Chapter 1

1. \([AB, CD] = ABCD - CDAB = ABCD + ACBD - ACDB + ACDB + CDB - CDB - CDAB = A(C, B)D - AC(D, B) + (C, A)DB - C(D, A)B.\)

2. (a) \(X = a_0 + \mathbf{I}a_x \sigma_x\), \(\text{tr}(X) = 2a_0\) because \(\text{tr}(\sigma_x) = 0\). Next evaluate 
\[
\text{tr}(\sigma_k X) = \text{tr}(\sum_L a_L \sigma_L \sigma_k) = \sum_L a_L 2 \delta_{Lk} = 2a_k \quad \text{(where we have used}
\]
\[
\text{tr}(\sigma_i \sigma_j) = \text{tr}(\lambda_i \sigma_i \sigma_j + \sigma_j \sigma_i) = 2 \delta_{ij}. \quad \text{Hence } a_0 = \frac{1}{4} \text{tr}(X), \quad a_k = \frac{1}{4} \text{tr}(\sigma_k X).\]

(b) \(a_0 = \frac{1}{2}(X_{11} + X_{22}), \) while \(a_k\) can be explicitly evaluated from \(a_k = \frac{1}{4} \text{tr}(\sigma_k X)\) with \(X = [X_{ij}]\) and \(i, j = 1, 2\). The result is \(a_1 = \frac{1}{4}(X_{11} + X_{21})\), \(a_2 = \frac{1}{2}(-X_{21} + X_{12})\), and \(a_3 = \frac{1}{2}(X_{11} - X_{22})\).

3. \(\hat{\sigma} \cdot \hat{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix},\)

\[
\text{det}(\hat{\sigma} \cdot \hat{a}) = -|\hat{a}|^2.\]

Without loss of generality, choose \(\hat{n}\) along positive \(z\)-direction, then \(\exp(\mathbf{i} \sigma_z \hat{n} \phi/2) = \mathbf{1} \cos \phi/2 + \mathbf{i} \sigma_z \sin \phi/2\), and if \(B\) is defined to be \(B = \cos \phi/2 + \mathbf{i} \sin \phi/2\), then
\[
\exp(\mathbf{i} \sigma_z \phi/2) \hat{\sigma} \cdot \hat{a} \exp(-\mathbf{i} \sigma_z \phi/2) = \begin{pmatrix} a_z B^* B & (a_x - ia_y) b^2 \\ (a_x + ia_y) b^2 & -a_z B^* B \end{pmatrix}.\]

Since \(B^* B = \cos^2 \phi/2 + \sin^2 \phi/2 = 1\), \(\det(\exp(\mathbf{i} \sigma_z \phi/2) \hat{\sigma} \cdot \hat{a} \exp(-\mathbf{i} \sigma_z \phi/2)) = - (a_z^2 + a_x^2 + a_y^2) = -|\hat{a}|^2\), that is determinant is
Invariant under specified operation. Next we note
\[
\begin{pmatrix}
    a_z' & a_{x'-ia_y}' \\
    a_{x'-ia_y}' & a_z'
\end{pmatrix}
\begin{pmatrix}
    a_z \\
    a_x + ia_y
\end{pmatrix}
= \begin{pmatrix}
    a_z \\
    a_x - ia_y
\end{pmatrix}
(\cos \phi + i \sin \phi)
\begin{pmatrix}
    \frac{a_z}{a_x - ia_y} \\
    \frac{a_{x'-ia_y}}{a_x + ia_y}
\end{pmatrix}
(\cos \phi - i \sin \phi)
\]

hence \( a_z' = a_z \), \( a_x' = a_x \cos \phi + a_y \sin \phi \), \( a_y' = a_x \cos \phi - a_y \sin \phi \). This is a
counter-clockwise rotation about \( z \)-axis through angle \( \phi \) in \( x-y \) plane.

4. (a) Note \( \text{tr}(XY) = \sum_a <a' | X | a''> <a'' | Y | a'> \) (by
closure property) = \( a', a'' <a'' | Y | a'> <a' | X | a''> \) (by rearrangement) =
\( \sum_a <a'' | X | a''> \). Since \( a'' \) is a dummy summation variable, relabel \( a'' = a' \),
\( \text{hence \( \text{tr}(XY) = \text{tr}(YX) \).} \)

(b) \( <(XY)^+a'|a''> = <a' | (XY)^+ | a''> = <a' | XY | a''> = <X^+a'|Y|a''> \)
\( = <Y^+X^+a'|a''>. \) Therefore \( (XY)^+ = Y^+X^+. \)

(c) Take \( \exp[i\theta(A)]a> = (1 + i\theta(A)_{\frac{\theta(A)}{2}!} + \ldots) |a> \)
\( = (1 + i\theta(a)_{\frac{\theta(a)}{2}!} + \ldots) |a> = \exp[i\theta(a)] |a>, \) where we
\( \text{assume that } A |a> = a |a>. \) Therefore \( \exp[i\theta(A)] = \)
\( \exp[i\theta(a)] |a><a|, \) where closure property of the complete set
\( \{|a>\} \) has been used.

(d) \( \sum_a \psi_a (x') \psi_a^*(x'') = \sum_a <x' | a'> <x'' | a'> = \sum_a a' |x'> x'' \)
\( <x'' | a'> = \sum_a <x'' | a'> a' |x'> = <x'' | x'>. \)

5. (a) \( |a><\beta| = \sum_a \sum_{a''} a' |a'><\beta|a''><a''|a'> \)
\( = \sum_a \sum_{a''} a' |a''><\beta|a''><a''|a'> \times
\( <a' |a'><a''|\beta>*). \) Hence \( |a><\beta| = [<a'|a> <a'|\beta>*], \) where
expression inside square bracket is the \((i,j)\) matrix element.

(b) \(|a> = |s_z = \frac{\hbar}{2} = |>\), \(|b> = |s_x = \frac{\hbar}{2} = \frac{1}{2}\chi(|> + |>)\).

Hence

\[
|a><b| = \begin{pmatrix} \langle +|a><b^* & \langle +|a><b^* \\ \langle -|a><b^* & \langle -|a><b^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]

6. Given \(A|> = a_1|>\) and \(A|j> = a_j|j>\). The normalized state vector \(|> + |j>\) is of form \(|\psi> = \frac{1}{2\chi}(|> + |j>)\). Hence \(A|\psi> = (1/\sqrt{2})[a_1|> - a_j|j>\) where \(a_1, a_j\) are real numbers if \(A\) is Hermitian; but for \(a_1 \neq a_j\) clearly r.h.s. is a state vector distinct from \(|\psi>\). However under the condition that \(|>\) and \(|j>\) are degenerate (i.e. \(a_1 = a_j = a\)), then \(A|\psi> = a[(1/\sqrt{2})(|> + |j>) = a|\psi>\) and \(|\psi>\) or \(|> + |j>\) is also an eigenket of \(A\).

