Chapter 1

1. $[AB,CD] = ABCD - CDAB = ABCD + ACBD - ACBD - ACDB + ACDB + CADB - CADB - CDAB = A\{C,B\}D - AC\{D,B\} + \{C,A\}DB - C\{D,A\}B.$

2. (a) $X = a_o + \Sigma a_\ell \sigma_\ell$, $tr(X) = 2a_o$ because $tr(\sigma_\ell) = 0$. Next evaluate $tr(\sigma_k X) = tr(\sum_\ell a_\ell \sigma_k \sigma_\ell) = \sum_\ell a_\ell 2\delta_{\ell k} = 2a_k$ (where we have used $tr(\sigma_i \sigma_j) = tr(\frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i)) = 2\delta_{ij}$). Hence $a_o = \frac{1}{2}\, tr(X)$, $a_k = \frac{1}{2}\, tr(\sigma_k X)$.

(b) $a_o = \frac{1}{2}(X_{11} + X_{22})$, while a_k can be explicitly evaluated from $a_k = \frac{1}{2}\, tr(\sigma_k X)$ with $X = [X_{ij}]$ and $i,j = 1,2$. The result is $a_1 = \frac{1}{2}(X_{12} + X_{21})$, $a_2 = \frac{i}{2}(-X_{21} + X_{12})$, and $a_3 = \frac{1}{2}(X_{11} - X_{22})$.

3.

$$\vec{\sigma}.\vec{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix},$$

$$\det(\vec{\sigma}.\vec{a}) = -|\vec{a}|^2.$$

Without loss of generality, choose $\hat{n}$ along positive z-direction, then $\exp(\pm i\vec{\sigma}.\hat{n}\phi/2) = \underline{1}\cos\phi/2 \pm i\sigma_z \sin\phi/2$, and if B is defined to be $B \equiv \cos\phi/2 + i\sin\phi/2$, then

$$\exp(i\sigma_z\phi/2)\vec{\sigma}.\vec{a}\exp(-i\sigma_z\phi/2) = \begin{pmatrix} a_z B^* B & (a_x - ia_y)B^2 \\ (a_x + ia_y)B^{*2} & -a_z B^* B \end{pmatrix}.$$

Since $B^*B = \cos^2\phi/2 + \sin^2\phi/2 = 1$, $\det[\exp(i\sigma_z\phi/2)\vec{\sigma}.\vec{a} \times \exp(-i\sigma_z\phi/2)] = -(a_z^2 + a_x^2 + a_y^2) = -|\vec{a}|^2$, that is determinant is

invariant under specified operation. Next we note

$$\vec{\sigma}.\vec{a}' = \begin{pmatrix} a'_z & a'_x - ia'_y \\ a'_x + ia'_y & -a'_z \end{pmatrix} =$$

$$= \begin{pmatrix} a_z & (a_x - ia_y)(\cos\phi + i\sin\phi) \\ (a_x + ia_y)(\cos\phi - i\sin\phi) & -a_z \end{pmatrix}$$

hence $a'_z = a_z$, $a'_x = a_x\cos\phi + a_y\sin\phi$, $a'_y = a_y\cos\phi - a_x\sin\phi$. This is a counter-clockwise rotation about z-axis through angle ϕ in x-y plane.

4. (a) Note $tr(XY) = \sum_{a'} \langle a'|XY|a'\rangle = \sum_{a',a''} \langle a'|X|a''\rangle\langle a''|Y|a'\rangle$ (by closure property) $= \sum_{a',a''} \langle a''|Y|a'\rangle\langle a'|X|a''\rangle$ (by rearrangement) $= \sum_{a''} \langle a''|YX|a''\rangle$. Since a" is a dummy summation variable, relabel $a'' = a'$, hence $tr(XY) = tr(YX)$.

(b) $\langle (XY)^\dagger a'|a''\rangle = \langle a'|[(XY)^\dagger]^\dagger|a''\rangle = \langle a'|XY|a''\rangle = \langle X^\dagger a'|Y|a''\rangle = \langle Y^\dagger X^\dagger a'|a''\rangle$. Therefore $(XY)^\dagger = Y^\dagger X^\dagger$.

(c) Take $\exp[if(A)]|\alpha\rangle = (1 + if(A) - \frac{[f(A)]^2}{2!} + \ldots\ldots)|\alpha\rangle$ $= (1 + if(\alpha) - \frac{[f(\alpha)]^2}{2!} + \ldots\ldots)|\alpha\rangle = \exp[if(\alpha)]|\alpha\rangle$, where we assume that $A|\alpha\rangle = \alpha|\alpha\rangle$. Therefore $\exp[if(A)] = \sum_\alpha \exp[if(\alpha)]|\alpha\rangle\langle\alpha|$, where closure property of the complete set $\{|\alpha\rangle\}$ has been used.

(d) $\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'') = \sum_{a'} \langle\vec{x}'|a'\rangle^*\langle\vec{x}''|a'\rangle = \sum_{a'} \langle a'|\vec{x}'\rangle \times \langle\vec{x}''|a'\rangle = \sum_{a'} \langle\vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle = \langle\vec{x}''|\vec{x}'\rangle$.

5. (a) $|\alpha\rangle\langle\beta| = \sum_{a'}\sum_{a''} |a'\rangle\langle a'|\alpha\rangle\langle\beta|a''\rangle\langle a''| = \sum_{a'}\sum_{a''} |a'\rangle\langle a''| \times (\langle a'|\alpha\rangle\langle a''|\beta\rangle^*)$. Hence $|\alpha\rangle\langle\beta| = [\langle a^{(i)}|\alpha\rangle\langle a^{(j)}|\beta\rangle^*]$, where

expression inside square bracket is the (i,j) matrix element.

(b) $|\alpha\rangle = |s_z = \hbar/2\rangle = |+\rangle$, $|\beta\rangle = |s_x = \hbar/2\rangle = \frac{1}{2^{1/2}}[|+\rangle + |-\rangle]$.

Hence

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle+|\alpha\rangle\langle+|\beta^*\rangle & \langle+|\alpha\rangle\langle-|\beta^*\rangle \\ \langle-|\alpha\rangle\langle+|\beta^*\rangle & \langle-|\alpha\rangle\langle-|\beta^*\rangle \end{pmatrix}$$

$$= 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

6. Given $A|i\rangle = a_i|i\rangle$ and $A|j\rangle = a_j|j\rangle$. The normalized state vector $|i\rangle + |j\rangle$ is of form $|\psi\rangle = \frac{1}{2^{1/2}}[|i\rangle + |j\rangle]$. Hence $A|\psi\rangle = (1/\sqrt{2})[a_i|i\rangle + a_j|j\rangle]$ where a_i, a_j are real numbers if A is Hermitian; but for $a_i \neq a_j$ clearly r.h.s. is a state vector distinct from $|\psi\rangle$. However under the condition that $|i\rangle$ and $|j\rangle$ <u>are degenerate</u> (i.e. $a_i = a_j = a$), then $A|\psi\rangle = a[(1/\sqrt{2})(|i\rangle + |j\rangle)] = a|\psi\rangle$ and $|\psi\rangle$ or $|i\rangle + |j\rangle$ is also an eigenket of A.

