**Problem Set 3 223B Condensed Matter Physics, Winter 2017**

Due: Thursday, February 23, 2017 by 11pm
Put homework in mailbox labelled 223B on 1st floor of Broida (by elevators).

1.) ANISOTROPIC ANTIFERROMAGNET

A simple model Hamiltonian for an anisotropic spin-$S$ antiferromagnet is,

$$\hat{\mathcal{H}} = \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} [J(\hat{S}^x_{\mathbf{r}} \hat{S}^x_{\mathbf{r}'} + \hat{S}^y_{\mathbf{r}} \hat{S}^y_{\mathbf{r}'}) + J_z \hat{S}^z_{\mathbf{r}} \hat{S}^z_{\mathbf{r}'}], \tag{1}$$

where the positive exchange couplings $J_z$ and $J$ are unequal. Here $\mathbf{r}$ and $\mathbf{r}'$ label sites of a d-dimensional (hyper-) cubic lattice, and the sum is over near neighbor sites. There are two interesting regimes,

$$(i) \;\; J_z > J \geq 0, \qquad\qquad (ii) \;\; J > J_z \geq 0, \tag{2}$$

corresponding to an easy-axis and an easy-plane magnet, respectively.

A.) Treating the spins as *classical* fixed length vectors, what is the ground state spin configuration for the easy-axis magnet? The easy-plane magnet?

When the spin $S$ is large, one expects the exact ground state to be "close" to the classical ground state. As discussed in class for the Heisenberg model, one can employ the Holstein-Primakoff transformation to systematically examine the quantum fluctuations in the large $S$ limit.

B.) For the large $S$ easy-axis magnet above, use the Holstein-Primakoff transformation to derive an effective magnon Hamiltonian (at leading non-trivial order in $1/S$), and show that it can be diagonalized (with a Bogoluibov transformation) and expressed in the form:

$$\hat{H}_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}, \tag{3}$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are magnon annihilation and creation operators. Sketch the resulting magnon dispersion relation, $\epsilon_{\mathbf{k}}$, (in the one-dimensional case for simplicity). How *many* gapless excitations (Goldstone modes) exist? Why?

C.) Repeat the above analysis for the easy-plane magnet, and sketch the resulting magnon dispersion in this case. [Hint: You might want to appropriately rotate the spins so that the classical ground state consists of all spins aligned along the z-axis.] How *many* Goldstone modes? Why?

D.) Show that the dispersion relations that you have derived for the easy-axis and easy-plane magnet coincide in the Heisenberg limit, $J_z = J$. How *many* Goldstone modes in this case? Why?

E.) Specialize to the XY easy-plane antiferromagnet, with $J_z = 0$. Due to quantum fluctuations the local moment is smaller than the classical value $S$. Using your magnon Hamiltonian, obtain a formula for the suppression of the local moment (to leading order in $1/S$) as an integral over the Brillouin zone. Show that in two-dimensions and at finite temperatures your integral is divergent. What does this imply?

2.) MEAN FIELD THEORY FOR MAGNETS

A.) Generalize the mean field theory described in class for the ferromagnet, to a Heisenberg spin 1/2 antiferromagnet with Hamiltonian,

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j. \tag{4}$$

Here $i$ and $j$ refer to sites of a d-dimensional (hyper-) cubic lattice and the sum is over all neighboring sites. (Hint: Consider the two sub-lattices separately, and introduce a staggered magnetic field which couples with opposite sign to the spins on the two sub-lattices.) Show that the self-consistent equation obtained in class,

$$M = (N/V)(g\mu_B/2) tanh[(g\mu_B/2k_B T)(H + \lambda M)], \tag{5}$$

with $\lambda = (V/N)(zJ/(g\mu_B)^2)$, also holds for the antiferromagnet, *provided* $M$ and $H$ are interpreted as the staggered magnetization and staggered magnetic field, respectively. Here $N$ denotes the number of sites, and $V$ is the volume.

B.) For the $S = 1/2$ ferromagnetic mean field theory, obtain an expression for the magnetization in the vicinity of the transition, $T_c$, and show that $M(T) \sim (T_c - T)^\beta$, with critical exponent $\beta = 1/2$.

C.) Within mean field theory for the $S = 1/2$ ferromagnetic, obtain an expression for $M(H, T_c)$, in the small field limit, and show that $M(H, T_c) \sim H^\delta$. What value do you find for the critical exponent $\delta$?

D.) Finally, calculate the leading low temperature correction to the spontaneous magnetization for the ferromagnet within mean field theory,

$$\Delta M(T) = M(T = 0) - M(T). \tag{6}$$

Why does the result differ from Bloch's $T^{3/2}$ law for $\Delta M(T)$?