7. (a) Let \(|\xi> \in \{a'>\}\) and \(A|a'> = a'|a'>\). Then since \(\Pi_a (A - a')|\xi>\) is a product over all eigenvalues, and \(|\xi> = \frac{1}{a'} |a'>a'\xi>\) must therefore satisfy \(\Pi_a (A - a')|\xi> = 0\). Hence \(\Pi_a (A - a')\) is the null operator.

(b) \(\Pi_a (A - a') \frac{(a'-a')}{(a'-a')} |a'> = \Pi_a \frac{(a'-a')}{(a'-a')} |a'> = |a'>\).

Hence \(\xi >= \frac{1}{a'} \frac{(A-a')}{(a'-a')} |a'>a'\xi> = |a'>a'\xi>\). The operator therefore projects out of ket \(|\xi>\), its \(|a'>\) component.
(c) Let \( \mathbf{A} = S_z \), then \( \mathbf{a}, (S_z - a') = (S_z - \mathbf{A}/2)(S_z + \mathbf{A}/2) \). Hence evidently

\[ a' \mathbf{A}/2 (S_z - a') |\pm> = 0. \]

This verifies (a) above. For case (b)

we have \( \theta_+ = (S_z + \mathbf{A}/2)/\mathbf{A}, \theta_- = -(S_z - \mathbf{A}/2)/\mathbf{A} \) and \( S_z = \mathbf{A}/2(\pm |+> + |-> \times

<+|) \) while ket \( |\xi> = |+><+||\xi> + |->><-||\xi> \). Hence \( \theta_+ |\xi> = <+<|\xi> \leftrightarrow \) and \( \theta_- |\xi> = <-<-|\xi> \leftrightarrow \) and \( \theta_0 \) are the projection operators of \( |\xi> \) to \(|\pm> \) states.

8.

The orthonormality property is \( <+|+> = <-|-> = 1, <+|-> = <-|+> = 0. \)

Hence using the explicit representations of \( S_i \) in terms of linear combinations of bra-ket products, we obtain by elementary calculation

\( [S_i, S_j] = ic_{ijk} S_k \) and \( \{S_i, S_j\} = (\mathbf{A}/2)\delta_{ij}. \)

9.

Let \( \mathbf{a} = n_x \mathbf{x} + n_y \mathbf{y} + n_z \mathbf{z} \), then \( n_x = \sin \beta \cos \alpha, n_y = \sin \beta \sin \alpha, n_z = \cos \beta \) and \( \mathbf{a} = \sin \beta \cos \alpha S_x + \sin \beta \sin \alpha S_y + \cos \beta S_z \). Also due to

completeness property of the ket space \( |\mathbf{a};+> = a|+> + b|-> \) where \( |a|^2 + |b|^2 = 1 \) (normalization). Therefore the relation \( |\mathbf{a};+> = (\mathbf{A}/2)|\mathbf{a};+> \) [taking advantage of explicit representations \( S_x = \mathbf{A}/2(\pm |+> \times

<+| + |-|\times|+|), S_y = \mathbf{A}/2(\pm |+| \times

<- + |->\times|+|), S_z = \mathbf{A}/2(\pm |+| \times

-> \times|->|) \)]

leads to:

\[
\begin{align*}
\sin \beta \cos \alpha - i \sin \beta \sin \alpha &= a \\
\sin \beta \cos \alpha + i \sin \beta \sin \alpha &= b
\end{align*}
\]

Together with the normalization condition \( |a|^2 + |b|^2 = 1 \), we find

\( a = \cos(\beta/2)e^{i\theta} \) and \( b = \sin(\beta/2)e^{i\theta} \). From equation (1a) we have

\( a = \frac{\sin \beta e^{-i\alpha}}{1 - \cos \beta} \), hence \( e^{i(\theta - \theta_a)} = e^{i\alpha} \). Choose \( \theta_a = 0 \), then \( \theta_b = \alpha \), and

\( |\mathbf{a};+> = \cos(\beta/2)|+> + \sin(\beta/2)e^{i\alpha}|-> \).
10. \( H = a(\lvert 1\rangle\langle 1\rvert - \lvert 2\rangle\langle 2\rvert + \lvert 1\rangle\langle 2\rvert + \lvert 2\rangle\langle 1\rvert) \). Let \( \lvert 1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lvert 2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lvert 1\rangle = (1,0) \) and \( \lvert 2\rangle = (0,1) \), \( H \) can be explicitly written using the outer product of matrices as \( H = a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \).

The eigenvalues and corresponding eigenkets are obtained from \( (H - \lambda I)X = 0 \) where \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) are eigenvectors and \( \lambda \) are corresponding eigenvalues determined from secular equation \( \det (H - \lambda I) = 0 \). This leads to \( \lambda = \pm \sqrt{2a} \) and \( x_2 = (\sqrt{2} - 1)x_1 \), hence \( X = x_1 \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} \) and by normalization of \( X \) we have \( x_1 = \frac{1}{\sqrt{2(2 - \sqrt{2})}} \). Thus eigenvectors and eigenvalues are

\[ \begin{align*}
|\psi_+\rangle &= \frac{\lvert 1\rangle + (\sqrt{2} - 1)\lvert 2\rangle}{\sqrt{2(2 - \sqrt{2})}}, \lambda = \sqrt{2a} \\
|\psi_-\rangle &= \frac{\lvert 1\rangle - (\sqrt{2} + 1)\lvert 2\rangle}{\sqrt{2(2 + \sqrt{2})}}, \lambda = -\sqrt{2a}
\end{align*} \]

