fact, show that $\hat{\mathcal{H}}|\Psi\rangle = 0$. Together with the positivity of $\hat{\mathcal{H}}$ this establishes that the valence bond solid $|\Psi\rangle$ is the *exact* ground state of $\hat{\mathcal{H}}$. It is possible to show that all spin correlations fall of exponentially in this state (even *at* $T = 0$!), so that the valence bond solid is sometimes referred to as a "quantum paramagnet". Quasi-one-dimensional $S = 1$ compounds have been found to exhibit this behavior.