7. (a) Let $|\xi\rangle \in \{|a'\rangle\}$ and $A|a'\rangle = a'|a'\rangle$. Then since $\prod_{a'}(A - a')|\xi\rangle$ is a product <u>over all eigenvalues</u>, and $|\xi\rangle = \sum_{a'}|a'\rangle\langle a'|\xi\rangle$ must therefore satisfy $\prod_{a'}(A-a')|\xi\rangle = 0$. Hence $\prod_{a'}(A-a')$ is the null operator.

(b) $\prod_{a''\neq a'} \frac{(A-a'')}{(a'-a'')}|a'\rangle = \prod_{a''\neq a'} \frac{(a'-a'')}{(a'-a'')}|a'\rangle = |a'\rangle$.

Hence $\theta|\xi\rangle = \prod_{a''\neq a'} \frac{(A-a'')}{(a'-a'')}|\xi\rangle = |a'\rangle\langle a'|\xi\rangle$. The operator therefore projects out of ket $|\xi\rangle$, its $|a'\rangle$ component.

(c) Let $A = S_z$, than $\prod_a, (S_z - a') = (S_z - \hbar/2)(S_z + \hbar/2)$. Hence evidently $\prod_{a'=\pm\hbar/2} (S_z - a')|\pm\rangle = 0$. This verifies (a) above. For case (b) we have $\theta_+ = (S_z + \hbar/2)/\hbar$, $\theta_- = -(S_z - \hbar/2)/\hbar$ and $S_z = \hbar/2(|+\rangle\langle+| - |-\rangle \times \langle-|)$ while ket $|\xi\rangle = |+\rangle\langle+|\xi\rangle + |-\rangle\langle-|\xi\rangle$. Hence $\theta_+|\xi\rangle = \langle+|\xi\rangle|+\rangle$ and $\theta_-|\xi\rangle = \langle-|\xi\rangle|-\rangle$ and $\theta_\pm$ are the projection operators of $|\xi\rangle$ to $|\pm\rangle$ states.

8. The orthonormality property is $\langle+|+\rangle = \langle-|-\rangle = 1$, $\langle+|-\rangle = \langle-|+\rangle = 0$. Hence using the explicit representations of S_i in terms of linear combinations of bra-ket products, we obtain by elementary calculation $[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$ and $\{S_i, S_j\} = (\hbar^2/2)\delta_{ij}$.

9. Let $\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$, then $n_x = \sin\beta\cos\alpha$, $n_y = \sin\beta\sin\alpha$, $n_z = \cos\beta$ and $\vec{S}.\hat{n} = \sin\beta\cos\alpha\, S_x + \sin\beta\sin\alpha\, S_y + \cos\beta\, S_z$. Also due to completeness property of the ket space $|\vec{S}.\hat{n};+\rangle = a|+\rangle + b|-\rangle$ where $|a|^2 + |b|^2 = 1$ (normalization). Therefore the relation $\vec{S}.\hat{n}|\vec{S}.\hat{n};+\rangle = (\hbar/2)|\vec{S}.\hat{n};+\rangle$ [taking advantage of explicit representations $S_x = \frac{\hbar}{2}(|+\rangle \times \langle-| + |-\rangle\langle+|)$, $S_y = \frac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|)$, $S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$] leads to :-

$$(\sin\beta\cos\alpha - i\sin\beta\sin\alpha)b + \cos\beta\, a = a \tag{1a}$$

$$(\sin\beta\cos\alpha + i\sin\beta\sin\alpha)a - \cos\beta\, b = b \tag{1b}$$

Together with the normalization condition $|a|^2 + |b|^2 = 1$, we find $a = \cos(\beta/2)e^{i\theta_a}$ and $b = \sin(\beta/2)e^{i\theta_b}$. From equation (1a) we have $a = \frac{\sin\beta\, e^{-i\alpha}b}{(1-\cos\beta)}$, hence $e^{i(\theta_b - \theta_a)} = e^{i\alpha}$. Choose $\theta_a = 0$, then $\theta_b = \alpha$, and $|\vec{S}.\hat{n};+\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)e^{i\alpha}|-\rangle$.

10. $H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$. Let $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, $\langle 1| = (1,0)$ and $\langle 2| = (0,1)$, H can be explicitly written using outer product of matrices as

$$H = a\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

The eigenvalues and corresponding eigenkets are obtained from $(H - \lambda I)X = 0$ where $X = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ are eigenvectors and λ are corresponding eigenvalues determined from secular equation $\det(H - \lambda I) = 0$. This leads to $\lambda = \pm\sqrt{2}a$ and $x_2 = (\pm\sqrt{2} - 1)x_1$, hence $X = x_1\begin{pmatrix}1\\ \pm\sqrt{2} - 1\end{pmatrix}$ and by normalization of X we have $x_1 = \frac{1}{\sqrt{2(2\mp\sqrt{2})}}$. Thus eigenvectors and eigenvalues are

$$|\psi_1\rangle = \frac{|1\rangle + (\sqrt{2} - 1)|2\rangle}{\sqrt{2(2 - \sqrt{2})}}, \quad \lambda = \sqrt{2}a$$

$$|\psi_2\rangle = \frac{|1\rangle - (\sqrt{2} + 1)|2\rangle}{\sqrt{2(2 + \sqrt{2})}}, \quad \lambda = -\sqrt{2}a$$

11. Rewrite H as $H = \frac{1}{2}(H_{11} + H_{22})(|1\rangle\langle 1| + |2\rangle\langle 2|) + \frac{1}{2}(H_{11} - H_{22})(|1\rangle \times \langle 1| - |2\rangle\langle 2|) + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|)$, where the three operator terms on r.h.s. behave like I, S_z, and S_x respectively. Note that $\frac{1}{2}(H_{11} + H_{22})$ is simply the "center of gravity" of the two levels. Because the identity operator I remains the same under any change of basis, we ignore the $\frac{1}{2}(H_{11} + H_{22})$ term for the moment. Compare now with the spin $\frac{1}{2}$ problem [problem 9 above]. $\vec{S}.\hat{n} = \frac{\hbar}{2} n_x(|+\rangle\langle -| + |-\rangle\langle +|) + \frac{\hbar}{2} n_y(-i|+\rangle \times \langle -| + i|-\rangle\langle +|) + \frac{\hbar}{2} n_z(|+\rangle\langle +| - |-\rangle\langle -|)$. The analogy is: $(\hbar/2)n_x \to H_{12}$,