11. Rewrite \( H \) as \( H = \frac{1}{2}(H_{11} + H_{22})(\lvert 1\rangle\langle 1\rvert + \lvert 2\rangle\langle 2\rvert) + \frac{1}{2}(H_{11} - H_{22})(\lvert 1\rangle\langle 2\rvert + \lvert 2\rangle\langle 1\rvert) \), where the three operator terms on the r.h.s. behave like \( I, S_z, \) and \( S_x \) respectively. Note that \( \frac{1}{2}(H_{11} + H_{22}) \) is simply the "center of gravity" of the two levels. Because the identity operator \( I \) remains the same under any change of basis, we ignore the \( \frac{1}{2}(H_{11} + H_{22}) \) term for the moment. Compare now with the spin \( \frac{1}{2} \) problem (Problem 9 above). \[ \hat{S}.\hat{n} = \frac{\hbar}{2} n_x (\lvert +\rangle\langle +\rvert + \lvert -\rangle\langle -\rvert) + \frac{\hbar}{2} n_y (\lvert -\rangle\langle +\rvert + \lvert +\rangle\langle -\rvert) + \frac{\hbar}{2} n_z (\lvert +\rangle\langle +\rvert - \lvert -\rangle\langle -\rvert). \] The analogy is: \( (\hbar/2)n_x \rightarrow H_{12}, \)
\[ \frac{\hbar}{2} \gamma \rightarrow 0 \ (\gamma = 0), \ \frac{\hbar}{2} n_z + H_1 (H_{11} - H_{22}). \] So one of the energy eigenkets is \( \cos(\beta/2) |1\rangle \) + \( \sin(\beta/2) |2\rangle \) where \( \beta \), analogous to \( \tan^{-1}(n_x/n_z) \), is given by \( \beta = \tan^{-1}\) \[ \frac{2H_{12}}{H_{11} - H_{22}}. \]

The other energy eigenket can be written down by the orthogonality requirement (or by letting \( \beta = \beta + \pi \)) as \( -\sin(\beta/2) |1\rangle + \cos(\beta/2) |2\rangle \). The energy eigenvalues can be obtained by diagonalizing

\[
\begin{pmatrix}
\frac{1}{2}(H_{11} - H_{22}) & H_{12} \\
H_{22} & -\frac{1}{2}(H_{11} - H_{22})
\end{pmatrix}.
\]

But they can also be obtained by comparing with the spin \( \frac{1}{2} \) problem:

\[
\left( \frac{\hbar}{2} n_x \right)^2 + \left( \frac{\hbar}{2} n_z \right)^2 = \hbar^2/4 \rightarrow \text{eigenvalue } \hbar/2,
\]

so by analogy the eigenvalue in our case is \( [\frac{1}{2}(H_{11} - H_{22})^2 + H_{12}^2]^{1/2} \). We must still add to this the center of gravity energy. The final answer is

\[
\frac{1}{2}(H_{11} + H_{22}) + [\frac{1}{2}(H_{11} - H_{22})^2 + H_{12}^2]^{1/2}
\]

where \( z \) is the analogue of parallel (anti-parallel) spin direction to \( \hat{n} \). For \( H_{12} = 0 \), we get \( \gamma = 0 \) or \( \pi \). The eigenvalues are \( \frac{1}{2}(H_{11} + H_{22}) \pm [\frac{1}{2}(H_{11} - H_{22})^2 + H_{12}^2]^{1/2} \)

a very reasonable result.

12. Here \( \hat{S} \cdot \hat{n} |\hat{n} \rangle \rangle = \frac{\hbar}{2} |\hat{n} \rangle \rangle \) and \( |\hat{n} \rangle \rangle = \cos(\gamma/2) |\rangle \rangle + \sin(\gamma/2) |\rangle \rangle = \left( \begin{array}{c}
\cos(\gamma/2) \\
\sin(\gamma/2)
\end{array} \right) \). It is easily seen that the eigenket of \( S_x \) belonging to eigenvalue \( \pm \hbar/2 \), is \( \frac{1}{2} \left( \begin{array}{c}
1 \\
1
\end{array} \right) \).

Thus (a) probability of getting \( \pm \hbar/2 \) when \( S_x \) is measured is \( \frac{1}{2} \left[ \frac{1}{2} \left( \begin{array}{c}
\cos(\gamma/2) \\
\sin(\gamma/2)
\end{array} \right) \right]^2 \)

\[
= \frac{1+\sin \gamma}{2}.
\]

(b) \( <S_x> = \frac{\hbar}{2} (\cos \gamma, \sin \gamma) \left( \begin{array}{c}
0 \\
1
\end{array} \right) \left( \begin{array}{c}
\cos(\gamma/2) \\
\sin(\gamma/2)
\end{array} \right) = \frac{\hbar}{2} \sin \gamma. \)

Hence \( <S_x^2> = <S_x>^2 = \frac{\hbar^2}{4} - (\frac{\hbar^2}{4}) \sin^2 \gamma = (\frac{\hbar^2}{4}) \cos^2 \gamma. \)

Answers are entirely reasonable for \( \gamma = 0, \pi \) (parallel and anti-parallel to \( OZ \)), and for \( \gamma = \pi/2 \) (along \( OX \)).
13. Choosing the $S_z$ diagonal basis, the first measurement corresponds to the operator $M(\mp) = |+\rangle\langle+|$. The second measurement is expressed by the operator $M(\mp;\hat{n}) = |+\rangle\langle+;\hat{n}|$ where $|+\rangle\langle+;\hat{n}| = \cos(\theta/2)|\rangle\langle+| + \sin(\theta/2)|\rightarrow\rangle\langle-|$ with $\theta = 0$. Therefore

$$M(\mp;\hat{n}) = (\cos^2_{\theta}|\rangle\langle+| + \sin^2_{\theta}|\rightarrow\rangle\langle-|) + \cos^2_{\theta}\sin^2_{\theta}|\rangle\langle+;\hat{n}| + \sin^2_{\theta}|\rightarrow\rangle\langle-;\hat{n}|.$$

The final measurement corresponds to the operator $M(-) = |\rightarrow\rangle\langle-|$, and the total measurement $M = M(-)M(\mp;\hat{n})M(\mp) = |\rightarrow\rangle\langle-| (\cos^2(\theta/2)|\rangle\langle+| + \sin^2(\theta/2)|-\rangle\langle-| + |\rangle\langle+| + \sin^2(\theta/2)|-\rangle\langle-|) |\rangle\langle+| = \cos^2_{\theta}\sin^2_{\theta}|\rangle\langle+|$. The intensity of the final $S_z = -\frac{\hbar}{2}$ beam, when the $S_z = \frac{\hbar}{2}$ beam surviving the first measurement is normalized to unity, is thus $\cos^2(\theta/2)\sin^2(\theta/2) = (\sin^2\theta)/4$. To maximize $S_z = -\frac{\hbar}{2}$ final beam, set $\theta = \pi/2$, i.e. along OX, and intensity is $\frac{1}{4}$.

14. (a) The eigenvalues and eigenvectors of $3\times3$ matrix representation

$$A = (1/\sqrt{2}) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

can be obtained by solving $\det[A - \lambda I] = 0$ and normalized eigenvectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $[A-\lambda I]\mathbf{x} = 0$ and $x_1^2 + x_2^2 + x_3^2 = 1$. The eigenvalues are $+1$, $0$, $-1$ and the corresponding eigenvectors are respectively

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

There is no degeneracy. (b) These are the eigenvalues and eigenvectors of $J_\perp = \hbar A$ for a spin 1 particle.

