Problem 1

Four problems from Chapter 2 of Sakurai/Napolitano, 2nd edition, 2.1, 2.3, 2.9, 2.10

Problem 2: Spin-1/2 particle in a time-dependent magnetic field

A single spin-1/2 particle (quantum two-level system) is subjected to a magnetic field, described by the Hamiltonian,

\[ \hat{H} = \mu \vec{b}(t) \cdot \vec{\sigma}, \]

which varies periodically with time:

\[ \vec{b}(t) \equiv (b_x(t), b_y(t), b_z(t)), \]

with \( b_0, b_1 \) (time independent) constants. Here, the Pauli operators satisfy \([\hat{\sigma}^\alpha, \hat{\sigma}^\beta] = 2i\epsilon_{\alpha\beta\gamma} \hat{\sigma}^\gamma\) with \( \alpha, \beta, \gamma \) running over the three components, \( x, y, z \), where the Einstein summation convention over the repeated index \( \gamma \) is implicit. The Pauli operators square to the identity, \((\hat{\sigma}^\alpha)^2 = 1\) (no summation over \( \alpha \)).

The time-dependent Schrödinger equation takes the form,

\[ i \hbar \frac{\partial}{\partial t} \ket{\Psi(t)} = \hat{H}(t) \ket{\Psi(t)}, \]

where the ket \( \ket{\Psi(t)} \) can be conveniently expanded in the eigenbasis of \( \hat{\sigma}^z \):

\[ \ket{\Psi(t)} = \psi_\uparrow(t) \ket{\uparrow} + \psi_\downarrow(t) \ket{\downarrow}, \]

with \( \hat{\sigma}^z \ket{\uparrow} = (+1) \ket{\uparrow} \) and \( \hat{\sigma}^z \ket{\downarrow} = (-1) \ket{\downarrow} \) The two-complex (c-number) wavefunctions, \( \psi_\uparrow \) and \( \psi_\downarrow \), are conventionally grouped into a two-component vector called a spinor:

\[ \vec{\Psi}(t) = \begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix}^T \]

with the superscript \( T \) denoting transpose.

(a) By projecting into the eigenbasis of \( \hat{\sigma}^z \), re-express the time-dependent Schrödinger equation in matrix-vector form:

\[ i \hbar \frac{\partial}{\partial t} \vec{\Psi}(t) = \hat{H}(t) \vec{\Psi}(t), \]

and obtain an expression for the 2 by 2 matrix \( \hat{H}(t) \).

(b) The time evolution operator can be defined by the relation:

\[ \vec{\Psi}(t) = \hat{U}(t) \vec{\Psi}(0). \]

Construct an explicit expression for the 2 by 2 time evolution matrix, \( \hat{U}(t) \). [Hint: It is convenient to first define a unitarily transformed spinor, \( \vec{\Psi}_0(t) = e^{i\omega t\hat{\sigma}^z/2} \vec{\Psi}(t) \), and show that it satisfies a (modified) Schrödinger equation,

\[ i \hbar \frac{\partial}{\partial t} \vec{\Psi}_0(t) = \hat{H}_0 \vec{\Psi}_0(t), \]

where the effective Hamiltonian \( \hat{H}_0 \) is independent of time.]

(c) Calculate the determinant of the time evolution operator, \( \det \hat{U}(t) \). Is this consistent with the conservation of probability?
(d) Assume that at time $t = 0$ the spin is pointing up, $\vec{\Psi}(0) = (1, 0)^T$. Consider the spin-flip probability, $P_{\text{flip}}(t)$, i.e. the probability that if the $z$-component of spin is measured at time $t$ it will be found pointing down. Compute $P_{\text{flip}}(t)$ as a function of time.

(e) Consider another initial condition, with the spin pointing in the x-direction at $t=0$. Find the exact time-dependent wavefunction, $\vec{\Psi}(t)$, at all times, $t$.

**Problem 3: Two interacting spin-1/2 particles**

Consider two spin-1/2 particles, labelled 1 and 2, each described by a triplet of Pauli operators, $\hat{\sigma}_j^\alpha$ with $\alpha = x, y, z$ labeling the 3 spin-components and $j = 1, 2$ the particle label. These operators satisfy the commutation relations,

$$[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta] = 2i\delta_{ij}\epsilon_{\alpha\beta\gamma}\hat{\sigma}_i^\gamma,$$

(8)

and square to the identity, $(\hat{\sigma}_j^\alpha)^2 = \hat{1}$ (no summation over $\alpha$ or $j$).

The two spins are taken to interact through a time-dependent Hamiltonian of the form,

$$\hat{H} = J(t) \hat{\sigma}_1^z \hat{\sigma}_2^y,$$

(9)

where $J(t)$ is a continuous real function of time that is non-zero only for $0 \leq t \leq t_f$ and satisfies $\int_0^{t_f} dt J(t) = \pi\hbar/4$.

Assume that at $t=0$ the first spin is in the state $(\psi_{1\up}, \psi_{1\down}) = (a, b)$ with complex numbers $a,b$ satisfying $|a|^2 + |b|^2 = 1$, and the second spin is pointing up, $(\psi_{2\up}, \psi_{2\down}) = (1, 0)$. Find the state vector for the two spin-1/2 particles at time $t_f$. 