$\frac{\hbar}{2}n_y \to 0$ $(\alpha=0)$, $\frac{\hbar}{2}n_z \to \frac{1}{2}(H_{11}-H_{22})$. So one of the energy eigenkets is $\cos(\beta/2)|1\rangle + \sin(\beta/2)|2\rangle$ where β, analogous to $\tan^{-1}(n_x/n_z)$, is given by $\beta = \tan^{-1}[\frac{2H_{12}}{(H_{11}-H_{22})}]$. The other energy eigenket can be written down by the orthogonality requirement (or by letting $\beta\to\beta+\pi$) as $-\sin(\beta/2)|1\rangle + \cos(\beta/2)|2\rangle$. The energy eigenvalues can be obtained by diagonalizing

$$\begin{pmatrix} \frac{1}{2}(H_{11}-H_{22}) & H_{12} \\ H_{12} & -\frac{1}{2}(H_{11}-H_{22}) \end{pmatrix}.$$

But they can also be obtained by comparing with the spin ½ problem:

$$(\frac{\hbar}{2}n_x)^2 + (\frac{\hbar}{2}n_z)^2 = \hbar^2/4 \to \text{eigenvalue } \hbar/2,$$

so by analogy the eigenvalue in our case is $[\frac{1}{4}(H_{11}-H_{22})^2 + H_{12}^2]^{\frac{1}{2}}$. We must still add to this the center of gravity energy. The final answer is

$$\frac{1}{2}(H_{11}+H_{22}) \pm [\frac{1}{4}(H_{11}-H_{22})^2 + H_{12}^2]^{\frac{1}{2}}$$

where $\pm$ is the analogue of parallel (anti-parallel) spin direction to $\hat{n}$. For $H_{12} = 0$, we get $\beta = 0$ or π. The eigenvalues are $\frac{1}{2}(H_{11}+H_{22}) \pm \frac{1}{2}(H_{11}-H_{22}) = \{ H_{11}, H_{22}$, a very reasonable result.

12. Here $\vec{S}.\hat{n}|\hat{n};+\rangle = \frac{\hbar}{2}|\hat{n};+\rangle$ and $|\hat{n};+\rangle = \cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle = \begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix}$. It is easily seen that the eigenket of S_x belonging to eigenvalue $+\hbar/2$, is $\frac{1}{2^{\frac{1}{2}}}\begin{pmatrix}1\\1\end{pmatrix}$. Thus (a) probability of getting $+\hbar/2$ when S_x is measured is $\left|\frac{1}{2^{\frac{1}{2}}}(1,1)\begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix}\right|^2 = \frac{1+\sin\gamma}{2}$. (b) $\langle S_x\rangle = \frac{\hbar}{2}(\cos\frac{\gamma}{2}, \sin\frac{\gamma}{2})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\gamma/2)\\ \sin(\gamma/2)\end{pmatrix} = \frac{\hbar}{2}\sin\gamma$. Hence $\langle(S_x-\langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = \hbar^2/4 - (\hbar^2/4)\sin^2\gamma = (\hbar^2/4)\cos^2\gamma$. Answers are entirely reasonable for $\gamma = 0,\pi$ (parallel and anti-parallel to OZ), and for $\gamma = \pi/2$ (along OX).

13. Choosing the S_z diagonal basis, the first measurement corresponds to the operator $M(+) = |+\rangle\langle+|$. The second measurement is expressed by the operator $M(+;\hat{n}) = |+;\hat{n}\rangle\langle+;\hat{n}|$ where $|+;\hat{n}\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle$ with $\alpha = 0$. Therefore

$$M(+;\hat{n}) = (\cos\tfrac{\beta}{2}|+\rangle + \sin\tfrac{\beta}{2}|-\rangle)(\cos\tfrac{\beta}{2}\langle+| + \sin\tfrac{\beta}{2}\langle-|)$$

$$= \cos^2(\beta/2)|+\rangle\langle+| + \cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2}(|+\rangle\langle-| + |-\rangle\langle+|) + \sin^2(\beta/2)|-\rangle\langle-|.$$

The final measurement corresponds to the operator $M(-) = |-\rangle\langle-|$, and the total measurement $M_T = M(-)M(+;\hat{n})M(+) = |-\rangle\langle-|\{\cos^2(\beta/2)|+\rangle\langle+| + \cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2}(|+\rangle\langle-| + |-\rangle\langle+|) + \sin^2(\beta/2)|-\rangle\langle-|\}|+\rangle\langle+| = \cos\tfrac{\beta}{2}\sin\tfrac{\beta}{2}|-\rangle\langle+|$. The intensity of the final $S_z = -\hbar/2$ beam, when the $S_z = \hbar/2$ beam surviving the first measurement is normalized to unity, is thus $\cos^2(\beta/2)\sin^2(\beta/2) = (\sin^2\beta)/4$. To maximize $S_z = -\hbar/2$ final beam, set $\beta = \pi/2$, i.e. along OX, and intensity is $\frac{1}{4}$.

14. (a) The eigenvalues and eigenvectors of 3×3 matrix representation

$$A = (1/\sqrt{2})\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

can be obtained by solving $\det[A - \lambda I] = 0$ and normalized eigenvectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $[A-\lambda I]x = 0$ and $x_1^2+x_2^2+x_3^2 = 1$. The eigenvalues are +1, 0, −1 and the and the corresponding eigenvectors are respectively

$$\tfrac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},\ (1/\sqrt{2})\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

There is no degeneracy. (b) These are the eigenvalues and eigenvectors of $J_x = \hbar A$ for a spin 1 particle.

15. Yes! Proof uses completeness and orthonormality of $\{|a',b'\rangle\}$, hence

$$[A,B] = \sum_{a',b'}\sum_{a'',b''}|a'',b''\rangle\langle a'',b''|(AB-BA)|a',b'\rangle\langle a',b'|;$$

but $(AB-BA)|a',b'\rangle = (a'b'-b'a')|a',b'\rangle = 0$, hence $[A,B] = 0$. An alternative

$a'',b''|0\rangle$ possible because $\{|a',b'\rangle\}$ form a complete orthonormal set. Then $[A,B]|\alpha\rangle = 0$ because $[A,B]|a',b'\rangle = 0$, but since $|\alpha\rangle$ is arbitrary, $[A,B] = 0$ must hold as an <u>operator</u> equation.

16. $\{A,B\} = AB+BA = 0$. This implies that $\langle a''|\{A,B\}|a'\rangle = \langle a''|AB|a'\rangle + \langle a''|BA|a'\rangle = (a''+a')\langle a''|B|a'\rangle = 0$. In general $a''+a' \neq 0$, so $\langle a''|B|a'\rangle$ must vanish for $a'' = a'$ as well as $a'' \neq a'$, hence it is not possible to have a simultaneous eigenket of A and B. The "trivial" case is when $a''+a' = 0$, then $\langle a''|B|a'\rangle \neq 0$ necessarily, and simultaneous eigenket of A and B would appear to be possible. But note $A|a',b'\rangle = a'|a',b'\rangle$, $B|a',b'\rangle = b'|a',b'\rangle \rightarrow (AB+BA)|a',b'\rangle = (a'b' + b'a')|a',b'\rangle = 0$. Hence $a' = 0$, or $b' = 0$, or $a' = b' = 0$. Thus nontrivial simultaneous eigenkets are possible but at the cost that the eigenvalues of one or the other (or both) of operators A and B are zero.