15. Yes! Proof uses completeness and orthonormality of $\{|a', b'>\}$, hence

$$[A, B] = \sum_{a', b', a'', b''} \epsilon_{a', b', a'', b''} (a'', b'')\langle a', b'| (AB-BA)|a', b'> <a', b'| (BA-AB)|a', b' > a', b' > = 0,$$

but $(AB-BA)|a', b'> = (a'b' - b'a')|a', b' > = 0$, hence $[A, B] = 0$. An alternative
16. \( (A, B) = AB + BA = 0 \). This implies that \( \langle a'' | \{A, B\} | a' \rangle = \langle a'' | AB | a' \rangle + \langle a'' | BA | a' \rangle = (a'' + a') \langle a'' | B | a' \rangle = 0 \). In general \( a'' + a' \neq 0 \), so \( \langle a'' | B | a' \rangle \) must vanish for \( a'' = a' \) as well as \( a'' \neq a' \), hence it is not possible to have a simultaneous eigenket of \( A \) and \( B \). The "trivial" case is when \( a'' + a' = 0 \), then \( \langle a'' | B | a' \rangle \neq 0 \) necessarily, and simultaneous eigenket of \( A \) and \( B \) would appear to be possible. But note \( A|a', b'\rangle = a'|a', b'\rangle, B|a', b'\rangle = b'|a', b'\rangle + (AB + BA)|a', b'\rangle = (a'b' + b'a')|a', b'\rangle = 0 \). Hence \( a' = 0 \), or \( b' = 0 \), or \( a' = b' = 0 \). Thus nontrivial simultaneous eigenkets are possible but at the cost that the eigenvalues of one or the other (or both) of operators \( A \) and \( B \) are zero.

17. No degeneracy implies \( |n\rangle \) defined by \( H |n\rangle = E_n |n\rangle \) is unique, i.e. only one energy eigenstate when \( E_n \) is given. Now \( [A_1, H] = 0 \Rightarrow [A_1, H] |n\rangle = 0 \) or \( H(A_1 |n\rangle) = E_n (A_1 |n\rangle) \), i.e. \( A_1 |n\rangle \) is an energy eigenket with eigenvalue \( E_n \). The non-degeneracy assumption then implies \( A_1 |n\rangle \) is proportional to \( |n\rangle \), viz. \( A_1 |n\rangle = a_1 |n\rangle \) and likewise \( A_2 |n\rangle = a_2 |n\rangle \). But we are given that \( [A_1, A_2] \neq 0 \), hence \( A_1 A_2 |n\rangle \neq A_2 A_1 |n\rangle \) or \( a_2 a_1 |n\rangle \neq a_2 a_1 |n\rangle \), and this is clearly impossible, hence energy eigenstates are, in general, degenerate. Note however this proof fails if \( A_1 |n\rangle = 0 \) (or \( A_2 |n\rangle = 0 \)). For \( H = \frac{p^2}{2m} + V(x) \), \( L_x \) and \( L_z \) both commute with \( H \) and \( [L_x, L_z] \neq 0 \), so energy eigenstates are usually degenerate (2\( z+1 \) fold degeneracy). The exception is for S-state (\( \ell = 0, m_\ell = 0 \)) where \( L_z |n, \ell = 0, m_\ell = 0\rangle = 0 \), hence there need not be degeneracy in this case.

18. (a) This is solved in (1.4.56) and (1.4.57) of text. Basically we set \( \lambda = -\langle \beta | a' \rangle / \langle \beta | \beta \rangle \) in \( \langle a' + \lambda \beta | \beta \rangle > 0 \), and obtain Schwarz inequality
\[ <a'|b>|b\rangle \cdot |a|b\rangle = |a|b\rangle^2. \]

(b) The generalized uncertainty relation (1.4.59) is \( <\Delta A>^2<\Delta B>^2 > |<\Delta A\Delta B>|^2 \)

where according to (1.4.63) \( |<\Delta A\Delta B>|^2 = k|<[A,B]>|^2 + k|<[\Delta A,\Delta B]>|^2 \). From

(1.4.50) we know that \( \Delta A = A - <A> \) and \( \Delta B = B - <B> \) and \( \Delta A|a> = \lambda \Delta B|a> = \lambda |a> \) as given.

An elementary calculation leads to \([A,B] = [\Delta A,\Delta B]\), hence \( <a|[A,B]|a> = <a| \times [\Delta A,\Delta B]|a> = \lambda^* <a|\Delta B|a> - \lambda <a|\Delta A|a> \). Choose next \( \lambda \) to be purely imaginary;

\[ <a|[A,B]|a> = -2\lambda <a|\Delta B|^2|a> \]

while \( k|<A|[A,B]|a>|^2 = |\lambda|^2 <a|\Delta B|^2|a> \). It is also evident that for \( \lambda \) imaginary \( <a|[\Delta A,\Delta B]|a> = 0 \), therefore from (1.4.63) and the recognition that \( <a|[\Delta A|^2|a> = <a|\Delta B|^2|a> = |\lambda|^2 <a|\Delta B|^2|a> \)

we have equality in the generalized uncertainty relation (1.4.59).

(c) Since \( \Delta x = x - <x> \), we may express \( <x'|\Delta x|a> \) as \( \int dx'' <x'|x''><x''|\Delta x|a> = \int dx'' \delta(x''-x')x''|a> = \int dx'' 4\pi \delta(x''-x')x''|a> \)

where normalization \( <x'|x''> = \delta(x''-x') \) is chosen. For \( \Delta p = p - <p> \) where \( p = -i\hbar \frac{\delta}{\delta x} \), we have \( <x'|\Delta p|a> = \int dx'' <x'|x''><x'''|\Delta p|a> \)

and \( <x''|p|a> = -(i\hbar \frac{\delta}{\delta x} |x''|a> \). Hence \( <x'|\Delta p|a> = \int dx'' 4\pi \delta(x''-x')x''|a> \)

\( \times (-i\hbar \frac{\delta}{\delta x} |x''|a> - <p>) \int dx'' 4\pi \delta(x''-x')x''|a> \). Use next explicit expression for \( <x''|a> = \frac{1}{\sqrt{2\pi}} \exp \left[ \frac{i<p|x''> - (x''-<x'>)^2}{\hbar} \right] \)

in above integral forms for \( <x'|\Delta x|a> \)

and \( <x'|\Delta p|a> \). We find

\[ <x'|\Delta x|a> = \Lambda <x'|\Delta p|a> \]

where \( \Lambda = -2i\hbar |p> \) an imaginary number.