17. No degeneracy implies $|n\rangle$ defined by $H|n\rangle = E_n|n\rangle$ is <u>unique</u>, i.e. only one energy eigenstate when E_n is given. Now $[A_1,H] = 0 \rightarrow [A_1, H]|n\rangle = 0$ or $H(A_1|n\rangle) = E_n(A_1|n\rangle)$, i.e. $A_1|n\rangle$ is an energy eigenket with eigenvalue E_n. The non-degeneracy assumption then implies $A_1|n\rangle$ is proportional to $|n\rangle$, viz. $A_1|n\rangle = a_1|n\rangle$ and likewise $A_2|n\rangle = a_2|n\rangle$. But we are given that $[A_1,A_2] \neq 0$, hence $A_1A_2|n\rangle \neq A_2A_1|n\rangle$ or $a_1a_2|n\rangle \neq a_2a_1|n\rangle$, and this is clearly impossible, hence energy eigenstates are, in general, degenerate. Note however this proof fails if $A_1|n\rangle = 0$ (or $A_2|n\rangle = 0$). For $H = \vec{p}^2/2m + V(r)$, L_x and L_z both commute with H and $[L_x,L_z] \neq 0$, so energy eigenstates are usually degenerate ($2\ell+1$ fold degeneracy). The exception is for S-state ($\ell=0$, $m_\ell = 0$) where $L_z|n,\ell=0,m_\ell=0\rangle = 0$, hence there need not be degeneracy in this case.

18. (a) This is solved in (1.4.56) and (1.4.57) of text. Basically we set $\lambda = -\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$ in $(\langle\alpha| + \lambda^*\langle\beta|).(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$, and obtain Schwarz inequality

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$.

(b) The generalized uncertainty relation (1.4.59) is $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\Delta B\rangle|^2$ where according to (1.4.63) $|\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4}|\langle [A,B]\rangle|^2 + \frac{1}{4}|\langle\{\Delta A,\Delta B\}\rangle|^2$. From (1.4.50) we know that $\Delta A = A - \langle A\rangle$ and $\Delta B = B - \langle B\rangle$ and $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ as given. An elementary calculation leads to $[A,B] = [\Delta A,\Delta B]$, hence $\langle\alpha|[A,B]|\alpha\rangle = \langle\alpha| \times [\Delta A,\Delta B]|\alpha\rangle = \lambda^*\langle\alpha|\Delta B\Delta B|\alpha\rangle - \lambda\langle\alpha|\Delta B\Delta B|\alpha\rangle$. Choose next λ to be purely imaginary; $\langle\alpha|[A,B]|\alpha\rangle = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$ while $\frac{1}{4}|\langle\alpha|[A,B]|\alpha\rangle|^2 = |\lambda|^2|\langle\alpha|(\Delta B)^2|\alpha\rangle|^2$. It is also evident that for λ imaginary $\langle\alpha|\{\Delta A,\Delta B\}|\alpha\rangle = 0$, therefore from (1.4.63) and the recognition that $\langle\alpha|(\Delta A)^2|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle = |\lambda|^2|\langle\alpha|(\Delta B)^2|\alpha\rangle|^2$, we have equality in the generalized uncertainty relation (1.4.59).

(c) Since $\Delta x = x - \langle x\rangle$, we may express $\langle x'|\Delta x|\alpha\rangle$ as $\int dx''\langle x'|x''\rangle\langle x''|\Delta x|\alpha\rangle = \int dx''\delta(x'-x'')x''\langle x''|\alpha\rangle - \int dx''\delta(x'-x'')\langle x\rangle\langle x''|\alpha\rangle$ where normalization $\langle x'|x''\rangle = \delta(x'-x'')$ is chosen. For $\Delta p = p - \langle p\rangle$ where $p = -i\hbar\frac{\partial}{\partial x}$, we have $\langle x'|\Delta p|\alpha\rangle = \int dx''\langle x'|x''\rangle\langle x''|\Delta p|\alpha\rangle$ and $\langle x''|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x''}\langle x''|\alpha\rangle$. Hence $\langle x'|\Delta p|\alpha\rangle = \int dx''\delta(x'-x'') \times (-i\hbar)\frac{\partial}{\partial x''}\langle x''|\alpha\rangle - \langle p\rangle\int dx''\delta(x'-x'')\langle x''|\alpha\rangle$. Use next explicit expression for $\langle x''|\alpha\rangle = (2\pi d^2)^{-\frac{1}{4}}\exp[\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}]$ in above integral forms for $\langle x'|\Delta x|\alpha\rangle$ and $\langle x'|\Delta p|\alpha\rangle$. We find

$$\langle x'|\Delta x|\alpha\rangle = A\langle x'|\Delta p|\alpha\rangle$$

where $A = -2id^2/\hbar$ an imaginary number.

9. (a) It is clear that $\langle\alpha|S_x|\alpha\rangle = \sum_{a''}\sum_{a'}\langle\alpha|a''\rangle\langle a''|S_x|a'\rangle\langle a'|\alpha\rangle = \sum_{a'}|\langle\alpha|a'\rangle|^2\langle a'|S_x|a'\rangle$ where $\{|a'\rangle\}$ is a complete set of base kets. Since $S_x = \frac{\hbar}{2}(|+\rangle\langle-|+|-\rangle\langle+|)$, evidently $S_x^2 = \frac{\hbar^2}{4}(|+\rangle\langle+|+|-\rangle\langle-|)$. Take $|\alpha\rangle = |+\rangle$ then $\langle+|S_x^2|+\rangle = \hbar^2/4$ and $\langle+|S_x|+\rangle = 0$. Therefore

$$\langle+|(\Delta S_x)^2|+\rangle = \langle+|S_x^2|+\rangle - \langle+|S_x|+\rangle^2 = \hbar^2/4.$$

Also from $S_y = \frac{i\hbar}{2}(-|+\rangle\langle-|+|-\rangle\langle+|)$, we have $S_y^2 = \frac{\hbar^2}{4}(|+\rangle\langle+|+|-\rangle\langle-|)$, hence it can be readily shown that $\langle+|S_y^2|+\rangle = \hbar^2/4$ and $\langle+|S_y|+\rangle = 0$. Therefore $\langle+|(\Delta S_y)^2|+\rangle$

$\times$ $\langle S_y\rangle$. [illegible] $\hbar/16$. But $[S_x,S_y] = i\hbar S_z$ and thus $\langle +|\times [S_x,S_y]|+\rangle = i\hbar\langle +|S_z|+\rangle = i\hbar^2/2$. The generalized uncertainty relation is therefore verified for the equality case.

(b) From $|\hat{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$ it follows for $\beta = \pi/2$ and $\alpha = 0$ we have $|S_x;+\rangle = \frac{1}{2}\sqrt{2}(|+\rangle + |-\rangle)$. Simple calculations lead to $\langle S_x;+|S_x|S_x;+\rangle = \hbar/2$ and $\langle S_x;+|S_x^2|S_x;+\rangle = \hbar^2/4$, therefore $\langle S_x;+|(\Delta S_x)^2|S_x;+\rangle = 0$. Again $[S_x,S_y] = i\hbar S_z$, hence $\langle S_x;+|S_z|S_x;+\rangle = 0$ and $\langle S_x;+|(\Delta S_x)^2|S_x;+\rangle\langle S_x;+|(\Delta S_y)^2|S_x;+\rangle = \frac{1}{4}|\langle S_x;+|\times [S_x,S_y]|S_x;+\rangle|^2$, both sides of generalized uncertainty relation being zero.

[Note explicit $\langle S_x;+|S_z|S_x;+\rangle = \frac{1}{2}\sqrt{2}(\langle +|+\langle -|)[\frac{\hbar}{2}(|+\rangle\langle +|-|-\rangle\langle -|)]\frac{1}{2}\sqrt{2}(|+\rangle+|-\rangle) = 0$ if we use systematically orthonormality conditions $\langle\pm|\pm\rangle = 1$, $\langle\pm|\mp\rangle = 0$.]

20. Take the normalized linear combination $|\ \rangle = \alpha|+\rangle + (1-\alpha^2)^{\frac{1}{2}}e^{i\beta}|-\rangle$, where α is real and $|\alpha| \leq 1$. Than elementary calculations yield $\langle\ |(\Delta S_x)^2|\ \rangle = \frac{\hbar^2}{4}[1-4\alpha^2(1-\alpha^2)\times\cos^2\beta]$ and $\langle\ |(\Delta S_y)^2|\ \rangle = \frac{\hbar^2}{4}(1-4\alpha^2(1-\alpha^2)\sin^2\beta)$. The product

$$\langle\ |(\Delta S_x)^2|\ \rangle\langle\ |(\Delta S_y)^2|\ \rangle = \frac{\hbar^4}{16}\{1-4\alpha^2(1-\alpha^2)+4\alpha^4(1-\alpha^2)^2\sin^2 2\beta\}.$$

Maximum for $\sin^2 2\beta$ is when $\beta = \pi/4$, and r.h.s. becomes $\frac{\hbar^4}{16}[1-2\alpha^2(1-\alpha^2)]^2$. It is clear that $\alpha^2 = \frac{1}{2}$ is a minimum, and the maximum value $\hbar^4/16$ is reached when $\alpha^2 = 0$, or $\alpha^2=1$. Hence the linear 'combination' that maximizes uncertainty product is $e^{i\pi/4}|-\rangle$ or $\pm|+\rangle$. That $\pm|+\rangle$ does not violate uncertainty relation has been proved in Problem 19(a) above. For the $e^{i\pi/4}|-\rangle$ case, we note that the phase factor $e^{i\pi/4}$ cancels out in the scalar product, and $\langle -|S_x|-\rangle = \langle -|S_y|-\rangle = 0$ while $\langle -|S_x^2|-\rangle = \langle -|S_y^2|-\rangle = \hbar^2/4$. Again $\langle -|[S_x,S_y]|-\rangle = \langle -|i\hbar S_z|-\rangle = i\hbar(-\hbar/2) = -i\hbar^2/2$. Hence explicitly we have $\langle -|(\Delta S_x)^2|-\rangle\langle -|(\Delta S_y)^2|-\rangle = \hbar^4/16 = \frac{1}{4}|\langle -|[S_x,S_y]|-\rangle|^2$, again no violation.

21. This is the rigid wall potential ("one-dimensional box"), c.f. (A.2.3) and (A.2.4) of Appendix A. The wave functions and energy eigenstates are $\psi_E(x) = \sqrt{2/a}\,\sin(n\pi x/a)$, $n=1,2,3,\ldots$, a [illegible] $= \hbar^2 n^2\pi^2/2ma^2$, $n=1$ is ground state $n>1$ are the excited states. Next note th[illegible]

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2, \quad \langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$$

where $p = \frac{\hbar\partial}{i\partial x}$ and $p^2 = -\hbar^2\partial^2/\partial x^2$. For rigid wall potential, we have

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2(n\pi x/a)\,[illegible] = 2a^2\left[\frac{1}{6} - \frac{1}{4n^2\pi^2}\right] = a^2[1/3 - 1/2n^2\pi^2]$$

$$\langle x\rangle = \frac{2}{a}\int_0^a x\sin^2(n\pi x/a)dx = a/2$$

$$\langle p^2\rangle = \frac{2}{a}\int_0^a \sin(\frac{n\pi x}{a})(-\hbar^2\partial^2/[illegible])\sin(\frac{n\pi x}{a})dx = \frac{\hbar^2}{a}2(n\pi)^2$$

$$\langle p\rangle = \frac{2}{a}\int_0^a \sin(\frac{n\pi x}{a})(\frac{\hbar\partial}{i\partial x})\sin(\frac{n\pi x}{a})dx = 0 .$$

Therefore the uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{a^2}{2}[1/6 - 1/n^2\pi^2]\frac{\hbar^2}{a}2(n\pi)^2 = \frac{\hbar^2}{2}[(n\pi)^2/6 - 1]$; for ground state $n=1$; for excited states $n>1$.

22. Assume that the ice pick is equivalent to a mass point m attached to a light rod of length L the other end of which is balanced on a fixed hard surface. For small angle θ departure of pick from vertical, the torque equation is $mL^2d^2\theta/dt^2 = mg\theta L$, and solution $\theta(t) = ae^{\sqrt{g/L}\,t} + be^{-\sqrt{g/L}\,t}$. The uncertainty relation at $t=0$ with $\Delta x = L\theta = (a+b)L$, $\Delta p = Lm\,d\theta/dt = \sqrt{g/L}(a-b)Lm = m\sqrt{gL}(a-b)$ is $\Delta x\Delta p = \hbar/2$ (best we can do and realized for Gaussian packet). Now $\Delta x\Delta p = \hbar/2$ implies $a^2 = b^2 + \hbar/(2m[gL^3]^{1/2})$. The displacement at later time t is minimized by making a and b as small as possible. So set $a = t\sqrt{\hbar/(2m[gL^3]^{1/2})}$, $b = 0$ (actually irrelevant for $t \gg \sqrt{L/g}$). Displacement becomes noticeable when θ becomes as large as $\theta_f = \pi/100 \cong 2^\circ$. We have $\theta_f = ae^{\sqrt{g/L}\,t}$ and taking for definiteness $a = +\sqrt{\hbar/(2m[gL^3]^{1/2})}$, $t_f = \sqrt{L/g}[\ln\theta_f + \frac{1}{2}\ln(\frac{m[gL^3]^{1/2}}{\hbar})]$. Use $L = 10$ cm, $m = 100$ gm, and $g = 980$ cm/sec^2, we have $t_f = 3.4$ sec. Actually this number is very insensitive

to m and θ_f. For any reasonable value, we get $t_f \sim 3$ sec.