9. (a) It is clear that

\[ <a|S_x|a> = \sum_a \sum_a a|a><a |S_x|a><a|a>=\frac{\hbar^2}{4} <a|a>|^2 <a|S_x|a> \]

where \( \{|a>\} \) is a complete set of base kets. Since \( S_x = \frac{\hbar}{2} (|+><-|+><+|) \), evidently \( S_x^2 = \frac{\hbar^2}{4} (|+><-|+><+|) \). Take \( |a> = |+> \) then \( |+|S_x^2|+> = \frac{\hbar^2}{4} \) and \( |+|S_x^2|+> = 0 \). Therefore

\[ ++|<\Delta S_x>|^2|++> = |++|S_x^2|++> - |++|S_x|++|^2 = \frac{\hbar^2}{4}. \]

Also from \( S_y = \frac{\hbar}{2} (|+><-|+><+|) \), we have \( S_y^2 = \frac{\hbar^2}{4} (|+><|+><-|) \), hence it can be readily shown that \( |+|S_y^2|+> = \frac{\hbar^2}{4} \) and \( |+|S_y|+> = 0 \). Therefore \( ++|<\Delta S_y>|^2|++> \)
\[ [S_x, S_y] \|+\| = i\hbar \|+\| = i\hbar^2/2. \] The generalized uncertainty relation is therefore verified for the equality case.

(b) From \( |\tilde{\alpha};\|+\| = \cos^{\frac{\beta}{2}} |+\| + e^{i\alpha} \sin^{\frac{\beta}{2}} |-\| \) it follows for \( \beta = \pi/2 \) and \( \alpha = 0 \) we have
\[ [S_x; \|+\| = \frac{1}{2}\hbar (|+\| + |-\|). \] Simple calculations lead to \( <S_x; |S_x|S_x; \|+\| = \hbar/2 \) and
\[ <S_x; |S_x|S_x; \|+\| = \hbar^2/4, \] therefore \( <S_x; |(\Delta S_x)^2|S_x; \|+\| = 0. \) Again \( [S_x, S_y] = i\hbar S_z \), hence \( <S_x; |S_y|S_x; \|+\| = 0 \) and \( <S_x; |(\Delta S_x)^2|S_x; \|+\| <S_x; |(\Delta S_y)^2|S_x; \|+\| \) both sides of generalized uncertainty relation being zero.

(20) Take the normalized linear combination \( |\tilde{\alpha};\|+\| = |\alpha;\|+\| + (1-\alpha^2)^{\frac{1}{2}} e^{i\beta} |\tilde{\alpha};\|+\| \), where \( \alpha \) is real and \( |\alpha| < 1 \). Then elementary calculations yield \( <(\Delta S_x)^2| = \frac{\hbar^2}{4} [1-4\alpha^2 (1-\alpha^2) \cos^2 \beta] \) and \( <(\Delta S_y)^2| = \frac{\hbar^2}{4} (1-4\alpha^2 (1-\alpha^2) \sin^2 \beta). \) The product
\[ <(\Delta S_x)^2| <(\Delta S_y)^2| = \frac{\hbar^4}{16} [1-2\alpha^2 (1-\alpha^2)]^2. \] Maximum for \( \sin^2 \beta \) is when \( \beta = \pi/4 \), and r.h.s. becomes \( \frac{\hbar^4}{16} [1-2\alpha^2 (1-\alpha^2)]^2. \) It is clear that \( \alpha^2 = \frac{1}{2} \) is a minimum, and the maximum value \( \frac{\hbar^4}{16} \) is reached when \( \alpha^2 = 0 \), or \( \alpha^2 = 1 \). Hence the linear 'combination' that maximizes uncertainty product is \( e^{i\pi/4} |\tilde{\alpha};\|+\| \) or \( \tilde{\alpha} = \|+. \) That \( \tilde{\alpha} = \|+ \) does not violate uncertainty relation has been proved in Problem 19(a) above. For the \( e^{i\pi/4} |\tilde{\alpha};\|+\| \) case, we note that the phase factor \( e^{i\pi/4} \) cancels out in the scalar product, and \( <\tilde{\alpha} = (<S_x = <S_y = 0 \) while
\[ <\tilde{S}^2_x| = <\tilde{S}^2_y| = \hbar^2/4. \] Again \( <\tilde{[S_x, S_y]| = <\tilde{i}\hbar S_z = \hbar/2 = -\hbar^2/2. \) Hence explicitly we have \( <\tilde{[S_x, S_y]| = <\tilde{[S_x, S_y]| = \hbar^4/16 = \hbar|<\tilde{[S_x, S_y]|^2, \) again no violation.
21. This is the rigid wall potential (one-dimensional box), c.f. (A.2.3) and (A.2.4) of Appendix A. The wave functions and energy eigenstates are \( \psi_n(x) = \sqrt{2/a} \sin(n\pi x/a), \ n=1,2,3,\ldots, \) and \( E_n = \frac{n^2\pi^2}{2m^2}, \ n=1 \) is ground state \( n>1 \) are the excited states. Next note that
\[
\langle \Delta x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2, \quad \langle \Delta p \rangle^2 = \langle p^2 \rangle - \langle p \rangle^2
\]
where \( p = \frac{\hbar}{i\partial_x} \) and \( p^2 = -\hbar^2\partial^2/\partial x^2. \) For rigid wall potential, we have
\[
\langle x^2 \rangle = \frac{2}{a} \int_0^a x^2 \sin^2(n\pi x/a) \, dx = 2a^2 \left[ \frac{1}{6} - \frac{1}{4n^2 \pi^2} \right] = a^2 \left[ \frac{1}{3} - \frac{1}{2n^2 \pi^2} \right]
\]
\[
\langle x \rangle = \frac{2}{a} \int_0^a x \sin(n\pi x/a) \, dx = \frac{a}{n\pi}
\]
\[
\langle p^2 \rangle = \frac{\hbar^2}{a} \int_0^a \sin^2(n\pi x/a) \left( -\frac{\hbar^2}{a^2 \partial_x^2} \right) \sin(n\pi x/a) \, dx = \frac{\hbar^2}{a^2} \left[ \frac{1}{3} \right]
\]
\[
\langle p \rangle = \frac{\hbar}{a} \int_0^a \sin(n\pi x/a) \left( -\frac{\hbar}{a^2} \right) \sin(n\pi x/a) \, dx = 0.
\]
Therefore the uncertainty product \( \langle \Delta x \rangle^2 \langle \Delta p \rangle^2 = \frac{a^2}{2} \left[ \frac{1}{6} - \frac{1}{2n^2 \pi^2} \right] \frac{\hbar^2}{a^2} \left[ \frac{1}{3} \right] = \frac{\hbar^2}{2} \left[ (n\pi)^2/6 - 1 \right] \); for ground state \( n=1 \) for excited states \( n>1 \).