23. (a) The characteristic equation $\det[B-\lambda I] = 0$, leads to $(\lambda-b)^2(\lambda+b) = 0$. Hence $\lambda = \pm b$ and $\lambda = b$ is a two-fold degenerate eigenvalue.

(b) Straightforward matrix multiplication gives

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = BA, \text{ hence } [A,B] = 0$$

(c) The eigenvectors (eigenkets) of B, together with $[A,B] = 0$, yield simultaneous eigenvectors of A and B. Let λ_i be eigenvalues of B, and corresponding eigenvectors are

$$u^i = \begin{pmatrix} u_1^i \\ u_2^i \\ u_3^i \end{pmatrix}, \text{ where } Bu^i = \lambda_i u^i, \; i=1,2,3.$$

For $\lambda_1 = b$, we have $bu_1^1 = bu_1^1$, $ibu_2^1 = bu_3^1$, and $iu_2^1 = u_3^1$. Choose $u_1^1 = 1$, $u_2^1 = u_3^1 = 0$ than

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle.$$

For the degenerate $\lambda_2 = b$, we have $bu_1^2 = bu_1^2$ and $iu_2^2 = u_3^2$. But u^2 must be orthogonal to u^1, hence $u_1^2 = 0$. Therefore we choose $u_1^2 = 0$, $u_2^2 = 1$, $u_3^2 = i$, and the normalized

$$u^2 = \frac{1}{2^{1/2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \frac{1}{2^{1/2}}(|2\rangle + i|3\rangle), \text{ where } |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For nondegenerate $\lambda_3 = -b$, again u^3 must be orthogonal to u^1 and u^2, therefore $u_1^3 = 0$ and relation $iu_2^3 = -u_3^3$ can be satisfied by choosing $u_2^3 = 1$, $u_3^3 = -i$. Together with normalization we have

$$u^3 = \frac{1}{2^{1/2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = \frac{1}{2^{1/2}}(|2\rangle - i|3\rangle).$$

In this new set u^i $(i=1,2,3)$, evidently $Au^1 = au^1$, $Au^2 = -au^2$, $Au^3 = -au^3$, and there

is two fold-degeneracy w.r.t. eigenvalue $-a$ of operator A.

24. (a) The rotation matrix [c.f. (3.2.44)] acting on a two-component spinor can be written as $\exp[-i\vec{\sigma}.\hat{n}\theta/2] = \underline{1}\cos\frac{\theta}{2} - i\vec{\sigma}.\hat{n}\sin\frac{\theta}{2}$. For clockwise rotation about x-axis through $-\pi/2$, we have $\theta = -\pi/2$, hence $\exp[-i\vec{\sigma}.\hat{n}\theta/2] = \frac{1}{2^{1/2}}(1+i\sigma_x)$.

(b) If we transform from base kets in S_z representation to eigenkets of S_y as base kets, i.e. rotate by angle $-\pi/2$ about x-axis, S_z is transformed into

$$\frac{\hbar}{2}(1/\sqrt{2})(1-i\sigma_x)\sigma_z(1/\sqrt{2})(1+i\sigma_x) = -\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

[This can be seen by noting that if $\{|c\rangle\}$ is S_y basis while $\{|\underline{b}\rangle\}$ is S_z basis, than transformation is

$$\langle c''|S_z|c'\rangle = \sum_{b',b''}\langle c''|b'\rangle\langle b'|S_z|b''\rangle\langle b''|c'\rangle.]$$

25. Given $\langle b'|A|b''\rangle$ is real. Take another basis $\{|c\rangle\}$, then $|c'\rangle = \sum_{b'}|b'\rangle\langle b'|c'\rangle$, hence $\langle c'|A|c''\rangle = (\sum_{b'}\langle c'|b'\rangle\langle b'|)A(\sum_{b''}\langle b''|c''\rangle|b''\rangle) = \sum_{b',b''}\langle c'|b'\rangle\langle b''|c''\rangle \times \langle b'|A|b''\rangle$. It is not necessary that $\langle c'|b'\rangle$ and $\langle b''|c''\rangle$ be real. Take the S_y and S_z cases of problem 24 above. Here $|b'\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |+\rangle$ while $|b''\rangle = \begin{pmatrix}0\\1\end{pmatrix} = |-\rangle$ for S_z and $|c'\rangle = \begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = |S_y;+\rangle$ while for S_y $|c''\rangle = \begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix} = |S_y;-\rangle$. Hence $\langle c'|b'\rangle = 1/\sqrt{2} = \langle c''|b'\rangle$, but $\langle c''|b''\rangle = i/\sqrt{2} = -\langle c'|b''\rangle$ are imaginary.

26. From problems 9 and 19, we have $|S_x;+\rangle = \frac{1}{2^{1/2}}(|+\rangle+|-\rangle)$, i.e. $\alpha = 0$, $\beta = \pi/2$ in $|\vec{S}.\hat{n};+\rangle$. Now $|S_x;-\rangle$ corresponds to axis of quantization in the $-x$ direction, i.e. $\alpha=\pi, \beta = \pi/2$, hence $|S_x;-\rangle = \frac{1}{2^{1/2}}(|+\rangle-|-\rangle)$. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}$ be the transformation matrix between S_z diagonal basis and S_x diagonal basis, i.e.

$$\begin{pmatrix}|S_x;+\rangle \\ |S_x;-\rangle\end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}|+\rangle \\ |-\rangle\end{pmatrix} = U|r\rangle$$

than evidently $U_{11} = U_{12} = 1/\sqrt{2}$ while $U_{21} = 1/\sqrt{2}$ and $U_{22} = -1/\sqrt{2}$. Take $|S_x;+\rangle$

$= U_{11}|+\rangle + U_{12}|-\rangle = \sum_a \langle a|S_x;+\rangle|a\rangle$ while $|S_x;-\rangle = U_{21}|+\rangle + U_{22}|-\rangle = \sum_b \langle b|S_x;-\rangle|b\rangle$ with $a,b = +,-$. Take the general form $U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|$, than $U|r\rangle = \sum_r |b^{(r)}\rangle \times \langle a^{(r)}|r\rangle$. Identify $|a\rangle$ or $|b\rangle$ with $|b^{(r)}\rangle$ and $\langle a^{(r)}|r\rangle$ with $\langle a|S_x;+\rangle$ or $\langle b|S_x;-\rangle$, we see that U can indeed be expressed as $U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|$.

27. (a) Matrix element $\langle b''|f(A)|b'\rangle = \sum_{a'} \langle b''|f(A)|a'\rangle\langle a'|b'\rangle = \sum_{a'} f(a')\langle b''|a'\rangle\langle a'|b'\rangle$ where $\langle a'|b'\rangle$ (likewise $\langle b''|a'\rangle$) is an element of the transformation matrix from the a' basis to the b' basis. (b) The matrix element $\langle \vec{p}''|F(\vec{r})|\vec{p}'\rangle = \int d\vec{r}'F(\vec{r}') \times \langle \vec{p}''|\vec{r}'\rangle\langle \vec{r}'|\vec{p}'\rangle$. Note that $\langle \vec{r}'|\vec{p}'\rangle = [1/(2\pi\hbar)^{3/2}]e^{i\vec{p}'\cdot\vec{r}'/\hbar}$, this implies that

$$\langle \vec{p}''|F(\vec{r})|\vec{p}'\rangle = [1/(2\pi\hbar)^3]\int d\vec{r}'F(\vec{r}')e^{i(\vec{p}'-\vec{p}'')\cdot\vec{r}'/\hbar}.$$

Suppose $F(\vec{r})$ is spherically symmetric $= F(r)$, than (choosing z-axis along $\vec{p}'-\vec{p}''$)

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle = \frac{2\pi}{(2\pi\hbar)^3}\int_{-1}^{+1} d(\cos\theta)\int_0^\infty r'^2dr'F(r')e^{iqr'\cos\theta/\hbar}$$

where $q = |\vec{p}'-\vec{p}''|$. Integrate out the $\cos\theta$ integration on r.h.s. we have

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle = \frac{1}{2\pi^2\hbar^2 q}\int_0^\infty r'\sin(qr'/\hbar)F(r')dr'$$

28. (a) $[x,F(p_x)]_{cl} = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F}{\partial x}$, but $\partial x/\partial p_x = 0$, hence $[x,F(p_x)]_{cl} = \partial F/\partial p_x$.

(b) Now $[x,F(p_x)]_{QM} = i\hbar[x,F(p_x)]_{cl}$, hence

$$[x,\exp(ip_xa/\hbar)]_{QM} = i\hbar\frac{\partial}{\partial p_x}\exp(ip_xa/\hbar) = -a\exp(ip_xa/\hbar).$$

(c) Using (b) we have

$$[x,\exp(ip_xa/\hbar)]|x'\rangle = -a\exp(ip_xa/\hbar)|x'\rangle.$$

Hence $x\exp(ip_xa/\hbar)|x'\rangle - \exp(ip_xa/\hbar)x|x'\rangle = -a\exp(ip_xa/\hbar)|x'\rangle$, and thence $x[\exp(ip_xa/\hbar)|x'\rangle] = (x'-a)[\exp(ip_xa/\hbar)|x'\rangle]$. This eigenvalue equation implies that $\exp(ip_xa/\hbar)|x'\rangle$ is an eigenstate of coordinate operator x, with corresponding eigenvalue $(x'-a)$.

29. (a) We assume that $G(\vec{p})$ and $F(\vec{x})$ can be expressed as a power series

$$G(\vec{p}) = \sum_{n,m,\ell} a_{nm\ell}p_i^n p_j^m p_k^\ell, \quad F(\vec{x}) = \sum_{n,m,\ell} b_{nm\ell}x_i^n x_j^m x_k^\ell.$$

An elementary calculation yields $[x_i, p_i^n p_j^m p_k^\ell] = ni\hbar p_i^{n-1} p_j^m p_k^\ell$ (use $[x_i, ABC] = [x_i,A]BC + A[x_i,B]C + (AB)[x_i,C]$) and $[p_i, x_i^n x_j^m x_k^\ell] = -ni\hbar x_i^{n-1} x_j^m x_k^\ell$, where the relationships $[x_i,p_i^n] = ni\hbar p_i^{n-1}$ and $[p_i,x_i^n] = -ni\hbar x_i^{n-1}$ can be easily proved by mathematical induction. Using the series form for $G(\vec{p})$ and $F(\vec{x})$ we get at once $[x_i,G(\vec{p})] = i\hbar\partial G/\partial p_i$ and $[p_i,F(\vec{x})] = -i\hbar\partial F/\partial x_i$.

(b) $[x^2,p^2] = [x^2,pp] = [x^2,p]p + p[x^2,p]$, but from (a) $[p,x^2] = -2i\hbar x$, so $[x^2,p^2] = 2i\hbar xp + 2i\hbar px = 2i\hbar\{x,p\}$. The classical P.B. for $[x^2,p^2]$ is evaluated via $[x^2,p^2]_{cl} = \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} = 2x(2p) = 4xp$. Since in the classical limit $\{x,p\} = 2xp$, we have $[x^2,p^2]_{QM} = i\hbar[x^2,p^2]_{cl}$.

30. (a) $[x_i,T(\vec{\ell})] = i\hbar\partial T(\vec{\ell})/\partial p_i = i\hbar\frac{\partial}{\partial p_i}\exp(-i\vec{p}.\vec{\ell}/\hbar) = i\hbar(-i\ell_i/\hbar)\exp(-i\vec{p}.\vec{\ell}/\hbar) = \ell_i T(\vec{\ell})$. (b) Noting that $\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle$ where $|\alpha\rangle$ is a general state ket, take expression $\langle\alpha|T^\dagger(\vec{\ell})[x_i,T(\vec{\ell})]|\alpha\rangle = \langle\alpha|T^\dagger(\vec{\ell})\ell_i T(\vec{\ell})|\alpha\rangle = \ell_i$. But we note that $\langle\alpha|T^\dagger(\vec{\ell})[x_i,T(\vec{\ell})]|\alpha\rangle = \langle\alpha|T^\dagger x_i T|\alpha\rangle - \langle\alpha|T^\dagger T x_i|\alpha\rangle$, hence

$$\langle x_i\rangle_{\text{translated}} = \langle x_i\rangle + \ell_i, \text{ and therefore } \langle\vec{x}\rangle_{\text{translated}} = \langle\vec{x}\rangle + \vec{\ell}.$$

31. Given $[\vec{x},T(d\vec{x}')] = d\vec{x}'$ or $\vec{x}T(d\vec{x}') = d\vec{x}' + T(d\vec{x}')\vec{x}$ and $[\vec{p},T(d\vec{x}')] = 0$ or $\vec{p}T(d\vec{x}') = T(d\vec{x}')\vec{p}$, we study $\langle\alpha|T^\dagger(d\vec{x}')\vec{x}T(d\vec{x}')|\alpha\rangle$ substituting as we did in problem 30. We find $\langle\vec{x}\rangle_{\text{translated}} = \langle\alpha|T^\dagger(d\vec{x}')(d\vec{x}' + T(d\vec{x}')\vec{x})|\alpha\rangle$. Now $T^\dagger(d\vec{x}') = 1 + i\vec{K}.d\vec{x}'$, hence $T^\dagger(d\vec{x}')d\vec{x}' = d\vec{x}'$ to first order in small quantity $d\vec{x}'$. Hence $\langle\vec{x}\rangle_{\text{translated}} = d\vec{x}' + \langle\vec{x}\rangle$. Using $\vec{p}T(d\vec{x}') = T(d\vec{x}')\vec{p}$, we find $\langle\vec{p}\rangle_{\text{translated}} = \langle\alpha|T^\dagger(d\vec{x}')\vec{p}T(d\vec{x}')\times|\alpha\rangle = \langle\alpha|T^\dagger(d\vec{x}')T(d\vec{x}')\vec{p}|\alpha\rangle = \langle\alpha|\vec{p}|\alpha\rangle = \langle\vec{p}\rangle$. Hence $\langle\vec{p}\rangle_{\text{translated}} = \langle\vec{p}\rangle$.