22. Assume that the ice pick is equivalent to a mass point \( m \) attached to a light rod of length \( L \) the other end of which is balanced on a fixed hard surface. For small angle \( \theta \) departure of pick from vertical, the torque equation is \( mL^2 d^2 \theta/dt^2 = mgL \), and solution \( \theta(t) = \alpha \sqrt{g/L} \sin \left( \sqrt{g/L} \right) t \). The uncertainty relation at \( t=0 \) with \( \Delta x = L \theta = (a-b)L \), \( \Delta p = m\omega \sqrt{L} \\theta = m\sqrt{gL}(a-b) \) is \( \Delta x \Delta p = \hbar/2 \) (best we can do and realized for Gaussian packet). Now \( \Delta x \Delta p = \hbar/2 \) implies \( a^2 = b^2 + \hbar/(2m[gL^3]^3) \). The displacement at later time \( t \) is minimized by making \( a \) and \( b \) as small as possible. So set \( a = \sqrt{\hbar/(2m[gL^3]^3)} \), \( b = 0 \) (actually irrelevant for \( t >> \sqrt{g/L} \)). Displacement becomes noticeable when \( \theta \) becomes as large as \( \theta_f = \pi/100 \approx 2^o \). We have \( \theta_f = \alpha \sqrt{g/L} \) and taking for definiteness \( a = \sqrt{\hbar/(2m[gL^3]^3)} \), \( t_f = \sqrt{g/L} \left[ \ln \theta_f + \sin^{-1}(\frac{gL^3}{\hbar}) \right] \). Use \( L = 10 \text{ cm}, m = 100 \text{ gm}, \) and \( g = 980 \text{ cm/sec}^2 \); we have \( t_f = 3.4 \text{ s} \). Actually this number is very insensitive.
to \( \lambda \) and \( \Theta_f \). For any reasonable value, we get \( t_f \approx 3 \) sec.

23. (a) The characteristic equation \( \det(b - \lambda I) = 0 \), leads to \((\lambda - b)^2(\lambda + b) = 0 \). Hence \( \lambda = \pm b \) and \( \lambda = b \) is a two-fold degenerate eigenvalue.

(b) Straightforward matrix multiplication gives

\[
AB = \begin{pmatrix}
ab & 0 & 0 \\
0 & 0 & \text{i}ab \\
0 & \text{i}ab & 0
\end{pmatrix}
\]

hence \([A, B] = 0\).

(c) The eigenvectors (eigenkets) of \( B \), together with \([A, B] = 0\), yield simultaneous eigenvectors of \( A \) and \( B \). Let \( \lambda_1 \) be eigenvalues of \( B \), and corresponding eigenvectors are

\[
u^1 = \begin{pmatrix} u^1_1 \\ u^1_2 \\ u^1_3 \end{pmatrix}, \text{ where } Bu^1_1 = \lambda_1 u^1_1, \text{ i.e., } u^1_{i=1,2,3}.
\]

For \( \lambda_1 = b \), we have \( Bu^1_1 = bu^1_1 \), \( iBu^2_2 = bu^1_3 \), and \( Bu^3_3 = u^1_1 \). Choose \( u^1_1 = 1, u^1_2 = u^1_3 = 0 \) as a basis.

For the degenerate \( \lambda_2 = b \), we have \( Bu^2_1 = bu^2_1 \) and \( iu^2_2 = u^2_3 \). But \( u^2 \) must be orthogonal to \( u^1 \), hence \( u^2_1 = 0 \). Therefore we choose \( u^2_1 = 0, u^2_2 = 1, u^2_3 = i \), and the normalized

\[
u^2 = \frac{1}{2\sqrt{i}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \frac{1}{2\sqrt{i}} (|2\rangle + i|3\rangle), \text{ where } |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

For nondegenerate \( \lambda_3 = -b \), again \( u^3 \) must be orthogonal to \( u^1 \) and \( u^2 \), therefore \( u^3_1 = 0 \) and relation \( iu^3_2 = -u^3_3 \) can be satisfied by choosing \( u^3_2 = 1, u^3_3 = -i \). Together with normalization we have

\[
u^3 = \frac{1}{2\sqrt{i}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = \frac{1}{2\sqrt{i}} (|2\rangle - i|3\rangle).
\]

In this new set \( u_i^k (i=1,2,3) \), evidently \( Au^1 = au^1, Au^2 = -au^2, Au^3 = -au^3 \), and there
is two fold-degeneracy w.r.t. eigenvalue $-\alpha$ of operator $A$.

24. (a) The rotation matrix [c.f. (3.2.44)] acting on a two-component spinor can be written as $\exp[-i\sigma_\alpha \hat{n}_d / 2] = \frac{1}{2} \cos \frac{\theta}{2} - i \sigma_\alpha \frac{\sin \frac{\theta}{2}}{2}$. For clockwise rotation about $x$-axis through $-\pi/2$, we have $\theta = -\pi/2$, hence $\exp[-i\sigma_x \hat{n}_d / 2] = \frac{1}{2} (1 - i \sigma_x)$.

(b) If we transform from base kets in $S_z$ representation to eigenkets of $S_y$ as base kets, i.e. rotate by angle $-\pi/2$ about $x$-axis, $S_z$ is transformed into

$$\frac{1}{2} (1 - i \sigma_x) \hat{n}_z (1 - i \sigma_x) \hat{n}_z = -\frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(This can be seen by noting that if $|c\rangle$ is $S_z$ basis while $|b\rangle$ is $S_y$ basis, than transformation is

$$\langle c''|S_z|c'\rangle = b''_c b''_c \langle c''|b''|S_z|b''\rangle.$$

25. Given $\langle b'|A|b''\rangle$ is real. Take another basis $|c\rangle$, then $|c\rangle = \sum b''_c |b''|c'\rangle$, hence $\langle c'|A|c''\rangle = (\sum b''_c \langle c'|b''\rangle A \sum b''_c \langle b''|c''\rangle) = b''_c \sum b''_c \langle c'|b''\rangle \langle b''|c''\rangle \times \langle b'|A|b''\rangle$. It is not necessary that $\langle c'|b'\rangle$ and $\langle b''|c''\rangle$ be real. Take the $S_y$ and $S_z$ cases of problem 24 above. Here $|b'\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$ while $|b''\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |--\rangle$. Hence $\langle c'|b'\rangle = 1/\sqrt{2} = \langle c''|b''\rangle$, but $\langle c''|b''\rangle = 1/\sqrt{2} = -\langle c'|b'\rangle$ are imaginary.