32. Use of $\langle x'|\alpha\rangle = \frac{1}{d^{1/2}\pi^{1/4}}\exp(ikx' - x'^2/2d^2)$, we find by elementary calculation

$$\frac{\partial}{\partial x'}\langle x'|\alpha\rangle = \frac{1}{d^{1/2}\pi^{1/4}}(ik - x'/d^2)\exp(ikx' - x'^2/2d^2),\ \frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right) = \frac{\partial^2}{\partial x'^2}(\langle x'|\alpha\rangle)$$

$$= \frac{1}{d^{1/2}\pi^{1/4}}\left[-k^2 - \frac{1}{d^2} - \frac{2ikx'}{d^2} + x'^2/d^4\right]\exp(ikx' - x'^2/2d^2).$$

(a) $\langle p\rangle = \int_{-\infty}^{+\infty}\langle\alpha|x'\rangle[-i\hbar\frac{\partial}{\partial x'}]\langle x'|\alpha\rangle dx' = \frac{-i\hbar}{d\pi^{1/2}}\int_{-\infty}^{+\infty}\exp(-x'^2/d^2)(ik-x'/d^2)dx'$.

The odd term of integrand vanishes, and $\langle p\rangle = [\hbar k/d\pi^{1/2}]\int_{-\infty}^{+\infty}\exp(-x'^2/d^2)dx' = \frac{\hbar k d\pi^{1/2}}{d\pi^{1/2}}$ $= \hbar k$. Likewise $\langle p^2\rangle = \int_{-\infty}^{+\infty}\langle\alpha|x'\rangle p^2\langle x'|\alpha\rangle dx' = \int_{-\infty}^{+\infty}\langle\alpha|x'\rangle(-\hbar^2)\frac{\partial^2}{\partial x'^2}\langle x'|\alpha\rangle dx' =$ $= -\frac{\hbar^2}{d\pi^{1/2}}\int_{-\infty}^{+\infty}\exp(-x^2/d^2)[x^2/d^4 - k^2 - 1/d^2 - 2ikx/d^2]dx = \hbar^2/2d^2 + \hbar^2k^2$, again dropping odd terms in integrand.

(b) $\langle p|\alpha\rangle = \frac{d^{1/2}}{\hbar^{1/2}\pi^{1/4}}\exp[-(p-\hbar k)^2d^2/2\hbar^2]$. The expectation value $\langle p\rangle$ using momentum space wave function is then

$$\langle p\rangle = \int_{-\infty}^{+\infty}\langle\alpha|p\rangle p\langle p|\alpha\rangle dp = \frac{d}{\hbar\pi^{1/2}}\int_{-\infty}^{+\infty}p\exp[-(p-\hbar k)^2d^2/\hbar^2]dp.$$

Change variable to $q = p-\hbar k$, we have $\langle p\rangle = (d/\hbar\pi^{1/2})\int_{-\infty}^{+\infty}(q+\hbar k)\exp[-q^2d^2/\hbar^2]dq$, and dropping the odd integration contribution

$$\langle p\rangle = (d/\hbar\pi^{1/2})\hbar k\,(\hbar\pi^{1/2}/d) = \hbar k.$$

Similarly

$$\langle p^2\rangle = \int_{-\infty}^{+\infty}(d/\hbar\pi^{1/2})p^2\exp[-(p-\hbar k)^2d^2/\hbar^2]dp$$

and changing variable to $q = p-\hbar k$ (hence $p^2 = q^2 + 2q\hbar k + \hbar^2k^2$), we have

$$\langle p^2\rangle = (d/\hbar\pi^{1/2})\int_{-\infty}^{+\infty}(q^2+2\hbar kq+\hbar^2k^2)\exp[-q^2d^2/\hbar^2]dq$$

$$= (d/\hbar\pi^{1/2})[\hbar^3\sqrt{\pi}/2d^3 + \hbar\sqrt{\pi}\,\hbar^2k^2/d] = \hbar^2/2d^2 + \hbar^2k^2.$$

33. (a) To prove (i) $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$, let us note that

$$\langle p'|x|p''\rangle = \int\langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx'$$

$$= [1/(2\pi\hbar)]\int dx'x'e^{-ix'\cdot(p'-p'')/\hbar}.$$

But $\delta(p'-p'') = [1/(2\pi\hbar)]\int dx'e^{-ix'\cdot(p'-p'')/\hbar}$, so $\frac{\partial}{\partial p'}\delta(p'-p'') = \int\frac{dx'}{(2\pi\hbar)}\frac{x'}{\hbar i}e^{\frac{-ix'\cdot(p'-p'')}{\hbar}}$,

hence $\langle p'|x|p''\rangle = i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$. Express now $\langle p'|x|\alpha\rangle = \int dp''\langle p'|x|p''\rangle\langle p''|\alpha\rangle =$ $\int dp''i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\langle p''|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$.

For (ii) we perform an analogous procedure. Write

$$\langle\beta|x|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|x|\alpha\rangle = \int dp'\langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$
$$= \int dp'\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p').$$

(b) Consider momentum eigenket with eigenvalue p'. Then $p|p'\rangle = p'|p'\rangle$. Now consider the ket $|p',\Xi\rangle = \exp[ix\Xi/\hbar]|p'\rangle$. Is this a momentum eigenket and if yes what is the value? To see this let's operate with p, than

$$p|p',\Xi\rangle = p(\exp[ix\Xi/\hbar])|p'\rangle = \{\exp(ix\Xi/\hbar)p + [p, \exp(ix\Xi/\hbar)]\}|p'\rangle$$

and $[p, \exp(ix\Xi/\hbar)] = -i\hbar\partial(\exp[ix\Xi/\hbar])/\partial x = -i\hbar(i\Xi/\hbar)\exp[ix\Xi/\hbar]$. So $p|p',\Xi\rangle = \exp(ix\Xi/\hbar)p'|p'\rangle + \Xi\exp(ix\Xi/\hbar)|p'\rangle = (p' + \Xi)\exp(ix\Xi/\hbar)|p'\rangle = (p' + \Xi)|p',\Xi\rangle$. Hence $|p',\Xi\rangle$ is eigenket of p with eigenvalue $p' + \Xi$ and operator $\exp(ix\Xi/\hbar)$ is momentum translation operator and x is the generator of momentum translations.