26. From problems 9 and 19, we have $|S_z;+\rangle = \frac{1}{2}(|\rangle+|--\rangle)$, i.e. $\alpha = 0, \beta = \pi/2$ in $|\hat{S}_z;\rangle$. Now $|S_z;--\rangle$ corresponds to axis of quantization in the $-x$ direction, i.e. $\alpha=\pi, \beta = \pi/2$, hence $|S_z;--\rangle = \frac{1}{\sqrt{2}}(|\rho\rangle+|--\rangle)$. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be the transformation matrix between $S_z$ diagonal basis and $S_x$ diagonal basis, i.e.

$$\begin{pmatrix} |S_z;+\rangle \\ |S_z;--\rangle \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |+\rangle \\ |--\rangle \end{pmatrix} = U|\rangle$$

than evidently $U_{11} = U_{12} = 1/\sqrt{2}$ while $U_{21} = 1/\sqrt{2}$ and $U_{22} = -1/\sqrt{2}$. Take $|S_z;+\rangle$
27. (a) Matrix element \( \langle b'' | f(A) | b' \rangle = \frac{\pi}{4}, \langle b'' | f(A) | a' \rangle = \frac{\pi}{4}, f(a') = b''|a'\rangle = b''|b'\rangle \)
where \( a'|b' \rangle \) (likewise \( b''|a'\rangle \)) is an element of the transformation matrix from the \( a' \) basis to the \( b' \) basis. (b) The matrix element \( \langle \vec{p}'' | F(\vec{r}) | \vec{p}' \rangle = -i \hbar^2 \psi(\vec{r})^* \psi(\vec{r}) \times \langle \vec{p}'' | \vec{r}' \rangle \langle \vec{r}' | \vec{p}' \rangle \). Note that \( \frac{\psi(\vec{r})^* \psi(\vec{r})}{\hbar^2} = \frac{1}{2(2\pi \hbar)^2} e^{i q \cdot \vec{r}' / \hbar} \).
Suppose \( F(\vec{r}) \) is spherically symmetric, then (choosing \( \vec{z} \)-axis along \( \vec{p}' - \vec{p}'' \))

\[
\langle \vec{p}'' | F(\vec{r}) | \vec{p}' \rangle = \frac{2\pi}{(2\pi \hbar)^3} \int_0^{\infty} r^2 dr \sin(q r) e^{i q r / \hbar} \frac{\psi(\vec{r})^* \psi(\vec{r})}{\hbar^2} \theta(\vec{r}') \psi(\vec{r}')^* \psi(\vec{r}')\]
where \( q = |\vec{p}' - \vec{p}''| \). Integrate out the \( \sin \theta \) integration on r.h.s. we have

\[
\langle \vec{p}'' | F(\vec{r}) | \vec{p}' \rangle = \frac{1}{2\pi \hbar^2 q} \int_0^{\infty} r \sin(q r / \hbar) F(r) dr
\]

28. (a) \([x, F(p_x)]_{CL} = \frac{\partial x}{\partial x} \frac{\partial F}{\partial p_x} - \frac{\partial x}{\partial p_x} \frac{\partial F}{\partial x} = 0\), hence \([x, F(p_x)]_{CL} = \frac{\partial F}{\partial p_x}\).
(b) Now \([x, F(p_x)]_{QM} = i\hbar [x, F(p_x)]_{CL}\), hence

\[
[x, \exp[ip_x a/\hbar)]_{QM} = i\hbar \frac{\partial}{\partial p_x} \exp[ip_x a/\hbar] = -a \exp[ip_x a/\hbar].
\]
(c) Using (b) we have

\([x, \exp[ip_x a/\hbar)]_{QM} = -a \exp[ip_x a/\hbar] x\).

Hence \(x \exp[ip_x a/\hbar)]x\) = \(i x a \exp[ip_x a/\hbar] x\), and hence \(x \exp[ip_x a/\hbar)]x\) = \(a x \exp[ip_x a/\hbar] x\). This equation implies that \(\exp[ip_x a/\hbar)]x\) is an eigenvalue of coordinate operator \(x\), with corresponding eigenvalue \((x-a)\).

29. (a) We assume that \(G(p)\) and \(F(x)\) can be expressed as a power series

\[
G(p) = \sum_{n,m} a_{nm} p^n m l, F(x) = \sum_{n,m} b_{nl} x^n m l
\]
An elementary calculation yields \[ x_1, p_1, \ldots, p_n \] = \[ n! Kp_1^{n-1} p_2 \ldots p_n \] (use \( x_1, A \)) = \[ x_1, A \] + \( A[x_1, B] + (AB)[x_1, C] \) and \[ p_1, x_1, \ldots, p_n \] = \[ -ni \hbar x_1, x_1^{n-1} p_2 \ldots p_n \]. These relationships \( x_1, p_1 \) = \[ ni \hbar x_1 \] and \( p_1, x_1 \) = \[ -ni \hbar x_1 \] can be easily proved by mathematical induction. Using the series form for \( G(\vec{r}) \) and \( F(\vec{r}) \), we get at once \( \{ x_1, C(\vec{r}) \} = i\hbar C(\vec{r}) / \partial x_1 \) and \( \{ p_1, F(\vec{r}) \} = -i\hbar F(\vec{r}) / \partial x_1 \).

(b) \( [x^2, p^2] = [x^2, pp] = [x^2, p]p + p[x^2, p] \), but from \( p, x^2 = -2i\hbar x \), so \( [x^2, p^2] = 2i\hbar xp + 2i\hbar px = 2i\hbar(x, p) \). The classical P.B. for \( [x^2, p^2] \) is evaluated via \( \{ x^2, p^2 \} = 3x^2 \partial^2 / \partial p^2 - 3x^2 \partial^2 / \partial p \partial x = 2x(2p) = 4xp \). Since in the classical limit \( x, p \) \[ x^2, p^2 \] \[ \{ x^2, p^2 \} = 2xp \), we have \( [x^2, p^2] = \{ x^2, p^2 \} \).

30. (a) \( \{ x_1, T(\vec{z}) \} = i\hbar T(\vec{z}) / \partial x_1 \) = \( i\hbar \) \[ \partial x_1 \] \[ \exp(-i\hbar \vec{z} / \hbar) \] = \( i\hbar (-i\hbar / \hbar) \exp(-i\hbar \vec{z} / \hbar) \) = \( \vec{z} \). (b) Noting that \( \{ x_1, a \} = a \{ x_1, \} \) where \( \{ a \} \) is a general state ket,

31. Given \( \{ x, T(d\vec{x}) \} = d\vec{x} \) or \( \{ x, T(d\vec{x}) \} = d\vec{x} + T(d\vec{x}) \), we study \( \{ a, T(d\vec{x}) \} \) substituting as we did in problem 30.

32. Use of \( \{ x', a \} = \frac{1}{d^2} \exp(ikx' - x'^2/2d^2) \), we find by elementary calculation

\[ \frac{2}{d^2} \frac{x'^2}{d^2} (1k-2x'/d^2) \exp(ikx' - x'^2/2d^2) + \frac{3}{d^2} \frac{x'^3}{d^2} \exp(ikx' - x'^2/2d^2) \] = \[ \frac{1}{d^2} \frac{x'^2}{d^2} (1k-2x'/d^2) \exp(ikx' - x'^2/2d^2). \]
(a) \( \langle p \rangle = \int_0^{\infty} \langle a | x' > [-i \hbar \frac{\partial}{\partial x'}] < x' | a \rangle dx' = -i \hbar \int_0^{\infty} \frac{dx}{d} \exp(-x'^2/d^2)(ikx'/d^2)dx' \).

The odd term of integrand vanishes, and \( \langle p \rangle = \left( \frac{\hbar k}{d} \right)^{1/2} \int_0^{\infty} \exp(-x'^2/d^2)dx' = \frac{\hbar k d}{d^{1/2}} \).

Likewise \( \langle p^2 \rangle = \int_0^{\infty} \langle a | x' > x'^2 < x' | a \rangle dx' = \int_0^{\infty} \langle a | x' > \left( -\hbar^2 \frac{\partial^2}{\partial x'^2} \right) 2x'^2 < x' | a \rangle dx' = -\frac{\hbar^2}{d} \int_0^{\infty} \exp(-x'^2/d^2)\left[ x'^2/d^4 - k^2 - 1/d^2 - 2ikx'/d^2 \right]dx' = \hbar^2/2d^2 + \hbar^2k^2 \), again dropping odd terms in integrand.

(b) \( \langle p | a \rangle = \frac{d}{\hbar k^{1/2}} \exp[-(p - \hbar k)^2 d^2/2\hbar^2] \). The expectation value \( \langle p \rangle \) using momentum space wave function is then:

\[ \langle p \rangle = \int_0^{\infty} \langle a | p \rangle \langle p | a \rangle dp = \frac{d}{\hbar k^{1/2}} \int_0^{\infty} \exp[-(p - \hbar k)^2 d^2/2\hbar^2] dp. \]

Change variable to \( q = p - \hbar k \), we have \( \langle p \rangle = \left( \frac{d}{\hbar k^{1/2}} \right) \int_0^{\infty} \exp[-q^2 d^2/2\hbar^2] dq \), and dropping the odd integration contribution

\[ \langle p \rangle = \left( \frac{d}{\hbar k^{1/2}} \right) \hbar k \left( \frac{d}{k^{1/2}} / d \right) = \hbar k. \]

Similarly

\[ \langle p^2 \rangle = \int_0^{\infty} \left( \frac{d}{\hbar k^{1/2}} \right) p^2 \exp[-(p - \hbar k)^2 d^2/2\hbar^2] dp \]

and changing variable to \( q = p - \hbar k \) (hence \( p^2 = q^2 + 2q \hbar k + \hbar^2 k^2 \)), we have

\[ \langle p^2 \rangle = \left( \frac{d}{\hbar k^{1/2}} \right) \int_0^{\infty} \exp[-q^2 d^2/2\hbar^2] dq \]

\[ = \left( \frac{d}{\hbar k^{1/2}} \right) \left[ \frac{k^3 \sqrt{\pi} / 2 d^3 + \hbar^2 \pi \hbar^2 k^2 / d!} = \hbar^2 / 2d^2 + \hbar^2 k^2. \]

33. (a) To prove (i) \( \langle p' | x | a \rangle = i \hbar \frac{\partial}{\partial p'}, \langle p' | a \rangle \), let us note that

\[ \langle p' | x | p'' \rangle = \int \langle p' | x | x' > < x' | p'' \rangle dx' = \int \langle x | cp' | x' > < x' | p'' \rangle dx' \]

\[ = \left[ 1/(2\pi \hbar) \right] / dx' x'e^{-i x' \cdot (p' - p'') / \hbar} \]

But \( \delta(p' - p'') = \left[ 1/(2\pi \hbar) \right] / dx' e^{-i x' \cdot (p' - p'') / \hbar} \), so \( \frac{\hbar}{3} \delta(p' - p'') = \int \frac{dx'}{2\pi \hbar} x'e^{-i x' \cdot (p' - p'') / \hbar} \).

hence \( \langle p' | x | p'' \rangle = i \hbar \frac{\partial}{\partial p'}, \delta(p' - p'') \). Express now \( \langle p' | x | a \rangle = \int dp'' \langle p' | x | p'' \rangle \langle p'' | a \rangle = \int dp'' i \hbar \frac{\partial}{\partial p'}, \delta(p' - p'') \rangle (p'' | a \rangle = i \hbar \frac{\partial}{\partial p'}, \langle p' | a \rangle. \)
For (ii) we perform an analogous procedure. Write

$$\langle \beta | x | \alpha \rangle = \int dp' \langle \beta | p' \rangle \langle p' | x | \alpha \rangle = \int dp' \langle \beta | p' \rangle \hat{W}^3_{p'} \langle p' | \alpha \rangle = \int dp' \phi^*_{\beta}(p') \hat{W}^3_{p'} \phi_{\alpha}(p').$$

(b) Consider momentum eigenket with eigenvalue $p'$. Then $p | p' \rangle = p' | p' \rangle$. Now consider the ket $| p', \Xi \rangle = \exp(i\Xi \hat{x}) | p' \rangle$. Is this a momentum eigenket and if yes what is the value? To see this let's operate with $p$, than

$$p | p', \Xi \rangle = p \exp(i\Xi \hat{x}) | p' \rangle = \{ \exp(i\Xi \hat{x})p + [p, \exp(i\Xi \hat{x})] \} | p' \rangle$$

and $[p, \exp(i\Xi \hat{x})] = -i\hat{W}(\exp(i\Xi \hat{x}))/\Xi x = -i\hat{W}(i\Xi \hat{x}) \exp(i\Xi \hat{x})$. So $p | p', \Xi \rangle = \exp(i\Xi \hat{x})p | p' \rangle + \Xi \exp(i\Xi \hat{x}) | p' \rangle = (p' + \Xi) | p', \Xi \rangle$. Hence $| p', \Xi \rangle$ is eigenket of $p$ with eigenvalue $p' + \Xi$ and operator $\exp(i\Xi \hat{x})$ is momentum translation operator and $x$ is the generator of momentum translations.